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A NOTE ON THE DIFFERENTIAL GEOMETRY OF MOVING PARTICLES IN SPECIAL RELATIVITY

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ABSTRACT. A moving particle in special relativity means a curve with a timelike unitary tangent vector. Consequently, the path of the mentioned particle corresponds to a timelike curve according to signature $(+,-,-,-)$. In this work, we introduce a method to determine Frenet-Serret vector fields and curvatures for a moving particle in special relativity in the light of the existing results of other metrics.

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1. INTRODUCTION

By the theory of relativity A. Einstein constructed a gap between motion of a charged particles and four dimensional non-Euclidean space. This special space generally called as Lorentz-Minkowski space and denoted by M_4, M_1^4, L^4, or E_1^4. These differences are commonly due to choosing the metric of the space. The nature of this special space is is generally studied by the mathematical physicists, which has the metric signature $(+,-,-,-)$ [3]. In special relativity, we know that, a particle means a curve with a timelike unitary tangent. So, this curve corresponds to a timelike path. And further, it can be said that [1], the curvatures of the timelike curve are also correspond to world lines of classical charged of the mentioned particle. There exists a vast literature on the subject, see [1], [3], [6].

Recently, the related studies on the subject of electromagnetic field of the charged particles are extended to the higher dimensions [5], and generalized to some special curves; null curves in [2]. It is safe to report that classical differential geometry of the curves are extended to higher spaces by the aid of the interesting paper dealt by Gluck [4]. This paper leads to computation of Frenet-Serret apparatus of the curves by the great method of Gram-Schmidt. However, an explicit method for regular curves in the Euclidean 4-space was expressed by Mağden [7], in his PhD dissertation, defining a special vector product of three vectors. This method was comphrensive and was easy to follow the equations. Therefore the researchers extended the method to the higher dimensions and non-Euclidean spaces. For instance, according to signature $(+,+,+,-)$ by [9];

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$(-,+,+,+)$ by [10] and [12]. And in recent papers there are extensions in E^5 [11] and in L^5 [8].

In this work, by the spirit of the papers [1-3,5-6], we observe that M_4 has a different metric therefore different signature from the existed papers published recently. So, we aim to adapt the method, originally expressed by Mağden [7] to timelike curves of M_4. We think of that this work is important in the area of moving particles, because determining of the curvatures of the timelike curve will give information about the path in spacetime.

2. Preliminaries

In this section, we briefly review some information about special relativity directly from the paper [1].

In special relativity, the 4-space is flat and hence the coordinate system $(x^r) = (ct, x, y, z)$, $r = 0, 1, 2, 3$ is such that the squared separation between the events x^r and $x^r + dx^r$ is given by the invariant (c is light velocity in the vacuum)

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = \eta_{ab}dx^a dx^b, \tag{2.1}$$

where we assume the dummy indices convention and

$$T^r = \frac{dx^r}{ds}, T_r T^r = \eta_{ab}T^a T^b = 1,$$

is the Minkowski metric tensor. $\overleftrightarrow{\eta}$ allows us to distinguish three types of vectors with respect to its proper inner product: the vector is null, spacelike or timelike if $\eta_{bc}A^b A^c$ is 0, < 0 or > 0, respectively. Furthermore, the vectors are orthogonal to each other if the inner product of them is equal to zero. A particle is in special relativity means a curve $x^r(s)$ with a timelike unitary tangent $T^i(s)$ defined by

$$T^r = \frac{dx^r}{ds}, T_r T^r = \eta_{ab}T^a T^b = 1,$$

therefore one has threee spacelike unitary normals $N^i_r(s)$, that is (there is no sum over r):

$$T_j N^j_r = 0, N^j_r N^j_r = -1, r = 1, 2, 3,$$

which together with the tangent give rise to the main tetrad $(T^r, N^r_1, N^r_2, N^r_3)$. Frenet-Serret formulae

$$\begin{aligned} \tfrac{dT^r}{ds} &= k_1 N^r_1, \\ \tfrac{dN^r_1}{ds} &= k_1 T^r + k_2 N^r_2, \\ \tfrac{dN^r_2}{ds} &= -k_2 N^r_1 + k_3 N^r_3, \\ \tfrac{dN^r_3}{ds} &= -k_3 N^r_2. \end{aligned} \tag{2.2}$$

Here $k_i(s), i = 1, 2, 3$ are the curvatures of the world line.

3. Main Results

In this section, we follow same procedure in the papers [7-12]. We first differentiate

$$\dot{x}^r = T^r(s). \tag{3.1}$$

Since we have the tangent vector. One more differentiating of (3), we write

$$\ddot{x}^r = \dot{T}^r(s) = k_1 N_1^r. \tag{3.2}$$

By (4), we have the first curvature

$$k_1 = \|\ddot{x}^r(s)\|.$$

Therefore it is easy to obtain N_1^r by

$$N_1^r = \frac{\ddot{x}^r(s)}{\|\ddot{x}^r(s)\|}. \tag{3.3}$$

Let us one more differentiate of (4);

$$\dddot{x}^r = k_1^2 T^r + k_1 N_1^r + k_1 k_2 N_2^r. \tag{3.4}$$

Now, we shall define a special vector by the aid of the following determinant:

$$\xi = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \begin{vmatrix} ce_1 & e_2 & e_3 & e_4 \\ ct_1 & x_1 & y_1 & z_1 \\ ct_2 & x_2 & y_2 & z_2 \\ ct_3 & x_3 & y_3 & z_3 \end{vmatrix},$$

where $\alpha_i i = 1,2,3$ are linearly independent vectors and $e_j, j = 1,2,3,4$ are the coordinate direction vectors of M_4 satisfying the equations

$$\begin{gathered} e_1 \wedge e_2 \wedge e_3 = -e_4, e_2 \wedge e_3 \wedge e_4 = -e_1, \\ e_3 \wedge e_4 \wedge e_1 = -e_2, e_4 \wedge e_1 \wedge e_2 = e_3. \end{gathered} \tag{3.5}$$

Lemma 3.1. The vector ξ is orthogonal to each vectors of $\alpha_i i = 1,2,3$ according to metric (1).

Proof. By the definition of the above, it can be easily proved. □

Here, the definition of the vector ξ obeys the rules of cross product. This product of three vectors in classical manner, it first introduced by Mağden [7] by the name of vector product. In this work, due to disagreement of the researchers about name of ″cross″ or ″vector″ product in higher dimensions, we avoid use of a specific name.

Now, we continue to compute Frenet-Serret apparatus of the timelike curve. The following product of the three vectors T^r, N_1^r and $\dddot{x}^r$ gives us

$$T^r \wedge N_1^r \wedge \dddot{x}^r = -k_1 k_2 N_3^r. \tag{3.6}$$

By taking the norm of the both sides of (8), we have

$$\|T^r \wedge N_1^r \wedge \dddot{x}^r\| = k_1 k_2.$$

Since we write the second curvature

$$k_2 = \frac{\|T^r \wedge N_1^r \wedge \dddot{x}^r\|}{\|\ddot{x}^r\|}.$$

So we have the third normal of the timelike path as the following:

$$N_3^r = \mu \frac{T^r \wedge N_1^r \wedge \dddot{x}^r}{\|T^r \wedge N_1^r \wedge \dddot{x}^r\|},$$

where μ is taken ± 1 to make determinant of $[T^r, N_1^r, N_2^r, N_3^r]$ matrix. By this way, the orthonormal frame is positively oriented. To compute the third curvature of the curve $x^r(s)$, we first differentiate (6) as follows:

$$\dddot{x}(s) = (3k_1\dot{k}_1)T^r + (k_1^3 + \ddot{k}_1 - k_1k_2^2)N_1^r + (2\dot{k}_1k_2 + k_1\dot{k}_2)N_2^r + (k_1k_2k_3)N_3^r. \tag{3.7}$$

By the aid of the inner product of (9) and N_3^r, we write

$$\ddddot{x}(s).N_3^r = k_1k_2k_3.$$

Since we write the third curvature as the following

$$k_3 = \frac{\ddddot{x}(s).N_3^r}{\left\| T^r \wedge N_1^r \wedge \dddot{x}^r \right\|}.$$

Finally, we have the second normal of the curve (in terms of (7)) by

$$N_2^r = N_3^r \wedge T^r \wedge N_1^r.$$

So we have calculated Frenet-Serret apparatus of the timelike curve.

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