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EXPANDING AND SOLVING ERDÖS-MORDELL'S GEOMETRIC EXTREMUM PROBLEM

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ABSTRACT. After transforming Erdös-Mordell's geometric inequality problem into a geometric extremum problem, we will expand and solve this extremum problem.

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1. INTRODUCTION

In 1935, the hungarian mathematician Erdös introduced a very intriguing geometric inequality problem [1].

Problem 1. *Let point M lie inside triangle ABC. H, K, L are the orthogonal projections of M on BC, CA, AB respectively. Prove that*

$$MA + MB + MC \geq 2\left(MH + MK + ML\right).$$

An initial solution to problem 1 was proposed by Louis Mordell in the same year [1]. After that, a series of other solutions are presented, such as Leo Bankoff's [2] using angular calculations through similar triangles, André Avez's [3] and Hojoo Lee [5] using Ptolemy theorem, V.Komornik's [4] using area inequality.
With regard to the inequality $MA + MB + MC \geq 2\left(MH + MK + ML\right)$, equality occurs if and only if triangle ABC is equilateral and M is its center. This means that M can only be defined such that $MA + MB + MC - 2\left(MH + MK + ML\right)$ is minimal if triangle ABC is equilateral. Naturally, the following geometric extremum problem is put out.

Problem 2. *Let M be a point inside triangle ABC. H, K, L are the orthogonal projections of M on BC, CA, AB respectively. Find a point M such that $MA + MB + MC - 2\left(MH + MK + ML\right)$ is minimal.*

Problem 2 is called Erdös-Mordell's geometric extremum problem. Owing to the concept of *signed* distance, problem 2 can be expanded as follow.

Problem 3. *Given a triangle ABC and a point M, denote H, K, L as the orthogonal projections of M on BC, CA, AB respectively. Find a point M such that*
$MA + MB + MC - 2\left(\overline{MH} + \overline{MK} + \overline{ML}\right)$ is minimal.

In problem 3, $\overline{MH}$, $\overline{MK}$, $\overline{ML}$ are used to denote the *signed* distances from M to lines BC, CA, AB respectively.
If M belongs to the half-plane with boundary line BC that contains A, then $\overline{MH} = MH$; if M belongs to the half-plane with boundary line BC that does not contain A, then $\overline{MH} = -MH$; if M belongs to line BC, then $\overline{MH} = 0$.
If M belongs to the half-plane with boundary line CA that contains B, then $\overline{MK} = MK$; if M belongs to the the half-plane with boundary line CA that does not contain B, then $\overline{MK} = -MK$; if M belongs to line CA, then $\overline{MK} = 0$.
If M belongs to the half-plane with boundary line AB that contains C, then $\overline{ML} = ML$; if M belongs to the half-plane with boundary line AB that does not contain C, then $\overline{ML} = -ML$; if M belongs to line AB, then $\overline{ML} = 0$.
The aim of this article is to solve problem 3.

2. Denotations and Lemmas

In the whole of this article:
· For a triangle ABC, α, β, γ are always the measures of angles $\widehat{BAC}$, $\widehat{CBA}$, $\widehat{ACB}$; (I) is always its inscribed circle; r is always the radius of (I); $D, E, F$are always the points of contact between (I) and sides BC, CA, AB respectively; Q is always the orthocenter of triangle DEF.
· Vector of unit$\dfrac{\overrightarrow{XY}}{XY}$ is denoted by $\mathbf{y_X}$.
· Vector $\overrightarrow{0}$ is denoted by $\mathbf{0}$.
· The similar direction of two vectors is denoted by $\uparrow\uparrow$.
· The opposite direction of two vectors is denoted by $\uparrow\downarrow$.
· The writing of sums is abbreviated:

$$\begin{aligned}
&\sum \cos\alpha = \cos\alpha + \cos\beta + \cos\gamma;\\
&\sum \sin\alpha = \sin\alpha + \sin\beta + \sin\gamma;\\
&\sum \sin^2\frac{\beta+\gamma}{2} = \sin^2\frac{\beta+\gamma}{2} + \sin^2\frac{\gamma+\alpha}{2} + \sin^2\frac{\alpha+\beta}{2};\\
&\sum \overrightarrow{ID} = \overrightarrow{ID} + \overrightarrow{IE} + \overrightarrow{IF};\\
&\sum EF^2 = EF^2 + FD^2 + DE^2.
\end{aligned}$$

In order to solve problem 3, we need a few lemmas as follows.

Lemma 1. *Given a triangle ABC and a point M, denote H, K, L as the orthogonal projections of M on BC, CA, AB respectively. Let*
$f(M) = MA + MB + MC - 2\left(\overline{MH} + \overline{MK} + \overline{ML}\right)$. *If* $|-(\mathbf{b_A} + \mathbf{c_A}) + 2(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I})| \le 1$, *then $f(M) \ge f(A)$. Equality occurs if and only if M coincides with A.*

Proof of lemma 1. Because $\overrightarrow{AC} \perp \mathbf{e_I}$; $\overrightarrow{AB} \perp \mathbf{f_I}$ and $|-(\mathbf{b_A} + \mathbf{c_A}) + 2(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I})| \le 1$,

$$\begin{aligned}
f(M) = &\ MA + MB + MC - 2\left(\overline{MH} + \overline{MK} + \overline{ML}\right) \\
= &\ MA + MB + MC - \left(\overrightarrow{MB} + \overrightarrow{MC}\right).\mathbf{d_I} - \left(\overrightarrow{MC} + \overrightarrow{MA}\right).\mathbf{e_I} - \left(\overrightarrow{MA} + \overrightarrow{MB}\right).\mathbf{f_I} \\
= &\ MA + \frac{MB.AB}{AB} + \frac{MC.AC}{AC} \\
&- \left(\overrightarrow{MB} + \overrightarrow{MC}\right).\mathbf{d_I} - \left(\overrightarrow{MC} + \overrightarrow{MA}\right).\mathbf{e_I} - \left(\overrightarrow{MA} + \overrightarrow{MB}\right).\mathbf{f_I} \\
\geq &\ MA + \frac{\overrightarrow{MB}.\overrightarrow{AB}}{AB} + \frac{\overrightarrow{MC}.\overrightarrow{AC}}{AC} \\
&- \left(\overrightarrow{MB} + \overrightarrow{MC}\right).\mathbf{d_I} - \left(\overrightarrow{MC} + \overrightarrow{MA}\right).\mathbf{e_I} - \left(\overrightarrow{MA} + \overrightarrow{MB}\right).\mathbf{f_I} \\
= &\ MA + \frac{\left(\overrightarrow{MA} + \overrightarrow{AB}\right).\overrightarrow{AB}}{AB} + \frac{\left(\overrightarrow{MA} + \overrightarrow{AC}\right).\overrightarrow{AC}}{AC} \\
&- \left(2\overrightarrow{MA} + \overrightarrow{AB} + \overrightarrow{AC}\right).\mathbf{d_I} - \left(2\overrightarrow{MA} + \overrightarrow{AC}\right).\mathbf{e_I} - \left(2\overrightarrow{MA} + \overrightarrow{AB}\right).\mathbf{f_I} \\
= &-\overrightarrow{MA}\left(-\left(\mathbf{b_A} + \mathbf{c_A}\right) + 2\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right)\right) + MA + AB + AC - \left(\overrightarrow{AB} + \overrightarrow{AC}\right).\mathbf{d_I} \\
\geq &- MA + MA + AB + AC - \left(\overrightarrow{AB} + \overrightarrow{AC}\right).\mathbf{d_I} \\
= &\ AB + AC - \left(\overrightarrow{AB} + \overrightarrow{AC}\right).\mathbf{d_I} \\
= &\ f(A).
\end{aligned}$$

Hence, $f(M) \geq f(A)$.

Equality occurs if and only if $\begin{cases} \overrightarrow{MB} \uparrow\uparrow \overrightarrow{AB} \\ \overrightarrow{MC} \uparrow\uparrow \overrightarrow{AC} \end{cases}$.

This means that M coincides with A. □

Lemma 2. *If $\alpha \geq \beta \geq \gamma$, then* $\left|-\left(\mathbf{c_B} + \mathbf{a_B}\right) + 2\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right)\right| > 1$.

Proof of lemma 2. Consider the function $g(x) = \cos x + 6\sin x$ defined on range $(0, \pi)$. We have $g'(x) = \sin x\,(-1 + 6\cot x)$.

Deduce that $g'(x)$ is positive in the range of $(0, \operatorname{arccot}\frac{1}{6})$, and it is negative in the range of $(\operatorname{arccot}\frac{1}{6}, \pi)$.

Therefore, $g(x)$ monotonically increases in the range of $(0, \operatorname{arccot}\frac{1}{6})$, and it monotonically decreases in the range of $(\operatorname{arccot}\frac{1}{6}, \pi)$.

On the other hand, we have

$$\begin{aligned}
&\left|-\left(\mathbf{b_A} + \mathbf{c_A}\right) + 2\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right)\right|^2 \\
&= \left(\mathbf{b_A} + \mathbf{c_A}\right)^2 + 4\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right)^2 - 4\left(\mathbf{b_A} + \mathbf{c_A}\right)\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right) \\
&= 2 + 2\cos\alpha + 4\left(3 - 2\left(\cos\alpha + \cos\beta + \cos\gamma\right)\right) - 4\left(\sin\beta - \sin\alpha + \sin\gamma - \sin\alpha\right) \\
&= 14 - 6\cos\alpha - 8\cos\beta - 8\cos\gamma + 8\sin\alpha - 4\sin\beta - 4\sin\gamma \\
&= 2g(\alpha) + 14 - 8\sum\cos\alpha - 4\sum\sin\alpha.
\end{aligned}$$

Similarly, $\left|-\left(\mathbf{c_B} + \mathbf{a_B}\right) + 2\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right)\right|^2 = 2g(\beta) + 14 - 8\sum\cos\alpha - 4\sum\sin\alpha$;

$$\left|-\left(\mathbf{a_C} + \mathbf{b_C}\right) + 2\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right)\right|^2 = 2g(\gamma) + 14 - 8\sum\cos\alpha - 4\sum\sin\alpha.$$

Hence, if β lies in the range of $(0, \text{arccot}\,\frac{1}{6})$, then $g(\beta) > g(\gamma)$; if β lies in the range of $(\text{arccot}\,\frac{1}{6}, \pi)$, then $g(\beta) > g(\alpha)$.

Deduce that $\left[\begin{array}{l} |-(\mathbf{c_B}+\mathbf{a_B})+2(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I})| \geq |-(\mathbf{a_C}+\mathbf{b_C})+2(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I})| \\ |-(\mathbf{c_B}+\mathbf{a_B})+2(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I})| \geq |-(\mathbf{b_A}+\mathbf{c_A})+2(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I})| \end{array} \right.$.

Therefore, if $|-(\mathbf{c_B}+\mathbf{a_B})+2(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I})| \leq 1$, then according to lemma 1, either B coincides with C or B coincides with A, contradiction.

Thus, $|-(\mathbf{c_B}+\mathbf{a_B})+2(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I})| > 1$.

□

Lemma 3. *Given a triangle ABC and a point M, denote H, K, L as the orthogonal projections of M on BC, CA, AB respectively. Let*

$$f(M) = MA + MB + MC - 2\left(\overline{MH} + \overline{MK} + \overline{ML}\right).$$

1) If M differs from A, B, C, then $\text{grad}\, f(M) = -(\mathbf{a_M}+\mathbf{b_M}+\mathbf{c_M})+2(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I})$.

2) If M_0 differs from A, B, C and $\text{grad}\, f(M_0) = \mathbf{0}$, *then $f(M) \geq f(M_0)\ \forall M$. Equality occurs if and only if M coincides with M_0.*

Proof of lemma 3. 1) Similar to the proof of lemma 1, we have

$$\begin{aligned} f(M) &= MA + MB + MC - 2\left(\overline{MH} + \overline{MK} + \overline{ML}\right) \\ &= MA + MB + MC - \left(\overrightarrow{MB} + \overrightarrow{MC}\right).\mathbf{d_I} + \left(\overrightarrow{MC} + \overrightarrow{MA}\right).\mathbf{e_I} + \left(\overrightarrow{MA} + \overrightarrow{MB}\right).\mathbf{f_I} \end{aligned}$$

From this, after a few simple calculations, we have:

$$\text{grad}\, f(M) = -(\mathbf{a_M}+\mathbf{b_M}+\mathbf{c_M})+2(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I}).$$

2) Because $\text{grad}\, f(M_0) = \mathbf{0}$, according to part 1, $-(\mathbf{a_{M_0}}+\mathbf{b_{M_0}}+\mathbf{c_{M_0}})+2(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I}) = \mathbf{0}$. From this, similar to the proof of lemma 1, we have

$$\begin{aligned} f(M) &= MA + MB + MC - 2\left(\overline{MH} + \overline{MK} + \overline{ML}\right) \\ &= MA + MB + MC - \left(\overrightarrow{MB} + \overrightarrow{MC}\right).\mathbf{d_I} - \left(\overrightarrow{MC} + \overrightarrow{MA}\right).\mathbf{e_I} - \left(\overrightarrow{MA} + \overrightarrow{MB}\right).\mathbf{f_I} \\ &= \frac{MA.M_0A}{M_0A} + \frac{MB.M_0B}{M_0B} + \frac{MC.M_0C}{M_0C} \\ &\quad - \left(\overrightarrow{MB} + \overrightarrow{MC}\right).\mathbf{d_I} - \left(\overrightarrow{MC} + \overrightarrow{MA}\right).\mathbf{e_I} - \left(\overrightarrow{MA} + \overrightarrow{MB}\right).\mathbf{f_I} \\ &\geq \frac{\overrightarrow{MA}.\overrightarrow{M_0A}}{M_0A} + \frac{\overrightarrow{MB}.\overrightarrow{M_0B}}{M_0B} + \frac{\overrightarrow{MC}.\overrightarrow{M_0C}}{M_0C} \\ &\quad - \left(\overrightarrow{MB} + \overrightarrow{MC}\right).\mathbf{d_I} - \left(\overrightarrow{MC} + \overrightarrow{MA}\right).\mathbf{e_I} - \left(\overrightarrow{MA} + \overrightarrow{MB}\right).\mathbf{f_I} \\ &= \frac{\left(\overrightarrow{MM_0} + \overrightarrow{M_0A}\right).\overrightarrow{M_0A}}{M_0A} + \frac{\left(\overrightarrow{MM_0} + \overrightarrow{M_0B}\right).\overrightarrow{M_0B}}{M_0B} + \frac{\left(\overrightarrow{MM_0} + \overrightarrow{M_0C}\right).\overrightarrow{M_0C}}{M_0C} \\ &\quad - \left(2\overrightarrow{MM_0} + \overrightarrow{M_0B} + \overrightarrow{M_0C}\right).\mathbf{d_I} \\ &\quad - \left(2\overrightarrow{MM_0} + \overrightarrow{M_0C} + \overrightarrow{M_0A}\right).\mathbf{e_I} \\ &\quad - \left(2\overrightarrow{MM_0} + \overrightarrow{M_0A} + \overrightarrow{M_0B}\right).\mathbf{f_I} \\ &= -\overrightarrow{MM_0}\left(-(\mathbf{a_{M_0}}+\mathbf{b_{M_0}}+\mathbf{c_{M_0}})+2(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I})\right) \\ &\quad + M_0A + M_0B + M_0C - \left(\overrightarrow{M_0B} + \overrightarrow{M_0C}\right).\mathbf{d_I} - \left(\overrightarrow{M_0C} + \overrightarrow{M_0A}\right).\mathbf{e_I} - \left(\overrightarrow{M_0A} + \overrightarrow{M_0B}\right).\mathbf{f_I} \\ &= f(M_0). \end{aligned}$$

Thus, $f(M) \geq f(M_0)$.

Equality occurs if and only if $\begin{cases} \overrightarrow{MA} \uparrow\uparrow \overrightarrow{M_0A} \\ \overrightarrow{MB} \uparrow\uparrow \overrightarrow{M_0B} \\ \overrightarrow{MC} \uparrow\uparrow \overrightarrow{M_0C} \end{cases}$

This means that M coincides with M_0.

□

Lemma 4. *Given a triangle ABC and a point M, denote H, K, L as the orthogonal projections of M on BC, CA, AB respectively. Let $f(M) = MA + MB + MC - 2\left(\overline{MH} + \overline{MK} + \overline{ML}\right)$. M_0 belongs to line segment BC and differs from B and C. Then,* $\operatorname{grad} f(M_0) = \mathbf{0}$ *if and only if $\overrightarrow{M_0A} \uparrow\uparrow \overrightarrow{IQ}$ and $\sum \cos\alpha = \frac{11}{8}$.*

Proof of lemma 4. Because M_0 belongs to segment BC and differs from B and C, $\mathbf{b_{M_0}} \uparrow\downarrow \mathbf{c_{M_0}}$. Then, noting that $|\mathbf{b_{M_0}}| = |\mathbf{c_{M_0}}|$, we can deduce that $\mathbf{b_{M_0}} + \mathbf{c_{M_0}} = \mathbf{0}$ (1).

Because (I) and Q are the circumscribed circle and the orthocenter of triangle DEF respectively, in accordance with familiar formulas, $\overrightarrow{IQ} = \sum \overrightarrow{ID}$ (2) and $IQ^2 = 9r^2 - \sum EF^2$ (3).

Because quadrilaterals $AFIE, BDIF, CEID$ are cyclic,

$$\widehat{EDF} = \frac{\beta+\gamma}{2};\ \widehat{FED} = \frac{\gamma+\alpha}{2};\ \widehat{DFE} = \frac{\alpha+\beta}{2}\ (4).$$

Thus, the following conditions are equivalent.

1) $\operatorname{grad} f(M_0) = \mathbf{0}$.

2) $-\left(\mathbf{a_{M_0}} + \mathbf{b_{M_0}} + \mathbf{c_{M_0}}\right) + 2\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right) = \mathbf{0}$.

3) $\mathbf{a_{M_0}} = 2\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right)$.

4) $\overrightarrow{M_0A} \uparrow\uparrow \overrightarrow{IQ}$ and $r = 2IQ$.

5) $\overrightarrow{M_0A} \uparrow\uparrow \overrightarrow{IQ}$ and $r^2 = 4\left(9r^2 - \sum EF^2\right)$.

6) $\overrightarrow{M_0A} \uparrow\uparrow \overrightarrow{IQ}$ and $\sum \sin^2 \frac{\beta+\gamma}{2} = \frac{35}{16}$.

7) $\overrightarrow{M_0A} \uparrow\uparrow \overrightarrow{IQ}$ and $\sum \cos\alpha = \frac{11}{8}$.

Note that, according to lemma 3, 1 ⇔ 2; because of (1), 2 ⇔ 3; because of (2), 3 ⇔ 4; because of (3), 4 ⇔ 5; because of (4) and law of sine, 5 ⇔ 6; and obviously, 6 ⇔ 7.

□

3. Solution to Problem 3

Let $f(M) = MA + MB + MC - 2\left(\overline{MH} + \overline{MK} + \overline{ML}\right)$.

It is easy to see that f is a continuous function.

Denote the set of the vertices of triangle ABC as Ω_0, the set of points on the sides of triangles ABC as Ω_1, the set of points in triangle ABC as Ω_2.

Obviously, $\Omega_0 \subset \Omega_1 \subset \Omega_2$.

Without the loss of generality, assume that $\alpha \geq \beta \geq \gamma$.

According to lemma 2, $\left|-\left(\mathbf{c_B} + \mathbf{a_B}\right) + 2\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right)\right| > 1$.

Therefore, there are three cases to consider.

Case 1. $\left|-\left(\mathbf{b_A} + \mathbf{c_A}\right) + 2\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right)\right| \leq 1$.

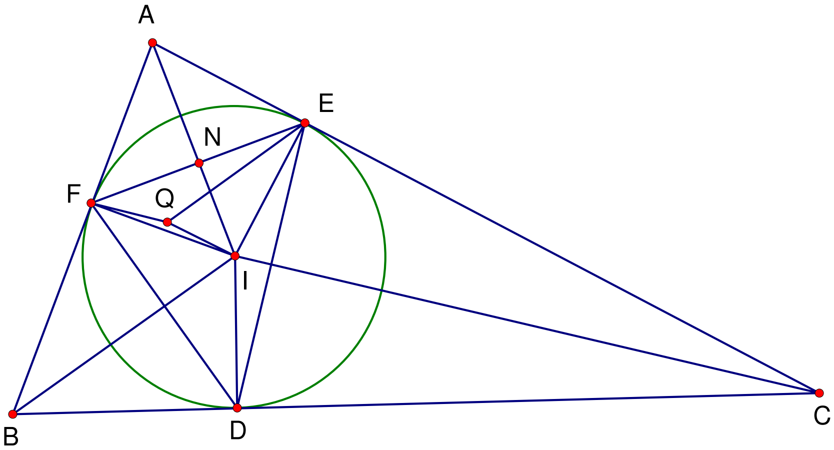
According to lemma 1, $f(M)$ is minimal if and only if M coincides with A.

Case 2. $\left|-\left(\mathbf{a_C}+\mathbf{b_C}\right)+2\left(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I}\right)\right| \leq 1$.
According to lemma 1, $f(M)$is minimal if and only if M coincides with C.
Cases 3. $\begin{cases} \left|-\left(\mathbf{b_A}+\mathbf{c_A}\right)+2\left(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I}\right)\right| > 1 \\ \left|-\left(\mathbf{a_C}+\mathbf{b_C}\right)+2\left(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I}\right)\right| > 1 \end{cases}$.
There are two sub-cases here.
Subcase 3.1. $\sum \cos\alpha \neq \frac{11}{8}$.
According to a well-known theorem of Weierstrass, continuous function $f(M)$ restricted to Ω_2 must have a minimal value at a point M_0 of Ω_2.
Let $\mathbf{v} = \dfrac{-\left(\mathbf{b_A}+\mathbf{c_A}\right)+2\left(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I}\right)}{\left|-\left(\mathbf{b_A}+\mathbf{c_A}\right)+2\left(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I}\right)\right|}$.
Consider the directional derivative of $f(M)$ in direction $\mathbf{v}$ at A.
Let $f_A(M) = MA - \overrightarrow{MA}\left(\mathbf{e_I}+\mathbf{f_I}\right)$;
$f_B(M) = MB - \overrightarrow{MB}\left(\mathbf{f_I}+\mathbf{d_I}\right)$;
$f_C(M) = MC - \overrightarrow{MC}\left(\mathbf{d_I}+\mathbf{e_I}\right)$.
It is easy to see that:
grad $f_B(A) = -\mathbf{b_A}+\mathbf{f_I}+\mathbf{d_I}$;
grad $f_C(A) = -\mathbf{c_A}+\mathbf{d_I}+\mathbf{e_I}$.
Hence, noting that $f(M) = f_A(M) + f_B(M) + f_C(M)$, we have

$$\begin{aligned} f'_{\mathbf{v}}(A) &= 1-\mathbf{v}.\left(\mathbf{e_I}+\mathbf{f_I}\right)+\mathbf{v}.\,\mathrm{grad}\, f_B(A)+\mathbf{v}.\,\mathrm{grad}\, f_C(A) \\ &= 1-\mathbf{v}.\left(-\left(\mathbf{b_A}+\mathbf{c_A}\right)+2\left(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I}\right)\right) \\ &= 1-\frac{\left(-\left(\mathbf{b_A}+\mathbf{c_A}\right)+2\left(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I}\right)\right)^2}{\left|-\left(\mathbf{b_A}+\mathbf{c_A}\right)+2\left(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I}\right)\right|} \\ &= 1-\left|-\left(\mathbf{b_A}+\mathbf{c_A}\right)+2\left(\mathbf{d_I}+\mathbf{e_I}+\mathbf{f_I}\right)\right| < 0. \end{aligned}$$

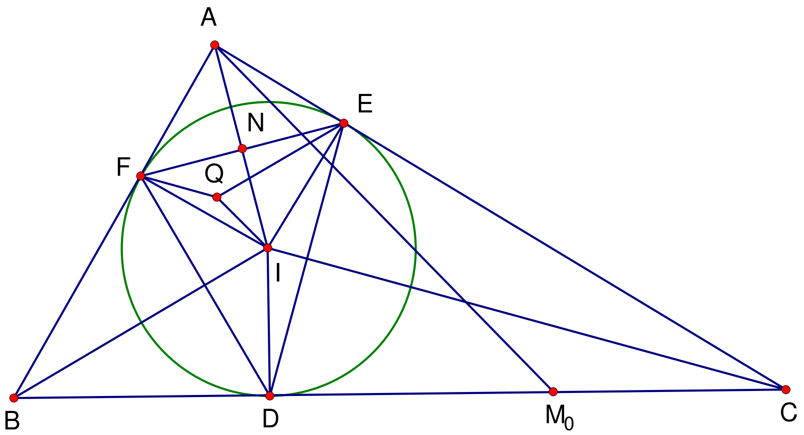
Therefore, $M_0 \neq A$.
Likewise, $M_0 \neq B$ and $M_0 \neq C$.
Hence, $M_0 \notin \Omega_0$.
Because $\sum \cos\alpha \neq \frac{11}{8}$, according to lemma 4, $M_0 \notin \Omega_1 \backslash \Omega_0$.
Therefore, $M_0 \in \Omega_2 \backslash \Omega_1$.
According to the standard theory of extremum values, M_0 must be the stationary point of f(M) at which grad $f(M_0) = \mathbf{0}$.
According to lemma 3, $f(M) \geq f(M_0)$.
Equality occurs if and only if M coincides with M_0.
Hence, $f(M)$ is minimal if and only if M coincides with M_0.
Subcase 3.2. $\sum \cos\alpha = \frac{11}{8}$.
Let N be the midpoint of EF (f.1, f.2, f.3).
Because $\alpha \geq \beta \geq \gamma$, $90^\circ - \frac{\beta+\gamma}{2} \geq 90^\circ - \frac{\alpha+\beta}{2}$; $90^\circ - \frac{\beta+\gamma}{2} \geq 90^\circ - \frac{\gamma+\alpha}{2}$.
On the other hand, it is easy to see that $\widehat{IEF} = 90^\circ - \frac{\beta+\gamma}{2}$; $\widehat{QEF} = 90^\circ - \frac{\alpha+\beta}{2}$; $\widehat{IFE} = 90^\circ - \frac{\beta+\gamma}{2}$; $\widehat{QFE} = 90^\circ - \frac{\gamma+\alpha}{2}$.
Hence, $\widehat{IEF} \geq \widehat{QEF}$; $\widehat{IFE} \geq \widehat{QFE}$.
Therefore, Q belongs to triangle IEF.
Together with $\beta \geq \gamma$, deduce that Q belongs to right triangle INF (1).

If $\cos\beta = \dfrac{3}{8}$, then $\cos\alpha + \cos\gamma = 1$ $(f.1)$.



(Figure 1.)

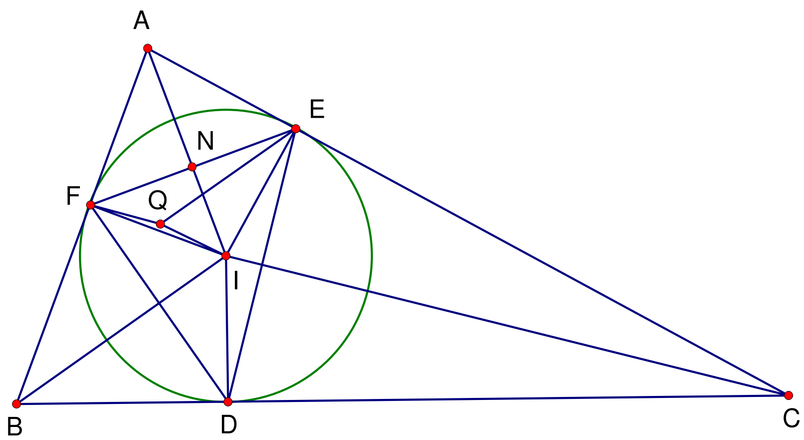
Deduce that $\overrightarrow{IQ}.\overrightarrow{IE} = r^2 \left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right).\mathbf{e_I} = r^2 \left(-\cos\gamma + 1 - \cos\alpha\right) = 0$.
Therefore $\widehat{QIE} = 90°$.
Together with (1), we have $\overrightarrow{IQ} \uparrow\uparrow \overrightarrow{CA}$.
This means that $2\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right) \uparrow\uparrow \mathbf{a_C}$.
On the other hand, because $\sum \cos\alpha = \dfrac{11}{8}$, similar to the proof of lemma 4, we have $2IQ = r$.
To put it another way, $\left|2\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right)\right| = \left|\mathbf{a_C}\right|$.
In short, $2\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right) = \mathbf{a_C}$.
Therefore, $\left|-\left(\mathbf{a_C} + \mathbf{b_C}\right) + 2\left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I}\right)\right| = \left|-\mathbf{b_C}\right| = 1$, contradiction.
Hence, there are two sub-subcases to consider.
Sub-subcase 3.2.1. $\cos\beta > \dfrac{3}{8}$ $(f.2)$.



(Figure 2.)

Because $\sum \cos\alpha = \dfrac{11}{8}$ and $\cos\beta > \dfrac{3}{8}$, $\cos\alpha + \cos\gamma < 1$.

Deduce that $\overrightarrow{IQ}.\overrightarrow{IE} = r^2 \left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I} \right) .\mathbf{e_I} = r^2 \left(- \cos\gamma + 1 - \cos\alpha \right) > 0$.
Therefore, $\widehat{QIE} < 90^\circ$ (2).
From (1) and (2), deduce that there is a point M_0 on segment DC such that $\overrightarrow{M_0A}$ shares the same direction with $\overrightarrow{IQ}$.
According to lemma 4, grad $f(M_0) = \mathbf{0}$.
According to lemma 3, $f(M) \geq f(M_0)$.
Equality occurs if and only if M coincides with M_0.
Hence, $f(M)$is minimal if and only if M coincides with M_0.
Sub-subcase 3.2.2. $\cos\beta < \frac{3}{8}$ $(f.3)$.
Similar to subcase 3.1, continuous function f(M) restricted on Ω_2 must have a minimal value at a point M_0 belonging to Ω_2 and $M_0 \notin \Omega_0$.
From (1), it is easy to deduce that there does not exist any point M on segment CA such that $\overrightarrow{MB}$ shares the same direction with $\overrightarrow{IQ}$ and there does not exist any point M on segment AB such that $\overrightarrow{MC}$ shares the same direction with $\overrightarrow{IQ}$.
Therefore, according to lemma 4, M_0 does not belong to segments CA and AB.
On the other hand, because $\sum \cos\alpha = \frac{11}{8}$ and $\cos\beta < \frac{3}{8}$, $\cos\alpha + \cos\gamma > 1$.
Deduce that $\overrightarrow{IQ}.\overrightarrow{IE} = r^2 \left(\mathbf{d_I} + \mathbf{e_I} + \mathbf{f_I} \right) .\mathbf{e_I} = r^2 \left(- \cos\gamma + 1 - \cos\alpha \right) < 0$.
Therefore, $\widehat{QIE} > 90^\circ$ (3).



(Figure 3.)

From (1) and (3), deduce that there does not exist any point M on segment BC such that $\overrightarrow{MA}$ shares with the same direction with $\overrightarrow{IQ}$.
Therefore, according to lemma 4, M_0 does not belong to segment BC.
In short, $M_0 \notin \Omega_1 \backslash \Omega_0$.
Hence, $M_0 \in \Omega_2 \backslash \Omega_1$.
Similar to subcase 3.1, M_0 must be the stationary point of f(M) at wich grad $f(M_0) = \mathbf{0}$.
According to lemma 3, $f(M) \geq f(M_0)$.
Equality occurs if and only if M coincides with M_0.
Hence, $f(M)$ is minimal if and only if M coincides with M_0.

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