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A CONCISE ELEMENTARY PROOF FOR THE PTOLEMY'S THEOREM

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ABSTRACT. In this paper a concise elementary Euclidean Geometric proof is divulged for the Ptolemy's Theorem of Cyclic Quadrilaterals using the isomorphic triangles while consequently being obtained the standard lengths of the diagonals and diagonal segments of a Cyclic Quadrilateral as collateral results.

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1. INTRODUCTION

In this short paper the author adduces a concise elementary proof for the Ptolemy's Theorem of cyclic quadrilaterals without being separately obtained the lengths of the diagonals of a cyclic quadrilateral by constructing some particular perpendiculars, as well as for the ratio of the lengths of the diagonals of a cyclic quadrilateral. Moreover the lengths of the main diagonals of a cyclic quadrilateral are obtained considering the Ptolemy's theorem and the ratio of two diagonals of a cyclic quadrilateral without the subsidy of trigonometry or vector algebra just a little bit. The Ptolemy's Theorem states that the multiple of the lengths of the diagonals of a Cyclic Quadrilateral is equal to the addition of separate multiples of the opposite side lengths of the Cyclic Quadrilateral.

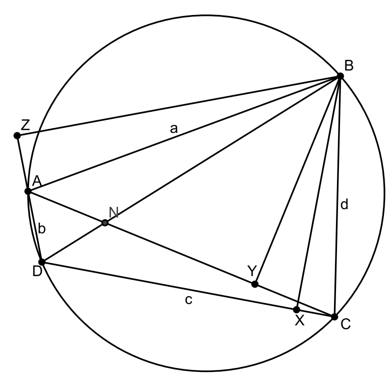

FIGURE 1. *A Cyclic Quadrilateral*

2. Main Results

Theorem 2.1. *Let ABCD be any cyclic quadrilateral such that AC and BD are its diagonals. Then $AC \cdot BD = AB \cdot DC + AD \cdot BC$ which is referred to as the Ptolemy's Theorem of Cyclic Quadrilaterals.*

Proof. **New Proof.** ABCD is a cyclic quadrilateral. Let $AB = a, BC = d, CD = c, AD = b$. BX, BY, BZ are perpendiculars drawn to DC, AC and extended DA respectively. Since ABCD is a cyclic quadrilateral $\widehat{BAC} = \widehat{BDC}, \widehat{ACB} = \widehat{ADB}, \widehat{BAZ} = \widehat{BCX}$ and $\widehat{AYB} = \widehat{DXB} = \widehat{DZB} = 90°$ (since BX,BY and BZ are perpendiculars). Hence triangle ABY is similar with triangle DBX, also triangle BYC is similar with triangle DBZ and triangle ABZ is similar with triangle BXC.
ΔABY and ΔDXB are similar, hence

$$\frac{AY}{DX} = \frac{AB}{BD} \tag{1}$$

ΔBYC and ΔDBZ are similar, hence

$$\frac{CY}{DZ} = \frac{BC}{BD} \tag{2}$$

The triangles ABZ and BXC are similar, hence $\frac{AZ}{CX} = \frac{AB}{BC}$, thereby

$$AZ \cdot BC = AB \cdot CX \tag{3}$$

$AY = AC - CY$, hence from (1),

$$AC \cdot BD = AB \cdot DC + BD \cdot CY - AB \cdot CX \tag{4}$$

DZ = AD + AZ, hence from (2),

$$BD \cdot CY = AD \cdot BC + AZ \cdot BC \tag{5}$$

By adding $(3), (4) and (5)$, one obtains that

$$AC \cdot BD = AB \cdot DC + AD \cdot BC = ac + bd \tag{6}$$

Hence the proof of the Ptolemy's Theorem is completed successfully in a prominent alternative manner as in (6).

□

Theorem 2.2. *Let ABCD be any cyclic quadrilateral such that AC and BD are its diagonals. Then AC/BD= (ab+cd)/(ad+bc) which is referred to as the standard ratio of the main diagonals of a cyclic quadrilateral.*

Proof. **New Proof**. Consider the figure 1. Since ABCD is a cyclic quadrilateral, the triangles AND and BNC are similar as well as ANB and NDC. Thus the following results are emerged.

$$\frac{AN}{NB} = \frac{b}{d} \tag{7}$$

$$\frac{AN}{ND} = \frac{a}{c} \tag{8}$$

$$\frac{NB}{NC} = \frac{a}{c} \tag{9}$$

$$\frac{ND}{NC} = \frac{b}{d} \tag{10}$$

From (7) and (8) $\frac{NB}{ND} = \frac{ad}{bc}$, so that $\frac{BD-ND}{ND} = \frac{ad}{bc}$, which intends that

$$\frac{BD}{ND} = \frac{ad+bc}{bc} \tag{11}$$

From (7) and (9) it can be obtained that $\frac{AN}{NC} = \frac{ab}{cd}$. So that $\frac{AC-NC}{NC} = \frac{ab}{cd}$ which intends that

$$\frac{AC}{NC} = \frac{ab+cd}{cd} \tag{12}$$

From (11) and (12), $\frac{AC}{BD} \cdot \frac{ND}{NC} = \frac{(ab+cd)}{(ad+bc)} \cdot \frac{b}{d}$. Substituting from (10) to $\frac{ND}{NC}$, one obtains:

$$\frac{AC}{BD} = \frac{ab+cd}{ad+bc} \tag{13}$$

□

Remark 1 This is the ratio of the lengths of two diagonals of a cyclic quadrilateral and it must be accented that in this particular paper this ratio is obtained without being found the lengths of the diagonals separately in a cyclic quadrilateral.

Theorem 2.3. *Let ABCD be any cyclic quadrilateral such that AC and BD are its diagonals. Then,*

$$AC^2 = \frac{(ac+bd)\cdot(ab+cd)}{ad+bc}$$

and

$$BD^2 = \frac{(ac+bd)\cdot(ad+bc)}{ab+cd}.$$

Proof. New Proof. It is not difficult to observe by (6) and (13) that, $AC^2 = \frac{(ac+bd)\cdot(ab+cd)}{ad+bc}$ as well as $BD^2 = \frac{(ac+bd)\cdot(ad+bc)}{ab+cd}$ easily. □

Remark 2 X, Y and Z points are collinear since XYZ is the Simpson line. Likewise this concise elementary proof for Ptolemy's theorem is based on the 3 perpendiculars on which the Simpson line is located.

Remark 3 Note that the above ratio of the main diagonals of a cyclic quadrilateral can be alternatively obtained by another method using collateral parallel lines as follows.

Another proof of Theorem 2.2:

In the following figure two lines are drawn at A and B such that they are parallel to BD and AC diagonals respectively while those lines intersect each other at Q and meet the extended CD and DC at P and R respectively. Using the principles of several similar triangles the ratio of the diagonals of a cyclic quadrilateral can be obtained as follows.

In the Figure 2 depicted above,

$\widehat{APD} = \widehat{BDC}$ (Since PQ is parallel with BD)

$\widehat{ABC} = \widehat{ADP}$ (Since ABCD is a cyclic quadrilateral)

Likewise triangles APD and ABC are similar. Whence

$$\frac{AP}{AC} = \frac{b}{d} = \frac{PD}{a} \tag{14}$$

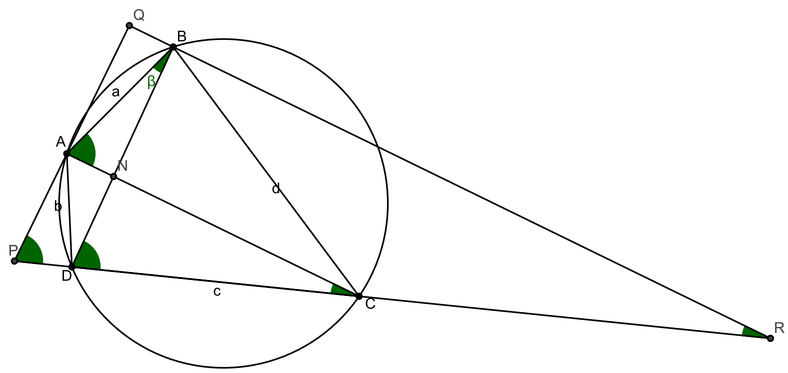

FIGURE 2. A Cyclic Quadrilateral with created parallel lines

$\widehat{BRC} = \widehat{ACD}$ (Since AC is parallel with QR)
$\widehat{BCR} = \widehat{DAB}$ (Since ABCD is a cyclic quadrilateral)
Likewise traingles BCR and ADB are similar. Whence

$$\frac{BR}{BD} = \frac{d}{b} = \frac{CR}{a} \tag{15}$$

$\widehat{APC} = \widehat{BDR}$ (Since PQ is parallel with BD)
$\widehat{ACP} = \widehat{BRC}$ (Since AC is parallel with QR)
Likewise, triangles BCR and ADB are similar. Whence

$$\frac{BR}{AC} = \frac{DR}{PC} = \frac{BD}{AP} \tag{16}$$

$PC = PD + c$, from (14) we have $PD = \frac{ab}{d}$, thus

$$PC = \frac{ab + cd}{d} \tag{17}$$

$DR = CR + c$ and from (15) we have $CR = \frac{ad}{b}$, thus

$$DR = \frac{ad + bc}{b} \tag{18}$$

By (14) and (16) we have $\frac{BD}{AC} = \frac{b \cdot DR}{d \cdot PC}$, whence using the values of (17) and (18) for DR and PC we have $\frac{BD}{AC} = \frac{ad+bc}{ab+cd}$, thus

$$\frac{AC}{BD} = \frac{ab + cd}{ad + bc} \tag{19}$$

This is in fact exactly the same result obtained in (13).

Remark 4 Using the geometry of the Figure 2, consequently the lengths of the segments generated by the diagonals of the cyclic quadrilateral can be easily adduced using the principles of similar triangles as follows.

3. TWO SIGNIFICANT LEMMAS ON CYCLIC QUADRILATERALS

Lemma 3.1. *The standard lengths of the diagonal segments CN and AN of the ABCD cyclic quadrilateral can be adduced such that*

$$CN = cd\sqrt{(\frac{(ac+bd)}{(ad+bc)(ab+cd)}}$$

and

$$AN = ab\sqrt{\frac{(ac+bd)}{(ad+bc)(ab+cd)}}$$

Proof. **New Proof of Lemma 3.1.** Since DN is parallel with AP, $\frac{c}{PD} = \frac{CN}{AN}$ and since $PD = \frac{ab}{d}$, $\frac{CN}{AN} = \frac{cd}{ab}$.
Thus $CN = \frac{cd \cdot AC}{ab+cd} = \frac{cd}{ab+cd}\sqrt{\frac{(ac+bd)(ad+bc)}{ad+bc}} = cd\sqrt{\frac{(ac+bd)}{(ad+bc)(ab+cd)}}$.
Hence $AN = ab\sqrt{\frac{(ac+bd)}{(ad+bc)(ab+cd)}}$ □

Lemma 3.2. *The standard lengths of the diagonal segments DN and NB of the ABCD cyclic quadrilateral can be adduced such that* $DN = bc\sqrt{\frac{(ac+bd)}{(ad+bc)(ab+cd)}}$ *and* $NB = ad\sqrt{\frac{(ac+bd)}{(ad+bc)(ab+cd)}}$

Proof. **New Proof of Lemma 3.2.** Since CN is parallel with RB, $\frac{c}{CR} = \frac{DN}{NB}$ and since $CR = \frac{ad}{b}$, $\frac{DN}{NB} = \frac{bc}{ad}$.
Thus $DN = \frac{bc \cdot BD}{ad+bc} = \frac{bc}{ad+bc}\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} = bc\sqrt{\frac{(ac+bd)}{(ad+bc)(ab+cd)}}$.
Hence, $NB = ad\sqrt{\frac{(ac+bd)}{(ad+bc)(ab+cd)}}$ □

4. CONCLUSION OF REMARKS

The Ptolemy's Theorem regarding cyclic quadrilaterals has been one of the most prominent and an admirable theorems in the stream of Elementary as well as the Advanced Euclidean Geometry throughout way back centuries ago, initially acquainted by Claudius Ptolemaeus who was a well known Greek mathematician adducing an elegant, succinct elementary proof for his theorem using the principles of similar triangles. More over although there have been some alternative proofs for the Ptolemy's Theorem and the lengths of the diagonals of cyclic quadrilaterals, most of those proofs are nearly consisted by the Cosine formulas particularly the one given by Brahmagupta(598-670 AD) who was an eminent mathematician of ancient India. Nevertheless this paper adduces a succinct elementary Euclidean geometric proof for the Ptolemy's Theorem using only the principles of isomorphic or similar triangles such as the original elegant proof given by Ptolemy for his felicitous Theorem while presenting the lengths of the diagonals and diagonal segments using only the isomorphism without further being referred to collateral intricate Euclidean or trigonometric length calculations.

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