EXPANDING AND SOLVING ERDÖS-MORDELL’S GEOMETRIC EXTREMUM PROBLEM

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ABSTRACT. After transforming Erdös-Mordell’s geometric inequality problem into a geometric extremum problem, we will expand and solve this extremum problem.

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1. INTRODUCTION

In 1935, the hungarian mathematician Erdös introduced a very intriguing geometric inequality problem [1].

Problem 1. Let point M lie inside triangle ABC. H, K, L are the orthogonal projections of M on BC, CA, AB respectively. Prove that

\[ MA + MB + MC \geq 2(MH + MK + ML). \]

An initial solution to problem 1 was proposed by Louis Mordell in the same year [1]. After that, a series of other solutions are presented, such as Leo Bankoff’s [2] using angular calculations through similar triangles, André Avez’s [3] and Hojoo Lee [5] using Ptolemy theorem, V. Komornik’s [4] using area inequality. With regard to the inequality \( MA + MB + MC \geq 2(MH + MK + ML) \), equality occurs if and only if triangle ABC is equilateral and M is its center. This means that M can only be defined such that \( MA + MB + MC - 2(MH + MK + ML) \) is minimal if triangle ABC is equilateral. Naturally, the following geometric extremum problem is put out.

Problem 2. Let M be a point inside triangle ABC. H, K, L are the orthogonal projections of M on BC, CA, AB respectively. Find a point M such that \( MA + MB + MC - 2(MH + MK + ML) \) is minimal.

Problem 2 is called Erdös-Mordell’s geometric extremum problem. Owing to the concept of signed distance, problem 2 can be expanded as follow.

Problem 3. Given a triangle ABC and a point M, denote \( H, K, L \) as the orthogonal projections of M on BC, CA, AB respectively. Find a point M such that \( MA + MB + MC - 2(\overline{MH} + \overline{MK} + \overline{ML}) \) is minimal.
In problem 3, $MH, MK, ML$ are used to denote the signed distances from $M$ to lines $BC, CA, AB$ respectively.

If $M$ belongs to the half-plane with boundary line $BC$ that contains $A$, then $MH = MH$; if $M$ belongs to the half-plane with boundary line $BC$ that does not contain $A$, then $MH = -MH$; if $M$ belongs to line $BC$, then $MH = 0$.

If $M$ belongs to the half-plane with boundary line $CA$ that contains $B$, then $MK = MK$; if $M$ belongs to the half-plane with boundary line $CA$ that does not contain $B$, then $MK = -MK$; if $M$ belongs to line $CA$, then $MK = 0$.

If $M$ belongs to the half-plane with boundary line $AB$ that contains $C$, then $ML = ML$; if $M$ belongs to the half-plane with boundary line $AB$ that does not contain $C$, then $ML = -ML$; if $M$ belongs to line $AB$, then $ML = 0$.

The aim of this article is to solve problem 3.

2. Denotations and Lemmas

In the whole of this article:

- For a triangle $ABC$, $\alpha, \beta, \gamma$ are always the measures of angles $\hat{BAC}, \hat{CBA}, \hat{ACB}$; $(I)$ is always its inscribed circle; $r$ is always the radius of $(I)$; $D, E, F$ are always the points of contact between $(I)$ and sides $BC, CA, AB$ respectively; $Q$ is always the orthocenter of triangle $DEF$.

- Vector of unit $\overrightarrow{XY}$ is denoted by $y_X$.

- Vector $\overrightarrow{0}$ is denoted by $0$.

- The similar direction of two vectors is denoted by $\uparrow\uparrow$.

- The opposite direction of two vectors is denoted by $\uparrow\downarrow$.

- The writing of sums is abbreviated:

\[
\sum \cos \alpha = \cos \alpha + \cos \beta + \cos \gamma;
\]
\[
\sum \sin \alpha = \sin \alpha + \sin \beta + \sin \gamma;
\]
\[
\sum \sin^2 \frac{\beta + \gamma}{2} = \sin^2 \frac{\beta + \gamma}{2} + \sin^2 \frac{\gamma + \alpha}{2} + \sin^2 \frac{\alpha + \beta}{2};
\]
\[
\sum \overrightarrow{ID} = \overrightarrow{ID} + \overrightarrow{IE} + \overrightarrow{IF};
\]
\[
\sum \overrightarrow{EF}^2 = \overrightarrow{EF}^2 + \overrightarrow{FD}^2 + \overrightarrow{DE}^2.
\]

In order to solve problem 3, we need a few lemmas as follows.

Lemma 1. Given a triangle $ABC$ and a point $M$, denote $H, K, L$ as the orthogonal projections of $M$ on $BC, CA, AB$ respectively. Let

$f(M) = MA + MB + MC - 2(MH + MK + ML)$. If $|-(b_A + c_A) + 2(d_1 + e_1 + f_1)| \leq 1$, then $f(M) \geq f(A)$. Equality occurs if and only if $M$ coincides with $A$.

Proof of lemma 1. Because $\overrightarrow{AC} \perp e_1$; $\overrightarrow{AB} \perp f_1$ and $|-(b_A + c_A) + 2(d_1 + e_1 + f_1)| \leq 1$, 

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Deduce that we have

\[ f(M) = MA + MB + MC - 2(MH + MK + ML) \]

\[ = MA + MB + MC - \left(MB + MC \right) \cdot d_1 - \left(MC + MA\right) \cdot e_1 - \left(MA + MB\right) \cdot f_1 \]

\[ = MA + \frac{MB \cdot AB}{AB} + MC \cdot AC \]

\[ - \left(MB + MC \right) \cdot d_1 - \left(MC + MA\right) \cdot e_1 - \left(MA + MB\right) \cdot f_1 \]

\[ \geq MA + \frac{MB \cdot AB}{AB} + \frac{AC \cdot MB + MC - MA}{d_1} \cdot \left(MA + AB \right) \cdot \left(A\right) - \left(MA + AB \right) \cdot \left(A\right) \cdot d_1 \]

\[ = AB + AC - \left(\frac{MA + AB}{AC}\right) \cdot d_1 \]

\[ = f(A). \]

Hence, \( f(M) \geq f(A). \)

Equality occurs if and only if \( \frac{MB}{MC} \uparrow \uparrow \frac{AB}{AC} \).

This means that \( M \) coincides with \( A \). \( \square \)

**Lemma 2.** If \( \alpha \geq \beta \geq \gamma \), then \( \left( c_B + a_B \right) + 2 \left( d_1 + e_1 + f_1 \right) > 1. \)

**Proof of lemma 2.** Consider the function \( g(x) = \cos x + 6 \sin x \) defined on range \((0, \pi)\). We have \( g'(x) = \sin x \left(-1 + 6 \cot x \right). \)

Deduce that \( g'(x) \) is positive in the range of \((0, \arccot \frac{1}{6})\), and it is negative in the range of \((\arccot \frac{1}{6}, \pi). \)

Therefore, \( g(x) \) monotonically increases in the range of \((0, \arccot \frac{1}{6})\), and it monotonically decreases in the range of \((\arccot \frac{1}{6}, \pi). \).

On the other hand, we have

\[ \left| \left( b_A + c_A \right) + 2 \left( d_1 + e_1 + f_1 \right) \right|^2 \]

\[ = \left( b_A + c_A \right)^2 + 4 \left( d_1 + e_1 + f_1 \right)^2 - 2 \left( b_A + c_A \right) \left( d_1 + e_1 + f_1 \right) \]

\[ = 2 + 2 \cos \alpha + 4 \left( 1 - 2 \cos \alpha \right) \left( 2 \cos \beta + 2 \cos \gamma \right) \]

\[ = 14 - 8 \cos \alpha - 8 \cos \beta - 8 \cos \gamma + 8 \sin \alpha - 4 \sin \beta - 4 \sin \gamma \]

\[ = 2g(\alpha) + 14 - 8 \cos \alpha - 4 \sum \sin \alpha. \]

Similarly, \( \left| \left( c_B + a_B \right) + 2 \left( d_1 + e_1 + f_1 \right) \right|^2 = 2g(\beta) + 14 - 8 \sum \cos \alpha - 4 \sum \sin \alpha; \)

\[ \left| \left( a_B + b_B \right) + 2 \left( d_1 + e_1 + f_1 \right) \right|^2 = 2g(\gamma) + 14 - 8 \sum \cos \alpha - 4 \sum \sin \alpha. \]
Hence, if $\beta$ lies in the range of $(0, \arccot \frac{1}{6})$, then $g(\beta) > g(\gamma)$; if $\beta$ lies in the range of $(\arccot \frac{1}{6}, \pi)$, then $g(\beta) > g(a)$.

Deduce that 

\[
| (c_B + a_B) + 2 (d_1 + e_1 + f_1) | \geq | (a_c + b_c) + 2 (d_1 + e_1 + f_1) |
\]

Therefore, if $| (c_B + a_B) + 2 (d_1 + e_1 + f_1) | \leq 1$, then according to lemma 1, either $B$ coincides with $C$ or $B$ coincides with $A$, contradiction. Thus, $| (c_B + a_B) + 2 (d_1 + e_1 + f_1) | > 1$.

\[ \square \]

**Lemma 3.** Given a triangle $ABC$ and a point $M$, denote $H, K, L$ as the orthogonal projections of $M$ on $BC, CA, AB$ respectively. Let

\[
f(M) = MA + MB + MC - 2 (MH + MK + ML)
\]

1) If $M$ differs from $A, B, C$, then \( \text{grad } f(M) = -(a_M + b_M + c_M) + 2 (d_1 + e_1 + f_1) \).

2) If $M_0$ differs from $A, B, C$ and $f(M_0) = 0$, then $f(M) \geq f(M_0) \forall M$. Equality occurs if and only if $M$ coincides with $M_0$.

**Proof of lemma 3.**

1) Similar to the proof of lemma 1, we have

\[
f(M) = MA + MB + MC - 2 (MH + MK + ML) = MA + MB + MC - (MB + MC) . d_1 + (MC + MA) . e_1 + (MA + MB) . f_1
\]

From this, after a few simple calculations, we have:

\[
\text{grad } f(M) = -(a_M + b_M + c_M) + 2 (d_1 + e_1 + f_1).
\]

2) Because $\text{grad } f(M_0) = 0$, according to part 1, $-(a_M + b_M + c_M) + 2 (d_1 + e_1 + f_1) = 0$. From this, similar to the proof of lemma 1, we have

\[
f(M) = MA + MB + MC - 2 (MH + MK + ML) = MA + MB + MC - (MB + MC) . d_1 - (MC + MA) . e_1 - (MA + MB) . f_1
\]

\[
\begin{align*}
&= MA.M_0.A + MB.M_0.B + MC.M_0.C - (MB + MC) . d_1 - (MC + MA) . e_1 - (MA + MB) . f_1 \\
&\geq \frac{M_0.A}{M_0.A} + \frac{M_0.B}{M_0.B} + \frac{M_0.C}{M_0.C} - (MB + MC) . d_1 - (MC + MA) . e_1 - (MA + MB) . f_1 \\
&= \frac{M_0.A}{M_0.A} \cdot M_0.A + \frac{M_0.B}{M_0.B} \cdot M_0.B + \frac{M_0.C}{M_0.C} \cdot M_0.C - (MB + MC) . d_1 - (MC + MA) . e_1 - (MA + MB) . f_1 \\
&= f(M_0).
\end{align*}
\]
Thus, \( f(M) \geq f(M_0) \).

Equality occurs if and only if

\[
\begin{align*}
&\overrightarrow{MA} \uparrow \uparrow \overrightarrow{M_0A} \\
&\overrightarrow{MB} \uparrow \uparrow \overrightarrow{M_0B} \\
&\overrightarrow{MC} \uparrow \uparrow \overrightarrow{M_0C}
\end{align*}
\]

This means that \( M \) coincides with \( M_0 \).

Lemma 4. Given a triangle \( ABC \) and a point \( M \), denote \( H, K, L \) as the orthogonal projections of \( M \) on \( BC, CA, AB \) respectively. Let \( f(M) = MA + MB + MC - 2(\overrightarrow{MH} + \overrightarrow{MK} + \overrightarrow{ML}) \).

\( M_0 \) belongs to line segment \( BC \) and differs from \( B \) and \( C \). Then, \( \text{grad} f(M_0) = 0 \) if and only if \( \overrightarrow{M_0A} \uparrow \uparrow \overrightarrow{IQ} \) and \( \sum \cos \alpha = 11/8 \).

Proof of lemma 4. Because \( M_0 \) belongs to segment \( BC \) and differs from \( B \) and \( C \), \( \overrightarrow{b_{M_0}} \uparrow \downarrow \overrightarrow{c_{M_0}} \). Then, noting that \( |\overrightarrow{b_{M_0}}| = |\overrightarrow{c_{M_0}}| \), we can deduce that \( \overrightarrow{b_{M_0}} + \overrightarrow{c_{M_0}} = 0 \) (1).

Because \( I \) and \( Q \) are the circumscribed circle and the orthocenter of triangle \( DEF \) respectively, in accordance with familiar formulas, \( \overrightarrow{IQ} = \overrightarrow{ID} \) (2) and \( IQ^2 = 9r^2 - \sum EF^2 \) (3).

Because quadrilaterals \( AFIE, BDIF, CEID \) are cyclic,

\[
\begin{align*}
\overrightarrow{EDF} &= \frac{\beta + \gamma}{2}; \\
\overrightarrow{FED} &= \frac{\gamma + \alpha}{2}; \\
\overrightarrow{DFE} &= \frac{\alpha + \beta}{2}
\end{align*}
\]

Thus, the following conditions are equivalent.

1) \( \text{grad} f(M_0) = 0 \).
2) \( - (\overrightarrow{a_{M_0}} + \overrightarrow{b_{M_0}} + \overrightarrow{c_{M_0}}) + 2(\overrightarrow{d_1} + \overrightarrow{e_1} + \overrightarrow{f_1}) = 0 \).
3) \( \overrightarrow{M_0A} = 2(\overrightarrow{d_1} + \overrightarrow{e_1} + \overrightarrow{f_1}) \).
4) \( \overrightarrow{M_0A} \uparrow \uparrow \overrightarrow{IQ} \) and \( r = 2IQ \).
5) \( \overrightarrow{M_0A} \uparrow \uparrow \overrightarrow{IQ} \) and \( r^2 = 4(9r^2 - \sum EF^2) \).
6) \( \overrightarrow{M_0A} \uparrow \uparrow \overrightarrow{IQ} \) and \( \sum \sin^2 \frac{\beta + \gamma}{2} = \frac{35}{16} \).
7) \( \overrightarrow{M_0A} \uparrow \uparrow \overrightarrow{IQ} \) and \( \sum \cos \alpha = 11/8 \).

Note that, according to lemma 3, 1 \( \iff \) 2; because of (1), 2 \( \iff \) 3; because of (2), 3 \( \iff \) 4; because of (3), 4 \( \iff \) 5; because of (4) and law of sine, 5 \( \iff \) 6; and obviously, 6 \( \iff \) 7.

\[\Box\]

3. Solution to Problem 3

Let \( f(M) = MA + MB + MC - 2(\overrightarrow{MH} + \overrightarrow{MK} + \overrightarrow{ML}) \).

It is easy to see that \( f \) is a continuous function.

Denote the set of the vertices of triangle \( ABC \) as \( \Omega_0 \), the set of points on the sides of triangles \( ABC \) as \( \Omega_1 \), the set of points in triangle \( ABC \) as \( \Omega_2 \).

Obviously, \( \Omega_0 \subset \Omega_1 \subset \Omega_2 \).

Without the loss of generality, assume that \( \alpha \geq \beta \geq \gamma \).

According to lemma 2, \( |-(\overrightarrow{c_{AB}} + \overrightarrow{a_{AB}}) + 2(\overrightarrow{d_1} + \overrightarrow{e_1} + \overrightarrow{f_1})| > 1 \).

Therefore, there are three cases to consider.

Case 1. \( |-(\overrightarrow{b_{A} + \overrightarrow{c_{A}}}) + 2(\overrightarrow{d_1} + \overrightarrow{e_1} + \overrightarrow{f_1})| \leq 1 \).

According to lemma 1, \( f(M) \) is minimal if and only if \( M \) coincides with \( A \).
Case 2. \(|-(\mathbf{a}_C + \mathbf{b}_C) + 2 (\mathbf{d}_I + \mathbf{e}_I + \mathbf{f}_I)| \leq 1\).

According to lemma 1, \(f(M)\) is minimal if and only if \(M\) coincides with \(C\).

Cases 3. \( \begin{cases} \ |-(\mathbf{a}_A + \mathbf{c}_A) + 2 (\mathbf{d}_I + \mathbf{e}_I + \mathbf{f}_I)| > 1 \, \text{or} \, \ |-(\mathbf{a}_C + \mathbf{b}_C) + 2 (\mathbf{d}_I + \mathbf{e}_I + \mathbf{f}_I)| > 1 \end{cases} \)

There are two sub-cases here.

Subcase 3.1. \(\sum \cos \alpha \neq \frac{11}{8}\).

According to a well-known theorem of Weierstrass, continuous function \(f(M)\) restricted to \(\Omega_2\) must have a minimal value at a point \(M_0\) of \(\Omega_2\).

Let \(v = \frac{-(\mathbf{b}_A + \mathbf{c}_A) + 2 (\mathbf{d}_I + \mathbf{e}_I + \mathbf{f}_I)}{\ |-(\mathbf{b}_A + \mathbf{c}_A) + 2 (\mathbf{d}_I + \mathbf{e}_I + \mathbf{f}_I)}|\).

Consider the directional derivative of \(f(M)\) in direction \(v\) at \(A\).

Let \(f_A(M) = MA - \overrightarrow{MA} (\mathbf{e}_I + \mathbf{f}_I)\);

\(f_B(M) = MB - \overrightarrow{MB} (\mathbf{f}_I + \mathbf{d}_I)\);

\(f_C(M) = MC - \overrightarrow{MC} (\mathbf{d}_I + \mathbf{e}_I)\).

It is easy to see that:

\[\text{grad } f_A(A) = -\mathbf{b}_A + \mathbf{f}_I + \mathbf{d}_I;\]

\[\text{grad } f_C(A) = -\mathbf{c}_A + \mathbf{d}_I + \mathbf{e}_I.\]

Hence, noting that \(f(M) = f_A(M) + f_B(M) + f_C(M)\), we have

\[f'_A(A) = 1 - v \cdot (\mathbf{e}_I + \mathbf{f}_I) + v \cdot \text{grad } f_B(A) + v \cdot \text{grad } f_C(A)\]

\[= 1 - v \cdot (- (\mathbf{b}_A + \mathbf{c}_A) + 2 (\mathbf{d}_I + \mathbf{e}_I + \mathbf{f}_I))\]

\[= 1 - \frac{-(\mathbf{b}_A + \mathbf{c}_A) + 2 (\mathbf{d}_I + \mathbf{e}_I + \mathbf{f}_I)}{\ |-(\mathbf{b}_A + \mathbf{c}_A) + 2 (\mathbf{d}_I + \mathbf{e}_I + \mathbf{f}_I)|} < 0.\]

Therefore, \(M_0 \neq A\).

Likewise, \(M_0 \neq B\) and \(M_0 \neq C\).

Hence, \(M_0 \notin \Omega_0\).

Because \(\sum \cos \alpha \neq \frac{11}{8}\), according to lemma 4, \(M_0 \notin \Omega_1 \backslash \Omega_0\).

Therefore, \(M_0 \in \Omega_2 \backslash \Omega_1\).

According to the standard theory of extremum values, \(M_0\) must be the stationary point of \(f(M)\) at which \(\text{grad } f(M_0) = 0\).

According to lemma 3, \(f(M) \geq f(M_0)\).

Equality occurs if and only if \(M\) coincides with \(M_0\).

Hence, \(f(M)\) is minimal if and only if \(M\) coincides with \(M_0\).

Subcase 3.2. \(\sum \cos \alpha = \frac{11}{8}\).

Let \(N\) be the midpoint of \(EF\) (f.1, f.2, f.3).

Because \(\alpha \geq \beta \geq \gamma, 90^\circ - \frac{\beta + \gamma}{2} \geq 90^\circ - \frac{\alpha + \beta}{2}; 90^\circ - \frac{\beta + \gamma}{2} \geq 90^\circ - \frac{\gamma + \alpha}{2}\).

On the other hand, it is easy to see that \(\overrightarrow{IEF} = 90^\circ - \frac{\beta + \gamma}{2}; \overrightarrow{QEF} = 90^\circ - \frac{\alpha + \beta}{2}; \overrightarrow{IFE} = 90^\circ - \frac{\beta + \gamma}{2}; \overrightarrow{QFE} = 90^\circ - \frac{\gamma + \alpha}{2}\).

Hence, \(\overrightarrow{IEF} \geq \overrightarrow{QEF}; \overrightarrow{IFE} \geq \overrightarrow{QFE}\).

Therefore, \(Q\) belongs to triangle \(IEF\).

Together with \(\beta \geq \gamma\), deduce that \(Q\) belongs to right triangle \(INF (1)\).
If $\cos \beta = \frac{3}{8}$, then $\cos \alpha + \cos \gamma = 1$ (f.1).

(Diagram 1.)

Deduce that $\vec{IQ} \cdot \vec{IE} = r^2 (\vec{d_I} + \vec{e_I} + \vec{f_I}) \cdot \vec{e_I} = r^2 (- \cos \gamma + 1 - \cos \alpha) = 0$.
Therefore $\angle QIE = 90^\circ$.
Together with (1), we have $\vec{IQ} \uparrow \uparrow \vec{CA}$.
This means that $2 (\vec{d_I} + \vec{e_I} + \vec{f_I}) \uparrow \uparrow \vec{a_C}$.

On the other hand, because $\sum \cos \alpha = \frac{11}{8}$, similar to the proof of lemma 4, we have $2 \vec{I}Q = r$.
To put it another way, $|2 (\vec{d_I} + \vec{e_I} + \vec{f_I})| = |\vec{a_C}|$.
In short, $2 (\vec{d_I} + \vec{e_I} + \vec{f_I}) = \vec{a_C}$.
Therefore, $|-(\vec{a_C} + \vec{b_C}) + 2 (\vec{d_I} + \vec{e_I} + \vec{f_I})| = |\vec{b_C}| = 1$, contradiction.
Hence, there are two sub-subcases to consider.

Sub-subcase 3.2.1. $\cos \beta > \frac{3}{8}$ (f.2).

(Diagram 2.)

Because $\sum \cos \alpha = \frac{11}{8}$ and $\cos \beta > \frac{3}{8}$, $\cos \alpha + \cos \gamma < 1$. 

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Deduce that \( \vec{I Q} \cdot \vec{I E} = r^2 (d_1 + e_1 + f_1) \cdot e_1 = r^2 (- \cos \gamma + 1 - \cos \alpha) > 0 \).

Therefore, \( \stackrel{\frown}{QE} < 90^\circ \) (2).

From (1) and (2), deduce that there is a point \( M_0 \) on segment DC such that \( \overrightarrow{M_0 A} \) shares the same direction with \( \overrightarrow{I Q} \).

According to lemma 4, \( \text{grad} f(M_0) = 0 \).

According to lemma 3, \( f(M) \geq f(M_0) \).

Equality occurs if and only if \( M \) coincides with \( M_0 \).

Hence, \( f(M) \) is minimal if and only if \( M \) coincides with \( M_0 \).

Sub-subcase 3.2.2. \( \cos \beta < \frac{3}{8} \) (f.3).

Similar to subcase 3.1, continuous function \( f(M) \) restricted on \( \Omega_2 \) must have a minimal value at a point \( M_0 \) belonging to \( \Omega_2 \) and \( M_0 \notin \Omega_0 \).

From (1), it is easy to deduce that there does not exist any point \( M \) on segment CA such that \( \overrightarrow{MB} \) shares the same direction with \( \overrightarrow{I Q} \) and there does not exist any point \( M \) on segment AB such that \( \overrightarrow{MC} \) shares the same direction with \( \overrightarrow{I Q} \).

Therefore, according to lemma 4, \( M_0 \) does not belong to segments CA and AB.

On the other hand, because \( \sum \cos \alpha = \frac{11}{8} \) and \( \cos \beta < \frac{3}{8} \), \( \cos \alpha + \cos \gamma > 1 \).

Deduce that \( \vec{I Q} \cdot \vec{I E} = r^2 (d_1 + e_1 + f_1) \cdot e_1 = r^2 (- \cos \gamma + 1 - \cos \alpha) < 0 \).

Therefore, \( \stackrel{\frown}{Q E} > 90^\circ \) (3).

From (1) and (3), deduce that there does not exist any point \( M \) on segment BC such that \( \overrightarrow{MA} \) shares the same direction with \( \overrightarrow{I Q} \).

Therefore, according to lemma 4, \( M_0 \) does not belong to segment BC.

In short, \( M_0 \notin \Omega_1 \setminus \Omega_0 \).

Hence, \( M_0 \in \Omega_2 \setminus \Omega_1 \).

Similar to subcase 3.1, \( M_0 \) must be the stationary point of \( f(M) \) at which \( \text{grad} f(M_0) = 0 \).

According to lemma 3, \( f(M) \geq f(M_0) \).

Equality occurs if and only if \( M \) coincides with \( M_0 \).

Hence, \( f(M) \) is minimal if and only if \( M \) coincides with \( M_0 \).
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