Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol. 13, (2024), Issue 1, pp. 90-103.

CENTROIDS OF BULGING TRIANGLES: A QUANTITATIVE APPROACH

NORIHIRO SOMEYAMA

ABSTRACT. There is a figure that is a kind of generalization of the Reuleaux triangle. It was introduced by Someyama (2021, [6]) and is called the bulging triangle. Roughly speaking, a Reuleaux triangle is a figure of an equilateral triangle with its sides inflated outward. On the other hand, a bulging triangle is a figure of an acute triangle with its sides inflated outward. The centroid of a bulging triangle obviously coincides with that of the original triangle. It is important to find the centroid of a figure in general, and so we find the centroid $G_{\widehat{\triangle}}$ of a bulging triangle $\widehat{\triangle}ABC$ quantitatively by computing the coordinates of $G_{\widehat{\triangle}}$ in this paper. To do so, we use that a bulging triangle can be divided into four parts: three crescents and one triangle. Moreover, we mention whether the position of $G_{\widehat{\triangle}}$ is displaced from that of the centroid of the original triangle $\triangle ABC$.

1. INTRODUCTION

Bulging triangles were introduced by Someyama (2021, [6]) as a generalization of Reuleaux triangles; we can see, e.g., Barrallo et al. [1], Conti et al. [2] and Hu et al. [4] for geometric properties and applications of Reuleaux triangles. Since then, a lot of properties of bulging triangles have been studied. Some of them will be discussed at the end of this section; see Someyama [6, 7, 8, 9] for more information.
We define bulging triangles only for acute or right triangles. By doing so, any bulging triangle becomes a convex figure. Indeed, obtuse triangles cannot generate convex bulging triangles; see Proposition 2.2 in Someyama [6]. However, there exists a way to define a convex generalized Reuleaux triangle made by bulging a triangle that can be obtuse; see Ridley [5]. A bulging triangle is made by bulging the sides without changing the vertices, whereas that convex generalized Reuleaux figure is made by changing the vertices; we mention for more information later. From beginning to end, we treat the former.

2010 *Mathematics Subject Classification.* 51M04, 51M05, 51N20.
Key words and phrases. Reuleaux triangle, Bulging triangle, Original triangle, Centroid, Weighted mean.

1.1. **Definition of Bulging Triangles.** In what follows, e.g., $\overline{\mathrm{AB}}$ and $|\overline{\mathrm{AB}}|$ means the segment and its length, respectively. Moreover, we always denote

$$a := |\overline{\mathrm{BC}}|, \quad b := |\overline{\mathrm{CA}}|, \quad c := |\overline{\mathrm{AB}}|$$

for $\triangle\mathrm{ABC}$. A bulging triangle is defined by the following procedures:

i) Consider an acute triangle $\triangle\mathrm{ABC}$ such that $c > a > b$; hereafter, we always assume this.
ii) Consider the perpendicular bisector ℓ_{AB} of $\overline{\mathrm{AB}}$, and find the intersection point P of ℓ_{AB} and $\overline{\mathrm{BC}}$. Note that P lies on $\overline{\mathrm{BC}}$ since $a > b$.
iii) Draw the arc $\overset{\frown}{\mathrm{AB}}$ with P as the center and $\overline{\mathrm{AP}}$, equivalently, $\overline{\mathrm{BP}}$ as the radius.
iv) Consider the perpendicular bisector ℓ_{BC} of $\overline{\mathrm{BC}}$, and find the intersection point Q of ℓ_{BC} and $\overline{\mathrm{AB}}$. Note that Q lies on $\overline{\mathrm{AB}}$ since $c > b$.
v) Draw the arc $\overset{\frown}{\mathrm{BC}}$ with Q as the center and $\overline{\mathrm{BQ}}$, equivalently, $\overline{\mathrm{CQ}}$ as the radius.
vi) Consider the perpendicular bisector ℓ_{CA} of $\overline{\mathrm{CA}}$, and find the intersection point R of ℓ_{CA} and $\overline{\mathrm{CA}}$. Note that R lies on $\overline{\mathrm{AB}}$ since $c > a$.
vii) Draw the arc $\overset{\frown}{\mathrm{CA}}$ with R as the center and $\overline{\mathrm{AR}}$, equivalently, $\overline{\mathrm{CR}}$ as the radius.
viii) The closed figure consisting of three arcs drawn according to the above procedure is called the **bulging triangle**, and is denoted by $\widehat{\triangle}\mathrm{ABC}$.

The Reuleaux triangle can be said to be a "bulging equilateral triangle." Because of this, we write $\widehat{\triangle}_{\mathrm{eq}}\mathrm{ABC}$ for the Reuleaux triangle ABC.

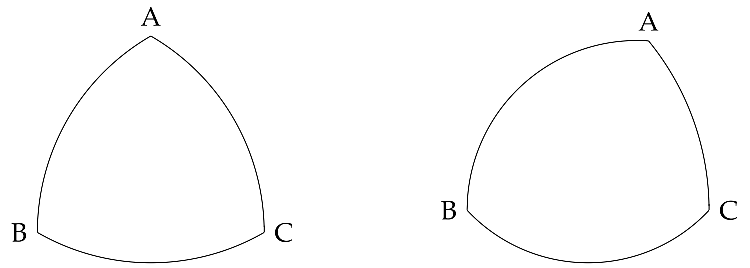


Figure 1. Reuleaux triangle and Bulging triangle

We set the following terms on a bulging triangle.

Definition 1.1. *Let* $\widehat{\triangle}\mathrm{ABC}$ *be a bulging triangle.*

1) $\triangle\mathrm{ABC}$ *is called the* **original triangle for** $\widehat{\triangle}\mathrm{ABC}$.
2) *The points* A, B *and* C *are called* **vertices** *of* $\widehat{\triangle}\mathrm{ABC}$.
3) *The arcs* $\overset{\frown}{\mathrm{AB}}$, $\overset{\frown}{\mathrm{BC}}$ *and* $\overset{\frown}{\mathrm{CA}}$ *are collectively called the* **edges** *of* $\widehat{\triangle}\mathrm{ABC}$, *and are rewritten by* $\widetilde{\mathrm{AB}}$, $\widetilde{\mathrm{BC}}$ *and* $\widetilde{\mathrm{CA}}$, *respectively.*
4) *The points* P, Q *and* R *are called the* AB**-center**, BC**-center** *and* CA**-center**, *respectively. These are collectively referred to as the* **edge-centers** *of* $\widehat{\triangle}\mathrm{ABC}$.
5) *The (lengths of) segments* $\overline{\mathrm{AP}}$ *(or,* $\overline{\mathrm{BP}}$*),* $\overline{\mathrm{BQ}}$ *(or,* $\overline{\mathrm{CQ}}$*) and* $\overline{\mathrm{CR}}$ *(or,* $\overline{\mathrm{AR}}$*) are called the* AB**-radius**, BC**-radius** *and* CA**-radius**, *respectively. These are collectively referred to as the* **edge-radii** *of* $\widehat{\triangle}\mathrm{ABC}$.

6) $\angle\mathrm{APB}$, $\angle\mathrm{BQC}$ *and* $\angle\mathrm{CRA}$ *are called the* AB-**central angle**, BC-**central angle** *and* CA-**central angle**, *respectively. These three angles are collectively referred to as the* **edge-central angles** *of* $\widehat{\triangle}\mathrm{ABC}$.

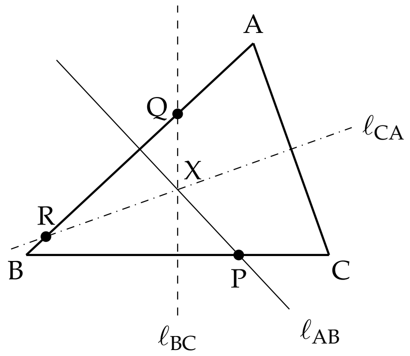


Figure 2. Edge-centers P, Q, R

Remark 1.1.

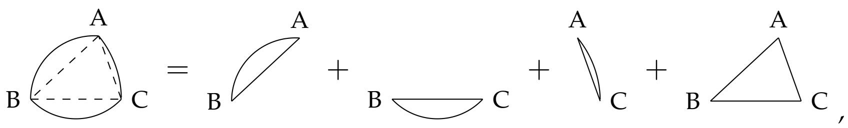
i) *It is well known that* ℓ_{AB}, ℓ_{BC} *and* ℓ_{CA} *intersect at one point* X *and it is the circumcenter of* $\triangle\mathrm{ABC}$, *since we have*
$$|\overline{\mathrm{AX}}| = |\overline{\mathrm{BX}}|, \quad |\overline{\mathrm{BX}}| = |\overline{\mathrm{CX}}|, \quad |\overline{\mathrm{CX}}| = |\overline{\mathrm{AX}}|.$$

ii) *As we can see from 5) of Definition 1.1, edge-radii of* $\widehat{\triangle}\mathrm{ABC}$ *are partial sides of* $\triangle\mathrm{ABC}$. *In case of a Reuleaux triangle* $\widehat{\triangle}_{\mathrm{eq}}\mathrm{ABC}$, *if the length of one side of the original equilateral triangle for it is* a, *then* $r_{\mathrm{AB}} = r_{\mathrm{BC}} = r_{\mathrm{CA}} = a$.

We always promise in this paper that $\widehat{\triangle}$ ABC (or, $\triangle\mathrm{ABC}$) satisfies $c > a > b$ and the notations P, Q and R are used as edge-centers of $\widehat{\triangle}\mathrm{ABC}$ without exception.
The "bulging areas" can be considered as the crescents, i.e., the area S_1 of sector ABP from which $\triangle\mathrm{ABP}$ is removed, the area S_2 of sector BCQ from which $\triangle\mathrm{BCQ}$ is removed and the area S_3 of sector CAR from which $\triangle\mathrm{CAR}$ is removed. These crescents are denoted by $\mathrm{A\overset{\frown}{B}A}$, $\mathrm{B\overset{\frown}{C}B}$ and $\mathrm{C\overset{\frown}{A}C}$, respectively.

1.2. **Areas of Bulging Triangles.** We use the following result, which is already known to find the formula of the centroid of a bulging triangle.

Theorem 1.1 (Someyama (2023, [8]), Theorem 4.1)**.** *Let* $\widehat{\triangle}\mathrm{ABC}$ *be a bulging triangle. Put*
$$u := \frac{1}{2}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}. \tag{1.1}$$
The area S *of* $\widehat{\triangle}\mathrm{ABC}$ *is given by*
$$S = S_1 + S_2 + S_3 + S_4 : \tag{1.2}$$



where

i) *if* $c \geq \sqrt{2}b$,

$$S_1 = \frac{c^4a^2}{2(a^2-b^2+c^2)^2}\left(\pi - \mathrm{Arcsin}\,\frac{(a^2-b^2+c^2)u}{c^2a^2} - \sin\frac{(a^2-b^2+c^2)u}{c^2a^2}\right),$$
$$S_2 = \frac{a^4c^2}{2(c^2+a^2-b^2)^2}\left(\pi - \mathrm{Arcsin}\,\frac{(c^2+a^2-b^2)u}{c^2a^2} - \sin\frac{(c^2+a^2-b^2)u}{c^2a^2}\right),$$
$$S_3 = \frac{b^4c^2}{2(b^2+c^2-a^2)^2}\left(\mathrm{Arcsin}\,\frac{(b^2+c^2-a^2)u}{b^2c^2} - \sin\frac{(b^2+c^2-a^2)u}{b^2c^2}\right),$$
$$S_4 = \frac{u}{2};$$

ii) *if* $c < \sqrt{2}b$,

$$S_1 = \frac{c^4a^2}{2(a^2-b^2+c^2)^2}\left(\mathrm{Arcsin}\,\frac{(a^2-b^2+c^2)u}{c^2a^2} - \sin\frac{(a^2-b^2+c^2)u}{c^2a^2}\right),$$
$$S_2 = \frac{a^4c^2}{2(c^2+a^2-b^2)^2}\left(\mathrm{Arcsin}\,\frac{(c^2+a^2-b^2)u}{c^2a^2} - \sin\frac{(c^2+a^2-b^2)u}{c^2a^2}\right),$$
$$S_3 = \frac{b^4c^2}{2(b^2+c^2-a^2)^2}\left(\mathrm{Arcsin}\,\frac{(b^2+c^2-a^2)u}{b^2c^2} - \sin\frac{(b^2+c^2-a^2)u}{b^2c^2}\right),$$
$$S_4 = \frac{u}{2}.$$

The four figures in Theorem 1.1, which are divisions of $\widehat{\triangle}\mathrm{ABC}$, appear frequently in this paper. For convenience, we write $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$ for $\mathrm{A\widehat{B}A}$, $\mathrm{B\widehat{C}B}$, $\mathrm{C\widehat{A}C}$ and $\triangle\mathrm{ABC}$, respectively.

1.3. **Our Aim of This Paper.** The centroid of a Reuleaux triangle is the same as that of its original triangle (i.e., equilateral triangle), since the Reuleaux triangle is the figure in which all sides of the equilateral triangle are bulged exactly the same way. More precisely, we can prove this as follows: Assume that the centroid $\mathrm{G_{eq}}$ of a Reuleaux triangle $\widehat{\triangle}_{\mathrm{eq}}\mathrm{ABC}$ is different from the centroid $\mathrm{G}_{\triangle}$ of its original equilateral triangle $\triangle\mathrm{ABC}$. Then, rotating $\widehat{\triangle}_{\mathrm{eq}}\mathrm{ABC}$ by 120°, $\mathrm{G_{eq}}$ moves to a different location than $\mathrm{G}_{\triangle}$ and itself. On the other hand, because of the symmetry of the Reuleaux triangle, the figure does not change when it is rotated. This is a contradiction, and hence $\mathrm{G_{eq}} = \mathrm{G}_{\triangle}$.
However, we can think the centroid of a bulging triangle $\widehat{\triangle}\mathrm{ABC}$ is generally different from that of its original triangle $\triangle\mathrm{ABC}$. If so, then it will be interesting to see how different the centroids are between $\triangle\mathrm{ABC}$ and $\widehat{\triangle}\mathrm{ABC}$. We want to investigate that in this paper.
Thus, our aim in this paper is to find (the coordinate of) the centroid of a bulging triangle, and we compute it quantitatively.

2. Properties of Edge-central Angles

As we will see later, we need to know all the edge-central angles so as to obtain the centroid of a bulging triangle. These, $\theta_1 := \angle\mathrm{APB}/2$, $\theta_2 := \angle\mathrm{BQC}/2$ and $\theta_3 := \angle\mathrm{CRA}/2$ are obtained by using Lemma 3.1 in Someyama [8]. However, since this Lemma 3.1 is somewhat incovenient, we shall derive formulas to facilitate the calculation of edge-central angles as follows. The following property, Eq. (2.1) is just a mathematically interesting result on triangles.

Proposition 2.1. *Let* $\widehat{\triangle}\,\mathrm{ABC}$ *be a bulging triangle. Suppose the original triangle* $\triangle\mathrm{ABC}$ *is acute and* $c > a > b$*. Then, the edge-central angles satisfy*

$$\angle\mathrm{APB} = \angle\mathrm{BQC}. \tag{2.1}$$

Moreover, when setting $\mathrm{A}(0, \alpha)$, $\mathrm{B}(\beta, 0)$ *and* $\mathrm{C}(\gamma, 0)$ *(*$\alpha, \gamma > 0$ *and* $\beta < 0$*) under the above assumptions, it follows that*

$$\angle\mathrm{APB} = \mathrm{Arccos}\,\frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}, \tag{2.2}$$

$$\angle\mathrm{BQC} = \mathrm{Arccos}\,\frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}, \tag{2.3}$$

$$\angle\mathrm{CRA} = \mathrm{Arccos}\,\frac{(\alpha^2 + \beta^2)(\alpha^2 + \gamma^2) - 2(\alpha^2 + \beta\gamma)^2}{(\alpha^2 + \beta^2)(\alpha^2 + \gamma^2)}. \tag{2.4}$$

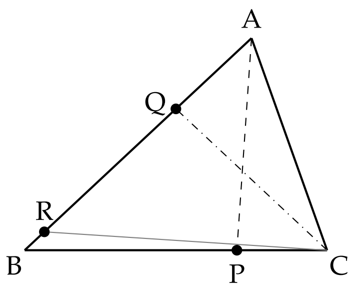


Figure 3. Edge-central angles

Proof. According to the assumption, set $\mathrm{A}(0, \alpha)$, $\mathrm{B}(\beta, 0)$ and $\mathrm{C}(\gamma, 0)$, where $\alpha, \gamma > 0$ and $\beta < 0$.
The straight line AB, $\ell(\mathrm{AB})$ is represented as $y = (-\alpha/\beta)x + \alpha$, and so the perpendicular bisector ℓ_1 of $\overline{\mathrm{AB}}$ is given as

$$\ell_1 :\ y = \frac{\beta}{\alpha}\left(x - \frac{\beta}{2}\right) + \frac{\alpha}{2}.$$

Since $|\overline{\mathrm{BC}}| > |\overline{\mathrm{CA}}|$ is assumed, the AB-center P lies on $\overline{\mathrm{BC}}$. We thus have

$$\mathrm{P}\left(-\frac{\alpha^2}{2\beta} + \frac{\beta}{2},\ 0\right)$$

by easy calculation. This implies

$$|\overline{\mathrm{AP}}| = |\overline{\mathrm{BP}}| = -\frac{\alpha^2}{2\beta} - \frac{\beta}{2},$$

and it is obvious that

$$|\overline{\mathrm{AB}}| = \sqrt{\alpha^2 + \beta^2}.$$

So we have

$$\begin{aligned}\cos \angle \mathrm{APB} &= \frac{|\overline{\mathrm{AP}}|^2 + |\overline{\mathrm{BP}}|^2 - |\overline{\mathrm{AB}}|^2}{2|\overline{\mathrm{AP}}||\overline{\mathrm{BP}}|} \\ &= \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2},\end{aligned}$$

and hence we obtain the desired Eq. (2.2).
From the assumption, $|\overline{\mathrm{AB}}| > |\overline{\mathrm{CA}}|$, the BC-center Q lies on $\overline{\mathrm{AB}}$. The perpendicular bisector ℓ_2 of $\overline{\mathrm{BC}}$ is given as $x = (\beta + \gamma)/2$, and so we have

$$\mathrm{Q}\left(\frac{\beta + \gamma}{2}, -\frac{\alpha(-\beta + \gamma)}{2\beta}\right)$$

by easy calculation. This implies

$$|\overline{\mathrm{BQ}}| = |\overline{\mathrm{CQ}}| = \frac{-\beta + \gamma}{2\beta}\sqrt{\alpha^2 + \beta^2},$$

and it is obvious that

$$|\overline{\mathrm{BC}}| = \gamma - \beta.$$

So we have

$$\begin{aligned}\cos \angle \mathrm{BQC} &= \frac{|\overline{\mathrm{BQ}}|^2 + |\overline{\mathrm{CQ}}|^2 - |\overline{\mathrm{BC}}|^2}{2|\overline{\mathrm{BQ}}||\overline{\mathrm{CQ}}|} \\ &= \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2},\end{aligned}$$

and hence we obtain the desired Eqs. (2.3) and (2.1).
The straight line CA, $\ell(\mathrm{CA})$ is represented as $y = (-\alpha/\gamma)x + \alpha$, and so the perpendicular bisector ℓ_3 of $\overline{\mathrm{CA}}$ is given as

$$\ell_3 : \; y = \frac{\gamma}{\alpha}\left(x - \frac{\gamma}{2}\right) + \frac{\alpha}{2}.$$

Since $|\overline{\mathrm{AB}}| > |\overline{\mathrm{BC}}|$ is assumed, the CA-center R lies on $\overline{\mathrm{AB}}$. We thus have

$$\mathrm{R}\left(\frac{\beta(\alpha^2 + \gamma^2)}{2(\alpha^2 + \beta\gamma)}, \frac{\alpha(\alpha^2 + 2\beta\gamma - \gamma^2)}{2(\alpha^2 + \beta\gamma)}\right)$$

by easy calculation. This implies

$$|\overline{\mathrm{CR}}| = |\overline{\mathrm{AR}}| = \frac{\alpha^2 + \gamma^2}{2(\alpha^2 + \beta\gamma)}\sqrt{\alpha^2 + \beta^2},$$

and it is obvious that

$$|\overline{\mathrm{CA}}| = \sqrt{\alpha^2 + \gamma^2}.$$

So we have

$$\begin{aligned}\cos\angle\mathrm{CRA} &= \frac{|\overline{\mathrm{CR}}|^2+|\overline{\mathrm{AR}}|^2-|\overline{\mathrm{CA}}|^2}{2|\overline{\mathrm{CR}}||\overline{\mathrm{AR}}|}\\ &= \frac{(\alpha^2+\beta^2)(\alpha^2+\gamma^2)-2(\alpha^2+\beta\gamma)^2}{(\alpha^2+\beta^2)(\alpha^2+\gamma^2)},\end{aligned}$$

and hence we obtain the desired Eq. (2.4).
This completes the proof. □

We have the following claim by virtue of this result and Lemma 3.1 in Someyama [8].

Corollary 2.1. *Make the same assumptions as in Proposition 2.1. If* $c \geq \sqrt{2}b$*, then the edge-central angles satisfy*

$$\angle\mathrm{APB} = \angle\mathrm{BQC} > \angle\mathrm{CRA}.$$

3. Centroids of Bulging Triangles

We want to investigate if the centroid of a bulging triangle coincides with that of its original triangle. In order to do that, let us find the coordinate of the centroid of a bulging triangle in this section.
Now, a bulging triangle can be regarded as a composite figure; see the figure in Eq. (1.2). The centroid of a composite figure should be given as a weighted mean for the centroids of parts of the composite figure; the centroid of a general figure is given by multiple integration, see, e.g., Friedman [3]. What to adopt for weights will be discussed later.
Recall first the following formula on the centroid of a crescent. We can know it in, e.g., Wikipedia [10].

Proposition 3.1. *Consider a crescent* $\widehat{\mathrm{STS}}$ *that is part of a semicircle whose center is the origin* O *and whose radius is* r*; see Figure 4. The centroid* $\mathrm{G_{cr}}$ *of* $\widehat{\mathrm{STS}}$ *is given by*

$$\mathrm{G_{cr}}\,(0,g)\,,\qquad g := \frac{2r^3\sin^3\theta}{3\mathrm{Ar}(\widehat{\mathrm{STS}})} = \frac{4r\sin^3\theta}{3(2\theta-\sin 2\theta)},$$

where $\theta := \angle\mathrm{SOG_{cr}} = \angle\mathrm{TOG_{cr}} \in (0,\pi/2]$ *and* $\mathrm{Ar}(F)$ *denotes the area of a figure* F*.*

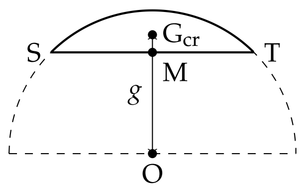


Figure 4. Centroid of a crescent

We rephrase this result for ease of use.

Lemma 3.1. *Consider the same crescent* $\widehat{\mathrm{STS}}$ *as in Proposition 3.1. Recall* $g = |\overline{\mathrm{OG_{cr}}}|$, $r = |\overline{\mathrm{OS}}| = |\overline{\mathrm{OT}}|$ *and* $\theta = \angle\mathrm{SOG_{cr}} = \angle\mathrm{TOG_{cr}} \in (0, \pi/2]$. *Denote by* M *the midpoint of* $\overline{\mathrm{ST}}$, *and write*

$$\widetilde{g} := g - |\overline{\mathrm{OM}}|.$$

Then, one has

$$\widetilde{g} = \frac{2r(-\sin^3\theta - 3\theta\cos\theta + 3\sin\theta)}{3(2\theta - \sin 2\theta)}.$$

In other words, if M *is the origin, the centroid of the crescent is given by* $\widetilde{\mathrm{G}}_{\mathrm{cr}}(0, \widetilde{g})$.

Proof. Since $|\overline{\mathrm{OM}}| = r\cos\theta$, we have

$$\begin{aligned}\widetilde{g} &= \frac{4r\sin^3\theta}{3(2\theta - \sin 2\theta)} - r\cos\theta \\ &= \frac{2r(-\sin^3\theta - 3\theta\cos\theta + 3\sin\theta)}{3(2\theta - \sin 2\theta)}\end{aligned}$$

by virtue of Proposition 3.1. □

Remark 3.1. *It must be held that* $\widetilde{g} > 0$, *but this is easily verified. In fact, first,* $\theta \in (0, \pi/2]$ *implies that* $2\theta - \sin 2\theta \geq 0$. *Next, putting* $f(\theta) := -\sin^3\theta - 3\theta\cos\theta + 3\sin\theta$, *this is monotone increasing since*

$$\frac{df}{d\theta}(\theta) = \frac{3}{2}(2\theta - \sin 2\theta)\sin\theta > 0,$$

and satisfies

$$\lim_{\theta\downarrow 0} f(\theta) = 0.$$

Hence, $f(\theta) > 0$ *for all* $\theta \in (0, \pi/2]$.

This paper also requires the following formula on coordinates of edge-centers and edge-radii.

Lemma 3.2. *Set* $\mathrm{A}(0, \alpha)$, $\mathrm{B}(\beta, 0)$ *and* $\mathrm{C}(\gamma, 0)$, *where* $\alpha, \gamma > 0$ *and* $\beta < 0$ *(See Figure 5). Then, it follows that*

$$\begin{aligned}&\mathrm{P}\left(-\frac{\alpha^2 - \beta^2}{2\beta}, 0\right), \\ &\mathrm{Q}\left(\frac{\beta + \gamma}{2}, \frac{\alpha(\beta - \gamma)}{2\beta}\right), \\ &\mathrm{R}\left(\frac{\beta(\alpha^2 - \gamma^2)}{2(\alpha^2 + \beta\gamma)}, \frac{\alpha(\alpha^2 + \gamma^2 + 2\beta\gamma)}{2(\alpha^2 + \beta\gamma)}\right),\end{aligned}$$

and

$$r_{\mathrm{AB}} = \frac{\alpha^2 + \beta^2}{2|\beta|}, \tag{3.1}$$

$$r_{\mathrm{BC}} = \frac{-\beta + \gamma}{2\beta}\sqrt{\alpha^2 + \beta^2}, \tag{3.2}$$

$$r_{\mathrm{CA}} = \frac{\alpha^2 - \gamma^2}{2(\alpha^2 + \beta\gamma)}\sqrt{\alpha^2 + \beta^2}. \tag{3.3}$$

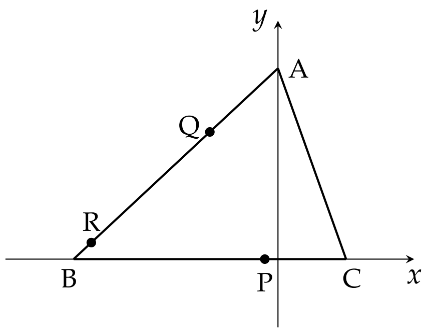


Figure 5

Proof. We leave the proof to the reader, because it is easy to see the claim. In Eq. (3.2), note that $\gamma - \beta > 0$ from the assumption that $\beta < 0$ and $\gamma > 0$. In Eq. (3.3), note that $\alpha^2 - \gamma^2 > 0$. Indeed, the assumption for $\triangle$ABC implies that

$$\angle \mathrm{ACB} > \angle \mathrm{BAC} > \angle \mathrm{OAC},$$

and hence we have

$$\alpha = |\overline{\mathrm{OA}}| > |\overline{\mathrm{OC}}| = \gamma.$$

This implies that $\alpha^2 > \gamma^2$, since $\alpha, \gamma > 0$. $\square$

Remark 3.2. *There are many other coordinate settings besides Lemma 3.2. It is not however wise to adopt any other setting, and so we shall henceforth set the coordinates in Lemma 3.2.*

By using the formulas in Lemma 3.2, we derive the following result.

Theorem 3.1. *Consider a* $\widehat{\triangle}$ABC *with coordinates* $\mathrm{A}(0, \alpha)$, $\mathrm{B}(\beta, 0)$ *and* $\mathrm{C}(\gamma, 0)$; $\alpha, \gamma > 0$ *and* $\beta < 0$. S *denotes the area of* $\widehat{\triangle}$ABC. *Recall that* S_j *is the area of* $\mathcal{S}_j$ *for* $j = 1, \ldots, 4$ *given in Theorem 1.1. Denote by* (x_j, y_j) *the centroid of* $\mathcal{S}_j$ *for* $j = 1, \ldots, 4$. *Then, the centroid* $\mathrm{G}_{\widehat{\triangle}}$ *of* $\widehat{\triangle}$ABC *is given by*

$$\mathrm{G}_{\widehat{\triangle}}\left(\frac{1}{S}\sum_{j=1}^{4} S_j x_j,\ \frac{1}{S}\sum_{j=1}^{4} S_j y_j\right); \tag{3.4}$$

$$
\begin{aligned}
x_1 &= \frac{\sqrt{\alpha^2+\beta^2}\cos\phi_1}{2} - \frac{(\alpha^2+\beta^2)(-\sin^3\theta_1 - 3\theta_1\cos\theta_1 + 3\sin\theta_1)\sin\phi_1}{-3\beta(2\theta_1 - \sin 2\theta_1)} + \beta,\\
y_1 &= \frac{\sqrt{\alpha^2+\beta^2}\sin\phi_1}{2} + \frac{(\alpha^2+\beta^2)(-\sin^3\theta_1 - 3\theta_1\cos\theta_1 + 3\sin\theta_1)\cos\phi_1}{-3\beta(2\theta_1 - \sin 2\theta_1)},\\
x_2 &= \frac{\beta+\gamma}{2},\\
y_2 &= \frac{(-\beta+\gamma)\sqrt{\alpha^2+\beta^2}(-\sin^3\theta_1 - 3\theta_1\cos\theta_1 + 3\sin\theta_1)}{-3\beta(2\theta_1 - \sin 2\theta_1)} \left(= \frac{-\beta+\gamma}{\sqrt{\alpha^2+\beta^2}} y_1\Big|_{\phi_1=0}\right),\\
x_3 &= -\frac{\sqrt{\alpha^2+\gamma^2}\cos\phi_3}{2} + \frac{(\alpha^2-\gamma^2)\sqrt{\alpha^2+\beta^2}(-\sin^3\theta_3 - 3\theta_3\cos\theta_3 + 3\sin\theta_3)\sin\phi_3}{3(\alpha^2+\beta\gamma)(2\theta_3 - \sin 2\theta_3)} + \gamma,\\
y_3 &= \frac{\sqrt{\alpha^2+\gamma^2}\sin\phi_3}{2} + \frac{(\alpha^2-\gamma^2)\sqrt{\alpha^2+\beta^2}(-\sin^3\theta_3 - 3\theta_3\cos\theta_3 + 3\sin\theta_3)\cos\phi_3}{3(\alpha^2+\beta\gamma)(2\theta_3 - \sin 2\theta_3)},\\
x_4 &= \frac{\beta+\gamma}{3},\\
y_4 &= \frac{\alpha}{3};
\end{aligned}
$$

and

$$\phi_1 = \operatorname{Arctan}\frac{\alpha}{|\beta|}, \quad \phi_3 = \operatorname{Arctan}\frac{\alpha}{\gamma}.$$

Furthermore, $2\theta_1$*,* $2\theta_2$ *and* $2\theta_3$ *are given as the* AB*-central angle,* BC*-central angle and* CA*-central angle of* $\widehat{\triangle}$ABC*, respectively.*

Proof. A bulging triangle can be generally regarded as a composite figure consisting of four figures, one is a triangle and the others are crescents. “A triangle” is $\triangle$ABC and “three crescents” are $\mathcal{S}_1 = \mathrm{A}\widehat{\mathrm{B}}\mathrm{A}$, $\mathcal{S}_2 = \mathrm{B}\widehat{\mathrm{C}}\mathrm{B}$ and $\mathcal{S}_3 = \mathrm{C}\widehat{\mathrm{A}}\mathrm{C}$; see the figure of Eq. (1.2).

The centroid $\mathrm{G}_4(x_4, y_4)$ of $\triangle$ABC is easily obtained:

$$x_4 = \frac{\beta+\gamma}{3}, \quad y_4 = \frac{\alpha}{3}.$$

We compute the centroids of $\mathrm{A}\widehat{\mathrm{B}}\mathrm{A}$, $\mathrm{B}\widehat{\mathrm{C}}\mathrm{B}$ and $\mathrm{C}\widehat{\mathrm{A}}\mathrm{C}$. They are denoted by $\mathrm{G}_1(x_1, y_1)$, $\mathrm{G}_2(x_2, y_2)$ and $\mathrm{G}_3(x_3, y_3)$, respectively.

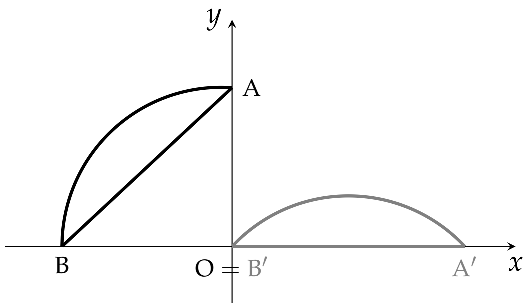


Figure 6

First, $\mathcal{S}_1$ is the figure rotated and translated the crescent $\mathrm{A}'\widehat{\mathrm{B}'}\mathrm{A}'$ with center at $\mathrm{B}' = \mathrm{O}$; see Figure 6. The centroid of $\mathrm{A}'\widehat{\mathrm{B}'}\mathrm{A}'$ is

$$\mathrm{G}_1'\left(\frac{\sqrt{\alpha^2+\beta^2}}{2},\ \frac{2r_{\mathrm{AB}}(-\sin^3\theta_1 - 3\theta_1\cos\theta_1 + 3\sin\theta_1)}{3(2\theta_1 - \sin 2\theta_1)}\right).$$

In fact, the x-coordinate is obtained since $|\overline{\mathrm{A}'\mathrm{B}'}| = |\overline{\mathrm{AB}}| = \sqrt{\alpha^2+\beta^2}$, and the y-coordinate is obtained from Lemma 3.1. Hence, G_1 is given by rotating G_1' counterclockwise by argument $\phi_1 := \mathrm{Arctan}(\alpha/|\beta|) = \angle\mathrm{ABC}$ and translating by $|\beta| = -\beta$ in the left direction. That is, computing

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{\alpha^2+\beta^2}}{2} \\ \frac{2r_{\mathrm{AB}}(-\sin^3\theta_1 - 3\theta_1\cos\theta_1 + 3\sin\theta_1)}{3(2\theta_1 - \sin 2\theta_1)} \end{pmatrix} - \begin{pmatrix} |\beta| \\ 0 \end{pmatrix}$$

and recalling Eq. (3.1), we have the coordinate of G_1.
Next, Lemma 3.1 implies that the coordinate of G_2 is given by

$$(x_2,\ y_2) = \left(\frac{\beta+\gamma}{2},\ \frac{2r_{\mathrm{BC}}(\sin^3\theta_2 + 3\theta_2\cos\theta_2 - 3\sin\theta_2)}{3(2\theta_2 - \sin 2\theta_2)}\right);$$

note that the sign of the y-coordinate has changed and Proposition 2.1 implies $\theta_2 = \theta_1$, and recall Eq. (3.2).
Finally, we find the centroid G_3 of $\mathcal{S}_3$ in the same way as G_1. $\mathcal{S}_3$ is the figure rotated and translated the crescent $\mathrm{C}'\widehat{\mathrm{A}'}\mathrm{C}'$ with center at $\mathrm{C}' = \mathrm{O}$; consider in the same way as $\mathcal{S}_1$. The centroid of $\mathrm{C}'\widehat{\mathrm{A}'}\mathrm{C}'$ is

$$\mathrm{G}_3'\left(-\frac{\sqrt{\alpha^2+\gamma^2}}{2},\ \frac{2r_{\mathrm{CA}}(-\sin^3\theta_3 - 3\theta_3\cos\theta_3 + 3\sin\theta_3)}{3(2\theta_3 - \sin 2\theta_3)}\right).$$

In fact, the x-coordinate is obtained since $|\overline{\mathrm{C}'\mathrm{A}'}| = |\overline{\mathrm{CA}}| = \sqrt{\alpha^2+\gamma^2}$, and the y-coordinate is obtained from Lemma 3.1. Hence, G_3 is given by rotating G_3' clockwise by argument $\phi_3 := \mathrm{Arctan}(\alpha/\gamma) = \angle\mathrm{BCA}$ and translating by γ in the right direction. That is, computing

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} \cos(-\phi_3) & -\sin(-\phi_3) \\ \sin(-\phi_3) & \cos(-\phi_3) \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{\alpha^2+\gamma^2}}{2} \\ \frac{2r_{\mathrm{CA}}(-\sin^3\theta_3 - 3\theta_3\cos\theta_3 + 3\sin\theta_3)}{3(2\theta_3 - \sin 2\theta_3)} \end{pmatrix} + \begin{pmatrix} \gamma \\ 0 \end{pmatrix}$$

and recalling Eq. (3.3), we have the coordinate of G_3.
See Section 2 for how to find θ_1, θ_2 and θ_3.
By the way, $\mathrm{G}_{\widehat{\triangle}}$ is given by considering the weighted mean of G_j ($j = 1, \ldots, 4$). It is then important to determine the weights by noting that the thickness and density of the object are uniform. For example, in case of water, 1 g can be considered equal to 1 mℓ. Also, 1 mℓ is equal to 1 cm^3. In other words, we may assume weights can be expressed by volumes. So, if we think of all $\mathcal{S}_j$ as three-dimensional figures with height 1, the volume of each $\mathcal{S}_j$ is equal to its area. Also, the area S of $\widehat{\triangle}\,\mathrm{ABC}$ is not 1, and so the weighted

sums $\sum S_j x_j$, $\sum S_j y_j$ should be divided by S. Therefore, we shall adopt weights as areas. From the above, we have the coordinate of $\mathrm{G}_{\widehat{\triangle}}$, i.e.,

$$\bar{x}_{\widehat{\triangle}} = \frac{1}{S}\sum_{j=1}^{4} S_j x_j, \quad \bar{y}_{\widehat{\triangle}} = \frac{1}{S}\sum_{j=1}^{4} S_j y_j.$$

This completes the proof. □

4. Verification by Examples

4.1. **Comparison with Reuleaux Triangles.** As mentioned in Section 1.3, that the centroid of a Reuleaux triangle is the same of that of the original triangle (=equilateral triangle) can be proved by a qualitative approach. Let us verify that the centroid of a Reuleaux triangle coincides with that of its original equilateral triangle by applying Theorem 3.1 to a Reuleaux triangle, i.e., by taking a quantitative approach.

Let $\widehat{\triangle}_{\mathrm{eq}}\mathrm{ABC}$ be a Reuleaux triangle. Since the original triangle $\triangle\mathrm{ABC}$ is an equilateral triangle, we can set $\mathrm{A}(0, \sqrt{3}\alpha)$, $\mathrm{B}(-\alpha, 0)$ and $\mathrm{C}(\alpha, 0)$ with $\alpha > 0$. Denote by G_1, G_2, G_3 and G_4 the centroids of $\mathrm{A}\widehat{\mathrm{B}}\mathrm{A}$, $\mathrm{B}\widehat{\mathrm{C}}\mathrm{B}$, $\mathrm{C}\widehat{\mathrm{A}}\mathrm{C}$ and $\triangle\mathrm{ABC}$, respectively. Note that

$$\begin{aligned} r_{\mathrm{AB}} &= r_{\mathrm{BC}} = r_{\mathrm{CA}} = 2\alpha, \\ \theta_1 &= \theta_2 = \theta_3 = \frac{\pi}{6}, \\ \phi_1 &= \phi_3 = \frac{\pi}{3}. \end{aligned}$$

Then, applying Theorem 3.1 for $\alpha \mapsto \sqrt{3}\alpha$, $\beta \mapsto -\alpha$ and $\gamma \mapsto \alpha$, we have

$$\begin{aligned} &\mathrm{G}_1\left(\frac{2\pi - 4\sqrt{3}}{2\pi - 3\sqrt{3}}\alpha,\ \frac{1}{2\pi - 3\sqrt{3}}\alpha\right), \\ &\mathrm{G}_2\left(0,\ \frac{2\sqrt{3}\pi - 11}{2\pi - 3\sqrt{3}}\alpha\right), \\ &\mathrm{G}_3\left(-\frac{2\pi - 4\sqrt{3}}{2\pi - 3\sqrt{3}}\alpha,\ \frac{1}{2\pi - 3\sqrt{3}}\alpha\right), \\ &\mathrm{G}_4\left(0,\ \frac{\sqrt{3}}{3}\alpha\right). \end{aligned}$$

(We can obtain G_3 from G_1 by virtue of symmetry with respect to the vertical axis.)
Easy calculation implies

$$\begin{aligned} S_{\mathrm{cr}} :=& \mathrm{Ar}(\mathrm{A}\widehat{\mathrm{B}}\mathrm{A}) = \mathrm{Ar}(\mathrm{B}\widehat{\mathrm{C}}\mathrm{B}) = \mathrm{Ar}(\mathrm{C}\widehat{\mathrm{A}}\mathrm{C}) = \frac{2\pi - 3\sqrt{3}}{3}\alpha^2, \\ S =& \mathrm{Ar}(\widehat{\triangle}\mathrm{ABC}) = 2(\pi - \sqrt{3})\alpha^2, \end{aligned}$$

and so the centroid $G(x, y)$ of $\widehat{\triangle}_{eq}ABC$ is obtained as follows: It is obviously that $x = 0$. Considering the weighted mean, we have

$$\begin{aligned} y &= S_{cr} \cdot \left(\frac{1}{2\pi - 3\sqrt{3}}\alpha + \frac{1}{2\pi - 3\sqrt{3}}\alpha + \frac{2\sqrt{3}\pi - 11}{2\pi - 3\sqrt{3}}\alpha \right) + \mathrm{Ar}(\triangle ABC) \cdot \frac{\sqrt{3}}{3}\alpha \\ &= \frac{2\sqrt{3}\pi - 6}{3}\alpha^3. \end{aligned}$$

From the above and Eq. (3.4), we gain

$$G\left(0,\ \frac{\sqrt{3}}{3}\alpha\right),$$

and hence this coincides with G_4. In other words, the centroid of a Reuleaux triangle is acutually equal to that of its original equilateral triangle.

Remark 4.1. *In case of Reuleaux triangles, it is not necessary to consider the weighted mean; it is sufficient to consider the arithmetic mean of* G_j, $j = 1, \ldots, 4$:

$$\begin{aligned} x &= \frac{1}{4}\left(\frac{2\pi - 4\sqrt{3}}{2\pi - 3\sqrt{3}}\alpha + 0 - \frac{2\pi - 4\sqrt{3}}{2\pi - 3\sqrt{3}}\alpha + 0 \right) = 0, \\ y &= \frac{1}{4}\left(\frac{1}{2\pi - 3\sqrt{3}}\alpha + \frac{2\sqrt{3}\pi - 11}{2\pi - 3\sqrt{3}}\alpha + \frac{1}{2\pi - 3\sqrt{3}}\alpha + \frac{\sqrt{3}}{3}\alpha \right) = \frac{\sqrt{3}}{3}\alpha. \end{aligned}$$

4.2. **Comparison with Original Triangles.** We see how much the centroid of the bulging triangle is shifted compared to the original triangle through a concrete example. Or, is the case where the centroid is equal to that of its original triangle limited to the case of the Reuleaux triangle?

Let us set $A(0, \sqrt{6})$, $B(-2, 0)$ and $C(1, 0)$. Then, note that this $\triangle ABC$ is acute and satisfies $c > a > b$. Denote by g_{x_1}, g_{x_2}, g_{x_3} and g_{x_4} the x-coordinates of G_1, G_2, G_3 and G_4, respectively. Moreover, we write g_x for the x-coordinate of $G_{\widehat{\triangle}}$.

First, it is obvious that

$$-2 < g_{x_1} < -1, \quad g_{x_2} = -\frac{1}{2}, \quad \frac{1}{2} < g_{x_3} < 1, \quad g_{x_4} = -\frac{1}{3}.$$

Therefore, we have

$$-2 < g_{x_1} + g_{x_2} + g_{x_3} < -\frac{1}{2}, \quad \text{i.e.,} \quad g_{x_1} + g_{x_2} + g_{x_3} < 0.$$

On the other hand, Theorem 1.1 implies $S_1, S_2 \geq S_3$ regardless of cases i) and ii). From these, it clearly holds that

$$g_x = \frac{1}{S}\sum_{j=1}^{4} S_j g_{x_j} < g_{x_4},$$

and so this says that the centroid of $\widehat{\triangle}ABC$ is shifted at least to the left of that of $\triangle ABC$. Hence, we can conclude that the centroid of a bulging triangle is generally different from that of its original triangle.

5. CONCLUSIONS

We have derived (the coordinate of) the centroid $G_{\widehat{\triangle}}$ of a bulging triangle and have compared it with (that of) the centroid $G_{\triangle}$ of its original triangle. As a result, we have found that $G_{\widehat{\triangle}}$ does not generally coincide with $G_{\triangle}$. It will be interesting to see if this result is a useful property in terms of application. Reuleaux triangles are applying to various fields, but among them, their application to technical engineering is remarkable. We hope that bulging triangles will be applied to technical engineering in the same way.
Whether the bulging triangle whose own centroid coincides with that of its original triangle is limited to Reuleaux triangle is our remaining research problem.
Moreover, in this paper, the centroid of a bulging triangle was given by a quantitative approach using analytic geometry. It is however expected to exist a way how to find the centroid of a bulging triangle by a qualitative approach.
We want to study them in the future.

REFERENCES

[1] Barrallo, J., González-Quintial, F. and Sánchez-Beitia, S., *An Introduction to the Vesica Piscis, the Reuleaux Triangle and Related Geometric Constructions in Modern Architecture*, Nexus Network Journal, 17 (2015), 671-684.

[2] Conti, G. and Paoletti, R., *Reuleaux Triangle in Architecture and Applications*, Faces of Geometry. From Agnesi to Mirzakhani (Springer, 2019), 79-89.

[3] Friedman, A., *Advanced Calculus*, Holt, Rinehart & Winston of Canada Ltd. (1971).

[4] Hu, X., Li, N. and Liu, BY., *Simulation and Application of Reuleaux Triangle In Geometric Measurement*, IOP Conference Series: Earth and Environmental Science, 310 (2019), 22-28.

[5] Ridley, J.N., *A Generalization of Reuleaux Triangle*,
http://frink.machighway.com/dynamicm/constant-diameter-curves.pdf,
2 pages.

[6] Someyama, N., *Bulging Triangles: Generalization of Reuleaux Triangles*, Global Journal of Advanced Research on Classical and Modern Geometries, Vol. 10 (2021), No. 2, 166-177.

[7] Someyama, N., *Bulging Triangles II: On Their Circumscribed Circles*, Global Journal of Advanced Research on Classical and Modern Geometries, Vol. 11 (2022), No. 1, 134-139.

[8] Someyama, N., *Lengths of Edges and Areas of Bulging Triangles*, International Journal of Geometry, Vol. 12 (2023), No. 1, 57-70.

[9] Someyama, N., *Qualitative Properties on Bulging Triangles and Their Original Triangles*, Journal of Advanced Studies in Topology, Vol. 12 (2022), No. 1-2, 57-65.

[10] Wikipedia, *List of Centroids*, https://en.wikipedia.org/wiki/List_of_centroids.

SHIN-YO-JI BUDDHIST TEMPLE, TOKYO, JAPAN., ORCID ID: HTTPS://ORCID.ORG/0000-0001-7579-5352
Email address: `philomatics@outlook.jp`