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A NEW APPEARANCE OF THE GOLDEN RATIO IN THE FOOTSTEPS OF ODOM

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ABSTRACT. We construct some golden ratios by Odom's construction

Let $\Delta\, ABC$ be an equilateral triangle and Γ its circumcircle (see Figure 1). Let K and H be the midpoints of the sides AC and AB, and extend KH to meet the circumcircle at P. In [1] Odom proved that H divides KP according to the golden section, that is

$$\frac{KP}{KH} = \phi.$$

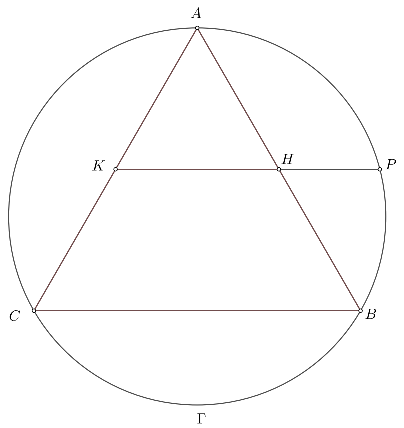

Figure 1

Now let us consider the equilateral triangle $\Delta\, A'B'C'$, which is symmetrical to triangle $\Delta\, ABC$ with respect to segment KH, and let Γ' be its circumcircle (see Figure 2).

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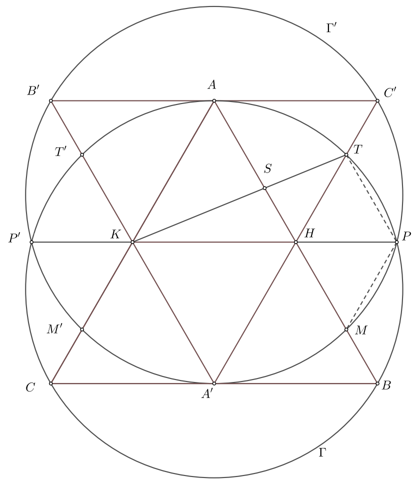

Figure 2

The points of intersection of Γ, Γ' are P, P' and the triangle $\Delta\, A'B'C'$ intersects Γ at T and T'. By the symmetry of the figure, one has $HC' = HB$ and $HT = HP$. Let us consider segment KT intersecting the side AB at point S.

Theorem. *S divides KT according to the golden section, that is*

$$\frac{KT}{KS} = \phi.$$

Proof. In the triangle $\Delta\, THK$ the segment HS is the bisector of $\measuredangle\, THK$, since $\measuredangle THS = \measuredangle\, SHK = \pi/3$. Since the angle bisector of an angle in a triangle divides the opposite side in the same ratio as the sides adjacent to the angle, we have $KS : ST = KH : HT$, from which, being $HT = HP$, one has: $(KS + ST) : KS = (KH + HP) : KH$. Therefore $KT : KS = KP : KH = \phi$. It follows that S divides segment KT in extreme and mean ratios.

□

Remark 1. *Furthermore, being $HC' = KH$ and $HT = HP$, one has:*

$$\frac{HC'}{HT} = \frac{KH}{HP} = \phi$$

Now, let us consider the two triangles $\Delta\, ASK$, $\Delta\, TSH$, which are similar (having congruent angles), so one has: $AS : SH = KS : ST = \phi$. This means that $AH/AS = \phi$ too. Symmetry means these relationships will be the same for the other identified segments. So, connecting point H to T' (Figure 3), one gets

$$\frac{HT'}{HS'} = \frac{AH}{AS'} = \frac{HP'}{HK} = \phi.$$

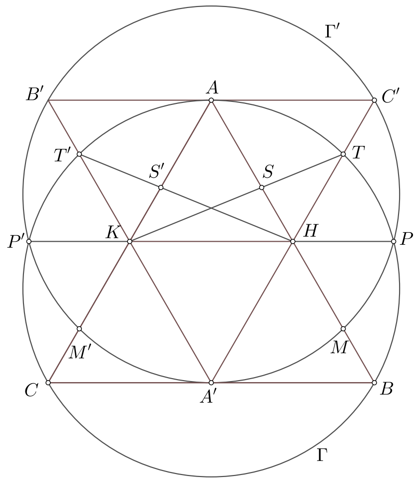

Figure 3

Furthermore, being $BH = HC$ and $MH = HT$, one has

$$\frac{HC'}{HT} = \frac{BH}{MH} = \phi$$

and symmetrically

$$\frac{KC}{KM'} = \phi$$

Remark 2. *The same proportions exist for other segments that complete the Figure 4.*

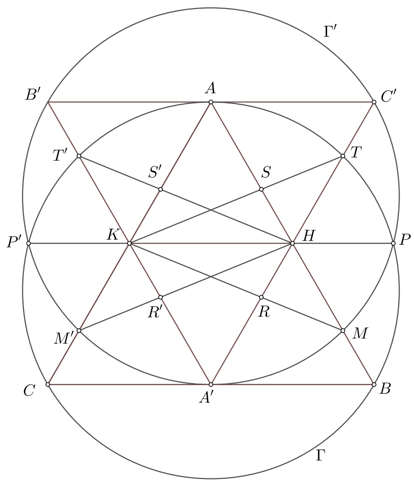

Figure 4

Remark 3. *It may be easily verified the existence of three ellipses* Σ, Σ' *and* Σ'' *(Figure 5) concentric, for whose focal distances* KH *and* PP' *one has:*

$$PP' = KH + P'K + PH = KH + 2PH = KH + 2\frac{KH}{\phi}$$

being $P'K = PH$.

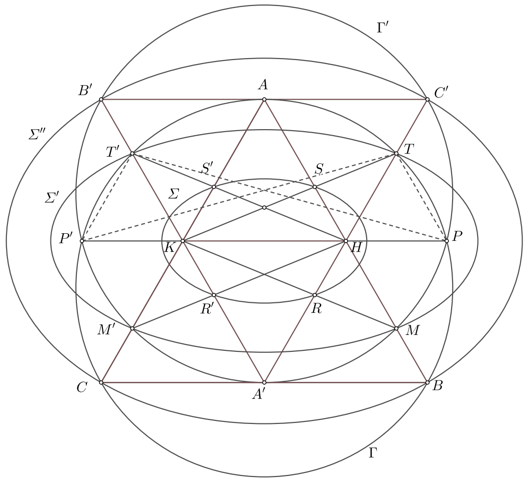

Figure 5

One gets

$$PP' = KH\,\frac{\phi + 2}{\phi}.$$

Hence, the ratio between focal distances is:

$$\frac{PP'}{KH} = \frac{\phi^2 + 1}{\phi} = \phi + \frac{1}{\phi} = \sqrt{5}\,.$$

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REFERENCES

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