Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.1, (2024), Issue 2, pp.83-89

NOTES ON COSYMPLECTIC MANIFOLDS

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ABSTRACT. In the present paper, we study a canonical cosymplectic manifold and some conditions on which the leaves of a characteristic foliation are lagrangian submanifold. We also study the reduction of cosymplectic manifolds from which arises the momentum mapping.

1. INTRODUCTION

An almost cosymplectic structure on a manifold M of odd dimension $2n+1$ is a pair (η, Ω), where η is a 1-form and Ω is a 2-form such that $\eta \wedge \Omega^n$ is a volume form on M. The structure is said to be cosymplectic if η and Ω are d-closed. Here d is the exterior differential operator.
The manifold (M, η, Ω) admits an atlas of canonical (Darboux) chart: in the neighborhood of every point, one can determine canonical coordinates $(t, x_1, ..., x_{2n})$ such that

$$\eta = dt, \ \Omega = \sum_{i=1}^{n} dx_i \wedge dx_{n+i}. \tag{1.1}$$

It well known that every almost cosymplectic structure (η, Ω) on M induces an isomorphism of $\mathcal{C}^\infty(M)$-modules

$$\flat_{(\eta,\Omega)} : \mathfrak{X}(M) \longrightarrow \Lambda^1(M), X \longmapsto i_X\Omega + \eta(X)\eta.$$

In terms of bundle (see [1]), one can write as

$$\flat_{(\eta,\Omega)} : TM \longrightarrow T^*M.$$

The Reeb vector field of the almost cosymplectic manifold (M, η, Ω) is determined by

$$\xi = \flat^{-1}_{(\eta,\Omega)}(\eta) \tag{1.2}$$

and characterized by

$$i_\xi\Omega = 0 \text{ and } \eta(\xi) = 1 \tag{1.3}$$

where 1 is the unity of $\mathcal{C}^\infty(M)$.
In an atlas of canonical chart, the Reeb vector field is given by

$$\xi = \frac{\partial}{\partial t}. \tag{1.4}$$

2020 *Mathematics Subject Classification.* 53C12, 53C25, 53D15, 53D20, 55N91.
Key words and phrases. Local gradient, momentum map, lagrangian submanifolds.

A smooth map $\psi : (M, \eta, \Omega) \longrightarrow (M', \eta', \Omega')$ between cosymplectic manifolds is said to be cosymplectic if

$$\psi^*\eta' = \eta \text{ and } \psi^*\Omega' = \Omega. \tag{1.5}$$

In this case, the Reeb vector field ξ of M is ψ-projectable and its projection is the Reeb vector field ξ' of M', that is,

$$T\psi \circ \xi = \xi' \circ \psi. \tag{1.6}$$

As in the symplectic case, a cosymplectic map is not in general a Poisson map. For a cosymplectic manifold (M, η, Ω) one can use the vector bundle isomorphism $\flat_{(\eta,\Omega)}$ to pull back the canonical symplectic 2-form Ω_M of the cotangent bundle T^*M to the tangent bundle TM by setting

$$\Omega_0 = \flat^*_{(\eta,\Omega)}\Omega_M. \tag{1.7}$$

Therefore the tangent bundle of a cosymplectic manifold is a symplectic manifold. Actually, the existence of the vector bundle isomorphism $\flat_{(\eta,\Omega)}$ means that the tangent bundle of a cosymplectic manifold has a Liouville structure (also known as special symplectic structure) in the sense of Tulczyjew. In [2] the authors obtain an explicit expression for the symplectic structure Ω_0 on the tangent bundle in terms of the cosymplectic structure (η, Ω), by using the notion of tangent derivation, again due to Tulczyjew [5].

The main content of the paper is divided into two sections. Section 2 is devoted to the notion of Hamiltonian vector field. On a cosymplectic manifold, three types of vector fields are associated with a differentiable function f, namely the gradient vector field, the Hamiltonian vector field and the evolution vector fields. These vector fields are defined and studied. Relevant relationships between them are investigated. Section 3 devoted to the analogue of the the momentum map of Hamiltonian action of Lie groups on symplectic manifolds. We focus on the analogue of the Weinstein-Marsden symplectic reduction.

2. HAMILTONIAN VECTOR FIELDS

Let M be a smooth manifold, finite dimensional and paracompact. Then the Lie algebra $\mathfrak{X}(M) = \Gamma(TM)$ of vector fields on M is a module over the commutative algebra $\mathcal{C}^\infty(M)$ of smooth functions on M, and $\mathfrak{X}(M)$ acts on $\mathcal{C}^\infty(M)$ as Lie algebra of derivations, via the map

$$\mathfrak{X}(M) \longrightarrow Der_{\mathbb{R}}\left[\mathcal{C}^\infty(M)\right].$$

Let N be another smooth manifold and $\rho : M \longrightarrow N$ be a diffeomorphism, then $\rho^* : \mathcal{C}^\infty(N) \longrightarrow \mathcal{C}^\infty(M)$ is an isomorphism of $\mathbb{R}$-algebras with inverse $(\rho^*)^{-1} : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(N)$. The map

$$\mathcal{C}^\infty(M) \times \mathfrak{X}(N) \longrightarrow \mathfrak{X}(N), (f, X) \longmapsto (\rho^*)^{-1}(f) \cdot X$$

endowes $\mathfrak{X}(N)$ with a $\mathcal{C}^\infty(M)$-module structure.

Proposition 2.1. *The map*

$$\rho_* : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(N), X \longmapsto \rho^T \circ X \circ \rho^{-1}$$

is simultaneously an isomorphism of $\mathcal{C}^\infty(M)$-modules and of $\mathbb{R}$-Lie algebras. Meorever,

$$\rho_*(f \cdot X) = (f \circ \rho^{-1}) \cdot \rho_* X,$$

for all $f \in \mathcal{C}^\infty(M), X \in \mathfrak{X}(M)$.

Proof. The proof is not difficult. □

The support of the vector field X on a smooth manifold M, is the set

$$Supp(X) = \overline{\{x \in M / X(x) \neq 0\}}.$$

If a function f has a compact support, so does X. Then X generates a flow $\{\varphi_t\}$ on M such that $\varphi_t^*\Omega = \Omega$ and $\varphi_t^*\eta = \eta$ [3].

Theorem 2.1. *Suppose that $\{\varphi_t\}$ is the flow of the vector field X on a cosymplectic manifold (M, η, Ω) and $\rho : M \longrightarrow N$ a diffeomorphism of M onto N. Then the flow of the vector field $\rho_* X$ is $\{\rho \circ \varphi_t \circ \rho^{-1}\}$.*

Proof. The family $\{\rho \circ \varphi_t \circ \rho^{-1}\}$ is the diffeomorphisms family of N onto N. The map $\mathbb{R} \longrightarrow N, t \longmapsto (\rho \circ \varphi_t \circ \rho^{-1})(\rho(x))$, for any $x \in M$ is smooth. We also verify that, for any $t, t' \in \mathbb{R}, \varphi_t \circ \varphi_{t'} = \varphi_{t+t'}$, that is, $\{\rho \circ \varphi_t \circ \rho^{-1}\}$ is a one parameter group of diffeomorphisms of N onto itself. Moreover, we get easily $\rho_* X = (\rho^{-1})^* \circ X \circ \rho^*$. □

Lemma 2.1. *Let X be a vector field on (M, η, Ω), i.e., $X = X_\eta + X_\Omega$, then we have the following assertions.*

1) *The bracket $[X_\eta, X_\Omega]$ vanishes.*
2) *The flow $\{\varphi_t\}$ generated by X decomposes as $\varphi_t = \varphi_t^\eta \circ \varphi_t^\Omega = \varphi_t^\Omega \circ \varphi_t^\eta$ where φ_t^η (resp. φ_t^Ω) is the flow generated by X_η (resp. X_Ω).*
3) *The bracket $[X, Y]$ is a vector field on (M, η, Ω) for Y a vector field on (M, η, Ω).*

Lemma 2.2. *Let X be a vector field on a cosymplectic manifold (M, η, Ω). if X has a compact support, then X generates a global one parameter group of diffeomorphisms of M onto itself.*

To each function $f \in \mathcal{C}^\infty(M)$ one can associate three vector fields on M:

(1) The gradient vector field $gradf$, which is defined by

$$gradf = \flat^{-1}_{(\eta,\Omega)}(df)$$

or equivalently,

$$i_{gradf}\Omega = df - \xi(f)\eta, \; i_{gradf}\eta = \xi(f).$$

(2) The hamiltonian vector field X_f according to

$$X_f = \flat^{-1}_{(\eta,\Omega)}(df - \xi(f)\eta)$$

or equivalently,

$$i_{X_f}\Omega = df - \xi(f)\eta, \; i_{X_f}\eta = 0.$$

(3) The evolution vector field $E_f = \xi + X_f$.

In Darboux coordinates, we find

$$grad f = \frac{\partial f}{\partial t}\cdot\frac{\partial}{\partial t} + \frac{\partial f}{\partial x_{n+i}}\cdot\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\cdot\frac{\partial}{\partial x_{n+i}} \tag{2.1}$$

$$X_f = \frac{\partial f}{\partial x_{n+i}}\cdot\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\cdot\frac{\partial}{\partial x_{n+i}} \tag{2.2}$$

$$E_f = \frac{\partial}{\partial t} + \frac{\partial f}{\partial x_{n+i}}\cdot\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\cdot\frac{\partial}{\partial x_{n+i}}. \tag{2.3}$$

Cosymplectic manifolds are a natural framework to develop the geometric formulation of time-dependent hamiltonian systems. The dynamics on a cosymplectic manifold (M, η, Ω) are introduced by giving a hamiltonian function $f \in \mathcal{C}^\infty(M)$. In fact, the integral curves of the evolution vector field E_f satisfy the Hamilton or motion equations corresponding to f:

$$\frac{dx_i}{dt} = \frac{\partial f}{\partial x_{n+i}},\ \frac{dx_{n+i}}{dt} = -\frac{\partial f}{\partial x_i}. \tag{2.4}$$

From (2.3) we deduce that the flow $\gamma(t, x)$ of E_f is characterized by

$$\frac{d}{dt}g(\gamma(t,x)) = \{g, f\}(\gamma(t,x)) + \xi(g)(\gamma(t,x)). \tag{2.5}$$

Notice that for any cosymplectic structure (η, Ω) on M, its modified structure is given by $(\eta, \Omega + df \wedge \eta)$. In this case, one notes that the Reeb vector field is equal to the evolution vector field.
On $\mathcal{C}^\infty(M)$ one can define a Poisson bracket

$$\{f, g\} = \Omega(grad\, f, grad\, g) = \Omega(X_f, X_g) = \Omega(E_f, E_g). \tag{2.6}$$

The two last egalities of (2.6) are due to the fact that $i_\xi\Omega = 0$. It is easy to prove that the distribution $ker\,\eta$ is integrable and this induces a foliation on which the leaves have symplectic structure.
A vector field X on a cosymplectic manifold (M, η, Ω) is called local gradient vector field if $\flat_{(\eta,\Omega)}(X)$ yields to a d-closed 1-form, i.e.,

$$d(i_X\Omega + (i_X\eta)\eta) = 0 \iff \mathcal{L}_X\Omega = \eta \wedge \mathcal{L}_X\eta,$$

where $\mathcal{L}_X$ is the Lie derivative with respect the vector field X.
Let us notice that

$$i_{[grad\, f, grad\, g]}\Omega = d(\Omega(grad\, f, grad\, g)), \tag{2.7}$$

for any $f, g \in \mathcal{C}^\infty(M)$

Proposition 2.2. *The Poisson bracket satisfies the following properties:*
i) $grad\{f, g\} = [grad\, f, grad\, g]$,
ii) $\{f, g\} = -\{g, f\}$,
iii) $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$,
iv) $\{f, g\cdot h\} = \{f, g\}\cdot h + g\cdot\{f, h\}$.

Proof. The properties i) is obvious, ii) follows straightforwardly from the defintion of the bracket, iii) holds by using, $d\Omega(grad\, f, grad\, g, grad\, h) = 0$ together with (2.6) and i). Finally the equation iv) presents no difficulty. □

Since iii) is the Jacobi identity, we have proven the fundamental fact that $(\mathcal{C}^\infty(M), \{,\})$ is a Lie algebra, called the Poisson-Lie algebra of the cosymplectic manifold (M, η, Ω). It plays a basic role in applications to mechanics.

Theorem 2.2. *A vector field X on a cosymplectic manifold (M, η, Ω) is a local gradient vector field if and only if $im(X)$ is a lagrangian submanifold of (TM, Ω_0). In local coordinates Ω_0 admits as expression*

$$\Omega_0 = \sum_{i=1}^{n} dx_{n+i} \wedge dv_i - \sum_{i=1}^{n} dx_i \wedge dw_i + dt \wedge du. \tag{2.8}$$

This construction goes back as far as [2].
The cosymplectic manifold $(M,\ \eta,\ \Omega)$ can be equipped with a Jacobi structure via the following map

$$\varphi_f : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M),\ g \longmapsto (grad\, f)(g) - \xi(g) \cdot \eta(grad\, f) + \xi(f) \cdot \eta(grad\, g). \tag{2.9}$$

The above map is called first order differential operator (for more details, see [4].

Theorem 2.3. *Let M be a manifold and η, Ω be two differential forms on M with degrees 1 and 2 respectively. Let $\pi_i, i = 1, 2$ denote the canonical projection from Q to the i-th factor. Consider $Q = M \times \mathbb{S}^1$ endowed with the 2-form $\overline{\Omega} = \pi_1^*(\Omega) + \pi_1^*(\eta) \wedge \pi_2^*(d\theta)$, where θ is the angular form on $\mathbb{S}^1$. The following statements are equivalent:*

(i) *the triple (M, η, Ω) is a cosymplectic manifold;*
(ii) *the pair $(Q, \overline{\Omega})$ is a symplectic manifold.*

Proof. Suppose that $dimQ = 2n$. We have $\overline{\Omega}^n = n \cdot \pi_1^*(\Omega^{n-1} \wedge \eta) \wedge d\theta$. Then $\Omega^{n-1} \wedge \eta$ is a volume form if and only if $\overline{\Omega}^n$ is a volume form. □

Corollary 2.1. *The first projection π_1 is a Poisson morphism.*

Proof. For any smooth functions $f,\ g$ on M, we have

$$\{f,\ g\}_M = \Omega(grad\, f,\ grad\, g) \text{ and } \{\pi_1^* f,\ \pi_1^* g\}_Q = \overline{\Omega}(grad\, \pi_1^* f,\ grad\, \pi_1^* g).$$

If $grad\, f$ is the solution of $grad\, f = \flat^{-1}_{(\eta,\Omega)}(df)$, then $grad\, \pi_1^* f$ is the solution of $i_{grad\, \pi_1^* f}\overline{\Omega} = d\, \pi_1^* f$. We verify that $X_{\pi_1^* f} = X_f - \xi(f)\frac{\partial}{\partial\theta}$. We deduce that $grad\, \pi_1^* f$ (resp. $X_{\pi_1^* f}$) projects onto M and its projection is just $grad\, f$ (resp. X_f). So

$$\pi_1^*\{f,\ g\}_M = \{\pi_1^* f,\ \pi_1^* g\}_Q,$$

as desired. □

We conclude that, a cosymplectic structure can always generate a symplectic structure, and vice-versa.

3. Reduction using the momentum

3.1. **Left action of Lie group on cosymplectic manifolds.** Let M be a smooth manifold and G a Lie group. The map $\Phi : G \times M \longrightarrow M$ is differentiable, for any $g \in G$, $\Phi_g : M \longrightarrow M,\ x \longmapsto \Phi(g, x)$. We say that Φ is a left action of G on M if the following assertions hold

1) $\Phi_g \circ \Phi_h = \Phi_{gh}, \forall g, h \in G$;
2) $\Phi_e = id_M$ where e is the unit of G.

The map $g \longmapsto \Phi_g$ is an homomorphism from G onto the set of diffeomorphisms of M. Let $\mathcal{G} = Lie(G)$ be a Lie algebra of G. For each $X \in \mathcal{G}$, one defines a fundamental vector field X_M on M associated with X, by

$$X_M(x) = \frac{d}{dt}\left[\Phi(exp(tX), x)\right] /_{t=0}, x \in M, X \in \mathcal{G}. \tag{3.1}$$

The mapping $\mathcal{G} \longrightarrow \mathfrak{X}(M), X \longmapsto X_M$ is an homomorphism of Lie algebras. It is very easy to show that $(\Phi_g)_*(X_M) = (Ad_g X)_M$. Moreover, the fundamental vector field X_M is complete and its flow is given by $(t, x) \longmapsto \Phi(exp(tX), x)$.
The action Φ is said to be:

a) Cosymplectic if for any $g \in G, (\Phi_g)^*\Omega = \Omega$ and $(\Phi_g)^*\eta = \eta$;
b) Hamiltonian if it is cosymplectic and there exists a linear mapping $\mathcal{G} \longrightarrow \mathcal{C}^\infty(M), X \longmapsto J_X$ such that $i_{X_M}\Omega = dJ_X$;
c) Strongly hamiltonian if its is hamiltonian and if one can choose $X \longmapsto J_X$ such that $\{J_X, J_Y\} = J_{[X,Y]}$, for any $X, Y \in \mathcal{G}$.

In the last case, $J : M \longrightarrow \mathcal{G}^*$ is such that $J_X(x) =< J(x), X >, x \in M, X \in \mathcal{G}$ and is called Φ-momentum hamiltonian action.
The sets $\mathcal{O}_x = \{\Phi(g, x), g \in G\}$ and $G_x = \{g \in G, \Phi(g, x) = x\}$ are respectively the orbit at x and the isotopy group of x, then one gets $ker(T_x J) = (T_x \mathcal{O}_x)^\perp$. The set $T_x J$ is called the annulator of $\mathcal{G}_x = Lie(G_x)$.

Theorem 3.1. *Let Φ be an hamiltonian action of the Lie group G on the cosymplectic manifold (M, η, Ω), of momentum J. Let H be an hamiltonian such that $H \circ \Phi_g = H$. Then J is constant along each integral curve of the vector field of hamiltonian H.*

3.2. **Reduction using the momentum.** Let $\epsilon \in \mathcal{G}^*$ a weakly regular value of J, G^0_ϵ the neutral component of the isotopy group of ϵ, and $\mathcal{G}_\epsilon$ its Lie algebra. Let $M_\epsilon = J^{-1}(\epsilon)$ a submanifold of M on which is applied the reduction method.

Theorem 3.2. *Let M_ϵ be a submanifold of constant rank of (M, η, Ω). The leaves of characteristic foliation are orbits of the action G^0_ϵ on M_ϵ which is restiction of the action Φ on the subgroup G_ϵ of G and on the submanifold M_ϵ of M. If the characteristic foliation is simple, there exists on $\widehat{M}_\epsilon$ a cosymplectic form $\widehat{\Omega}_\epsilon$ and a unique reduced hamiltonian $\widehat{H}_\epsilon$ such that $i^*_{M_\epsilon}\Omega = \pi_{\widehat{M}_\epsilon}\widehat{\Omega}_\epsilon, H/M_\epsilon = \widehat{H}_\epsilon \circ \pi_{\widehat{M}_\epsilon}$.*

Proof. By straightforward calculation. □

Proposition 3.1. *Let (M, η_M, Ω_M) and (P, η_P, Ω_P) two cosymplectic manifolds and let $J : N \longrightarrow P$ a cosymplectic reduction from a coisotropic submanifold N of M intersecting transversely. The intersection $L \cap N$ is both a submanifold of L, N and M and $J/L \cap N : L \cap N \longrightarrow (P, \eta_P, \Omega_P)$ is a lagrangian immersion.*

Proof. By the fact that $L \cap N$ is a submanifold of L and N, and due to the fact that M is a consequence of the transvesalty of the intersection of L and N. Then $T_x(L \cap N) = T_x L \cap T_x N$ for any $x \in L \cap N$. Since J is a cosymplectic reduction, one has $ker(T_x J) = T_x J \cap (T_x J)^\perp$. Moreover, $T_x J = (T_x J)^\perp$ and $(T_x N)^\perp \subset T_x N$. Thus

$$\begin{aligned} ker(T_x J / L \cap N) &= T_x(L \cap N) \cap ker(T_x J) \\ &= (T_x M)^\perp \\ &= \{O\}. \end{aligned}$$

We conclude that $T_x J/L \cap N$ is a Lagrangian immersion. $\square$

Theorem 3.3. *Let $\lambda : G \times M \longrightarrow$ M be an action on the cosymplectic manifold (M, η, Ω). Suppose that λ is a hamiltonian action on a cosymplectic manifold (M, η, Ω). If $J : M \longrightarrow \mathcal{G}^*$ is equivariant momentum, then*

(i) $\flat_{(\eta,\Omega)}$ *is G-equivariant with respect to the actions $\lambda^{TM} : G \times TM \longrightarrow TM$ and $\lambda^{T^*M} : G \times T^*M \longrightarrow T^*M$.*

(ii) J_{TM} *satisfies $J_{TM} = -J_{T^*M} \circ \flat_{(\eta,\Omega)}$, where $J_{T^*M} : T^*M \longrightarrow \mathcal{G}^*$ is the momentum map associated with the cosymplectic action λ^{T^*M}.*

Proof. (i) Consider the cosymplectic action $\lambda : G \times M \longrightarrow M$. For any $x \in M, g \in \mathcal{G}$, we have $\flat_{(\eta,\Omega)}(\lambda_g(x)) \circ T_x\lambda_g = T^*_{\lambda_g(x)}\lambda_g^{-1} \circ \flat_{(\eta,\Omega)}(x)$.
(ii)Define $J_{T^*M} : T^*M \longrightarrow \mathcal{G}^*$ by $\langle J_{T^*M}(\alpha_x), \xi\rangle = \langle \alpha_x, \xi_M(x)\rangle$, for $\alpha_x \in T^*M, \xi \in \mathcal{G}$. On the other hand, $\langle J_{TM}(v_x), \xi\rangle = v_x(J_\xi) = -\langle \flat_{(\eta,\Omega)}(v_x), \xi_M(x)\rangle$, for all $v_x \in TM, \xi \in \mathcal{G}$. Thus $J_{TM} = -J_{T^*M} \circ \flat_{(\eta,\Omega)}$ as desired. $\square$

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