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ON TRANS-SASAKIAN SPACE FORM AND SOME RIEMANN SOLITONS

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ABSTRACT. The Riemann flow is an evolution equation for metrics on the Riemannian manifold defined with the aid of the bialternate product Riemannian metric and the Riemannian curvature tensor, which is one of the generalizations of Ricci flow. These types of evolutions are novel and quite natural for comprehending specific natural flows and the geometry of evolving manifolds. It has a lot of intriguing qualities from a mathematical and physical standpoint.

Our aim is to study the Riemann soliton (the self-similar solution of the Riemann flow) on trans-Sasakian space form. We find the nature of solitons and the φ-sectional curvature if a trans-Sasakian space form admits Riemann soliton and the conditions of potential vector field when almost Riemann soliton reduces to Riemann soliton.

1. INTRODUCTION AND MOTIVATIONS

In this section we present the motivation and some well known results related to the paper.
The concept of Riemann flow was introduced in [9] as a natural generalization of Ricci flow [4, 5]. In the paper [9], the author defined the Riemann flow by

$$\frac{\partial}{\partial t}G(t) = -2R(g(t)), \quad t \in [0, I], \tag{1.1}$$

where $G = \frac{1}{2}g \odot g$, R is the Riemann curvature tensor associated to the metric g and $\odot$ is Kulkarni-Nomizu product [1]. The Riemann soliton is the self-similar solution of Riemann flow studied in [10] where the authors showed as fixed points of the Riemann flow, as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms. The authors in [8] characterize the Riemann soliton in terms of infinitesimal harmonic transformation. A Riemannian manifold (M, g) admits a Riemann soliton if there exists a C^∞ vector field V

$$2R + \lambda G + g \odot \mathcal{L}_V g = 0, \tag{1.2}$$

where $\mathcal{L}_V$ denotes the Lie-derivative along the vector field V and λ is a real scalar. A Riemann soliton is called shrinking when $\lambda < 0$, steady when $\lambda = 0$ and expanding $\lambda > 0$. If V is a gradient of some scalar function f on M i.e., $V = Df$, where D denote the gradient operator, then the soliton is called gradient Riemann soliton. We call the

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vector field V the potential field of the Riemann soliton and the function f the potential function. In this case the equation (1.2) can be written as

$$2R + \lambda G + g \odot \nabla^2 f = 0, \tag{1.3}$$

where $\nabla^2 f$ denotes the Hessian of f.
If λ in (1.2) is a differentiable function on M , then the soliton is called an almost-Riemann soliton. In [11, 3], authors studied almost Riemann solitons within the context of a contact manifold, in particular on almost Kenmotsu manifolds and proved some inportant results. If the potential vector field is the gradient of some smooth function on M in almost-Riemann soliton then it is called gradient almost-Riemann soliton.

In this article, we first derive the curvature tensor and Ricci tensor of trans-Sasakian space form in section-2.1. In section-2.2, we have studied the nature of Riemann soliton on trans-Sasakian space form and find condition of $\varphi-$sectional curvature and potential vector field.

Before proving the main results the following properties are required for the next section.
Let M be a $(2n+1)$ dimensional (denoted by M^{2n+1}) manifold having almost contact structure (φ, ξ, η) i.e.,

$$\eta(\xi) = 1,\ \varphi^2 = -I + \eta \otimes \xi,\ \varphi(\xi) = 0,\ \eta \circ \varphi = 0. \tag{1.4}$$

where φ is a (1, 1)-tensor field, ξ a contravariant vector field, η a one form.
A Riemannian metric g is said to be compatible with the structure (φ, ξ, η) if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{1.5}$$

If the manifold M^{2n+1} equipped with an almost contact structure (φ, ξ, η) and a compatible Riemannian metric g, is called almost contact metric manifold. From the equations (1.4) and (1.5), we have the following results

$$g(X, \xi) = \eta(X), \quad g(\xi, \xi) = 1 \quad \text{and} \tag{1.6}$$

$$g(X, \varphi Y) = -g(\varphi X, Y). \tag{1.7}$$

Recall the four tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ in almost contact manifold, which are defined by

$$\begin{cases} N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi, \\ N^{(2)}(X, Y) = (\mathcal{L}_{\varphi X}\eta)Y - (\mathcal{L}_{\varphi Y}\eta)X, \\ N^{(3)}(X) = (\mathcal{L}_{\xi}\varphi)X, \\ N^{(4)}(X) = (\mathcal{L}_{\xi}\eta)X. \end{cases}$$

It is evedent that the almost contact structure (φ, ξ, η) is normal if and only if the four tensors $N^{(1)}, N^{(2)}, N^{(3)}$ and $N^{(4)}$ vanish.
In an almost contact manifold M with almost contact metric structure (φ, ξ, η, g), the tensor Φ defined by

$$\Phi(X, Y) = g(X, \varphi Y) \tag{1.8}$$

for all vector field X, Y on M, called the the fundamental 2-form.

Proposition 1.1. [2] *For an almost contact metric structure (φ, ξ, η, g), the covariant derivative of φ is given by*

$$2g\left((\nabla_X \varphi)Y, Z\right) = 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) + g\left(N^{(1)}(Y, Z), \varphi X\right)$$

$$+N^{(2)}(Y,Z)\eta(X) + 2d\eta(\varphi Y, X)\eta(Z) - 2d\eta(\varphi Z, X)\eta(Y). \tag{1.9}$$

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be trans-Sasakian [7] if and only if it is normal and

$$d\Phi = 2\beta\eta \wedge \Phi, \quad d\eta = \alpha\Phi, \tag{1.10}$$

where α, β are smooth functions on M. Thus the tuple $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ called trans-Sasakian manifold of type (α, β) and the Trans-Sasakian manifolds of type $(0,0)$, $(\alpha, 0)$ and $(0, \beta)$ are called cosymplectic, α-Sasakian, and β-Kenmotsu manifolds respectively. From (1.9) and (1.10), we have the following properties

$$(\nabla_X \varphi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X]. \tag{1.11}$$

$$\nabla_X \xi = -\alpha\varphi X - \beta\varphi^2 X. \tag{1.12}$$

$$(\nabla_X \eta)Y = \alpha g(X, \varphi Y) + \beta g(\varphi X, \varphi Y). \tag{1.13}$$

We will use the formulas by Yano [12] which are helpful to prove our main results:

$$\mathcal{L}_X \nabla_Y Z - \nabla_Y \mathcal{L}_X Z - \nabla_{[X,Y]} Z = (\mathcal{L}_X \nabla)(Y, Z), \tag{1.14}$$

$$(\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z) = (\mathcal{L}_V R)(X, Y)Z. \tag{1.15}$$

Using the symmetry of $\mathcal{L}_V \nabla$ in the above formula, we obtain

$$(\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V,X]} g)(Y, Z) = -g((\mathcal{L}_V \nabla)(X, Y), Z) - g((\mathcal{L}_V \nabla)(X, Z), Y). \tag{1.16}$$

We also infer

$$2g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_X \mathcal{L}_V g)(Y, Z) + (\nabla_Y \mathcal{L}_V g)(Z, X) - (\nabla_Z \mathcal{L}_V g)(X, Y). \tag{1.17}$$

2. MAIN RESULTS

2.1. **Trans-Sasakian Space form.** A trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ of constant φ-sectional curvature is called trans-Sasakian space form. In the present section we denote this φ-sectional curvature by c.

Lemma 2.1. *Let $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ be a trans-Sasakian manifold with consideration of α and β constants and R its curvature tensor. Then for any vector fields X, Y, Z orthogonal to ξ,*

(1)

$$\left.\begin{aligned} &R(X,Y)\varphi Z - \varphi R(X,Y)Z \\ &\quad = \alpha^2[g(X,Z)\varphi Y - g(Y,Z)\varphi X + g(Y,\varphi Z)X - g(X,\varphi Z)Y] \\ &\quad\; +2\alpha\beta[g(Y,Z)X - g(X,Z)Y + g(Y,\varphi Z)\varphi X - g(X,\varphi Z)\varphi Y] \\ &\quad\; +\beta^2[g(Y,Z)\varphi X - g(X,Z)\varphi Y + g(\varphi Y,Z)X - g(\varphi X,Z)Y] \end{aligned}\right\} \tag{2.1}$$

(2)

$$\left.\begin{aligned} &R(\varphi X,\varphi Y)Z - R(X,Y)Z \\ &\quad = \alpha^2[g(X,Z)Y - g(Y,Z)X + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y] \\ &\quad\; +2\alpha\beta[g(\varphi Y,Z)X - g(Z,\varphi X)Y + g(Y,Z)\varphi X - g(X,Z)\varphi Y] \\ &\quad\; +\beta^2[g(Y,Z)X - g(X,Z)Y + g(\varphi X,Z)\varphi Y - g(\varphi Y,Z)\varphi X] \end{aligned}\right\} \tag{2.2}$$

(3)
$$\left.\begin{aligned} &R(X,Y,\varphi X,\varphi Y)-R(X,Y,X,Y)\\ &\quad=\alpha^2[g(X,X)g(Y,Y)-g(X,Y)^2-g(X,\varphi Y)^2]\\ &\quad+\beta^2[g(X,Y)^2-g(X,X)g(Y,Y)+g(X,\varphi Y)^2].\end{aligned}\right\} \tag{2.3}$$

(4)
$$\left.\begin{aligned} R(X,\varphi X,Y,\varphi Y)&=R(X,\varphi Y,Y,\varphi X)+R(X,Y,X,Y)\\ &+\alpha^2[g(X,X)g(Y,Y)-g(X,Y)^2-g(X,\varphi Y)^2]\\ &+\beta^2[g(X,Y)^2-g(X,X)g(Y,Y)-g(X,\varphi Y)^2].\end{aligned}\right\} \tag{2.4}$$

(5)
$$\left.\begin{aligned} &R(X,\varphi Y,X,\varphi Y)-R(X,\varphi Y,Y,\varphi X)\\ &\quad=\alpha^2[g(X,Y)^2-g(Y,Y)g(X,X)+g(\varphi Y,X)^2]\\ &\quad+\beta^2[g(X,X)g(Y,Y)-g(X,Y)^2-g(\varphi Y,X)^2].\end{aligned}\right\} \tag{2.5}$$

(6)
$$\left.\begin{aligned} &R(Y,\varphi X,Y,\varphi X)-R(X,\varphi Y,Y,\varphi X)\\ &\quad=\alpha^2[g(X,Y)^2+g(\varphi X,Y)^2-g(X,X)g(Y,Y)]\\ &\quad+\beta^2[g(X,X)g(Y,Y)-g(X,Y)^2-g(\varphi X,Y)^2].\end{aligned}\right\} \tag{2.6}$$

Proof. (1) The Ricci identity is given by
$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z. \tag{2.7}$$
Replacing Z by φZ in the previous equation, we have
$$R(X,Y)\varphi Z=\nabla_X\nabla_Y\varphi Z-\nabla_Y\nabla_X\varphi Z-\nabla_{[X,Y]}\varphi Z. \tag{2.8}$$
With the help of equations (2.7) and (2.8), we obtain
$$\begin{aligned} &R(X,Y)\varphi Z-\varphi R(X,Y)Z\\ &\quad=\nabla_X[(\nabla_Y\varphi)Z]-\nabla_Y[(\nabla_X\varphi)Z]+(\nabla_X\varphi)\nabla_Y Z-(\nabla_Y\varphi)\nabla_X Z-(\nabla_{[X,Y]}\varphi)Z.\end{aligned}$$
Using (1.11), we have
$$\begin{aligned} &R(X,Y)\varphi Z-\varphi R(X,Y)Z\\ &\quad=\nabla_X[\alpha[g(Y,Z)\xi-\eta(Z)Y]+\beta[g(\varphi Y,Z)\xi-\eta(Z)\varphi Y]]\\ &\quad-\nabla_Y[\alpha[g(X,Z)\xi-\eta(Z)X]+\beta[g(\varphi X,Z)\xi-\eta(Z)\varphi X]]\\ &\quad+\alpha[g(X,\nabla_Y Z)\xi-\eta(\nabla_Y Z)X]+\beta[g(\varphi X,\nabla_Y Z)\xi-\eta(\nabla_Y Z)\varphi X]\\ &\quad-\alpha[g(Y,\nabla_X Z)\xi-\eta(\nabla_X Z)Y]-\beta[g(\varphi Y,\nabla_X Z)\xi-\eta(\nabla_X Z)\varphi Y]\\ &\quad-\alpha[g([X,Y],Z)\xi-\eta(Z)[X,Y]]-\beta[g(\varphi[X,Y],Z)\xi-\eta(Z)\varphi[X,Y]]\\ &\quad=\alpha[g(Y,Z)\nabla_X\xi-g(X,Z)\nabla_Y\xi+[(\nabla_Y\eta)Z]X-[(\nabla_X\eta)Z]Y]\\ &\quad+\beta[[(\nabla_Y\eta)Z]\varphi X-[(\nabla_X\eta)Z]\varphi Y+g(\varphi Y,Z)\nabla_X\xi-g(\varphi X,Z)\nabla_Y\xi\\ &\quad+g((\nabla_X\varphi)Y,Z)\xi-g((\nabla_Y\varphi)X,Z)\xi+\eta(Z)(\nabla_Y\varphi)X-\eta(Z)(\nabla_X\varphi)Y].\end{aligned}$$
Using (1.12) and (1.13), we get result
$$\begin{aligned} &R(X,Y)\varphi Z-\varphi R(X,Y)Z\\ &\quad=\alpha[g(Y,Z)[-\alpha\varphi X-\beta\varphi^2 X]-g(X,Z)[-\alpha\varphi Y-\beta\varphi^2 Y]\\ &\quad+[\alpha g(Y,\varphi Z)+\beta g(\varphi Y,\varphi Z)]X-[\alpha g(X,\varphi Z)+\beta g(\varphi X,\varphi Z)]Y\end{aligned}$$

$$
\begin{aligned}
&+\beta[[\alpha g(Y,\varphi Z)+\beta g(\varphi Y,\varphi Z)]\varphi X-[\alpha g(X,\varphi Z)+\beta g(\varphi X,\varphi Z)]\varphi Y\\
&+g(\varphi Y,Z)[-\alpha\varphi X-\beta\varphi^2X]-g(\varphi X,Z)[-\alpha\varphi Y-\beta\varphi^2Y]\\
&+g(\alpha[g(X,Y)\xi-\eta(Y)X]+\beta[g(\varphi X,Y)\xi-\eta(Y)\varphi X],Z)\xi\\
&-g(\alpha[g(X,Y)\xi-\eta(X)Y]+\beta[g(\varphi Y,X)\xi-\eta(X)\varphi Y],Z)\xi\\
&+\eta(Z)[\alpha[g(X,Y)\xi-\eta(X)Y]+\beta[g(\varphi Y,X)\xi-\eta(X)\varphi Y]]\\
&-\eta(Z)[\alpha[g(X,Y)\xi-\eta(Y)X]+\beta[g(\varphi X,Y)\xi-\eta(Y)\varphi X]]]\\
=\;&\alpha^2[g(X,Z)\varphi Y-g(Y,Z)\varphi X+g(Y,\varphi Z)X-g(X,\varphi Z)Y]\\
&+2\alpha\beta[g(Y,Z)X-g(X,Z)Y+g(Y,\varphi Z)\varphi X-g(X,\varphi Z)\varphi Y]\\
&+\beta^2[g(Y,Z)\varphi X-g(X,Z)\varphi Y+g(\varphi Y,Z)X-g(\varphi X,Z)Y].
\end{aligned}
$$

Since X, Y, Z orthogonal to ξ, we have $\eta(X)=\eta(Y)=\eta(Z)=0$. Therefore

$$
\begin{aligned}
R(X,Y)\varphi Z-\varphi R(X,Y)Z=\;&\alpha^2[g(X,Z)\varphi Y-g(Y,Z)\varphi X+g(Y,\varphi Z)X-g(X,\varphi Z)Y]\\
&+2\alpha\beta[g(Y,Z)X-g(X,Z)Y+g(Y,\varphi Z)\varphi X-g(X,\varphi Z)\varphi Y]\\
&+\beta^2[g(Y,Z)\varphi X-g(X,Z)\varphi Y+g(\varphi Y,Z)X-g(\varphi X,Z)Y].
\end{aligned}
$$

(2) Taking inner product of the equation (2.1) with φW and then by standard calculation we get the result (2.2).

$$
\begin{aligned}
&R(X,Y,\varphi Z,\varphi W)-g(\varphi R(X,Y)Z,\varphi W)\\
&=\alpha^2[g(X,Z)g(\varphi Y,\varphi W)-g(Y,Z)g(\varphi X,\varphi W)+g(Y,\varphi Z)g(X,\varphi W)\\
&-g(X,\varphi Z)g(Y,\varphi W)]+2\alpha\beta[g(Y,Z)g(X,\varphi W)-g(X,Z)g(Y,\varphi W)\\
&+g(Y,\varphi Z)g(\varphi X,\varphi W)-g(X,\varphi Z)g(\varphi Y,\varphi W)]+\beta^2[g(Y,Z)g(\varphi X,\varphi W)\\
&-g(X,Z)g(\varphi Y,\varphi W)+g(\varphi Y,Z)g(X,\varphi W)-g(\varphi X,Z)g(Y,\varphi W)].
\end{aligned}
$$

This implies

$$
\begin{aligned}
&R(\varphi Z,\varphi W)X-R(Z,W)X\\
&=\alpha^2[g(X,Z)W-g(X,W)Z+g(X,\varphi W)\varphi Z-g(X,\varphi Z)\varphi W]\\
&+2\alpha\beta[g(X,\varphi W)Z-g(X,Z)\varphi W+g(X,W)\varphi Z-g(X,\varphi Z)W]\\
&+\beta^2[g(X,W)Z-g(X,Z)W-g(X,\varphi W)\varphi Z-g(\varphi X,Z)\varphi W].
\end{aligned}
$$

Replacing W by Y and swaping Z and X, we get the result, we get

$$
\begin{aligned}
&R(\varphi X,\varphi Y)Z-R(X,Y)Z\\
&=\alpha^2[g(X,Z)Y-g(Y,Z)X+g(\varphi Y,Z)\varphi X-g(\varphi X,Z)\varphi Y]\\
&+2\alpha\beta[g(\varphi Y,Z)X-g(Z,\varphi X)Y+g(Y,Z)\varphi X-g(X,Z)\varphi Y]\\
&+\beta^2[g(Y,Z)X-g(X,Z)Y+g(\varphi X,Z)\varphi Y-g(\varphi Y,Z)\varphi X].
\end{aligned}
$$

(3) Taking inner product of (2.1) with φW and replacing Z, W by X, Y respectively, we obtain

$$
\begin{aligned}
R(X,Y,\varphi X,\varphi Y)-R(X,Y,X,Y)=\;&\alpha^2[g(X,X)g(Y,Y)-g(X,Y)^2-g(X,\varphi Y)^2]\\
&+\beta^2[g(X,Y)^2-g(X,X)g(Y,Y)+g(X,\varphi Y)^2].
\end{aligned}
$$

(4) Taking inner product of (2.1) with W and using Bianchi's identity, we get

$$\begin{aligned}-R(X,Y,W,\varphi Z) &= R(Z,X,Y,\varphi W)+R(Y,Z,X,\varphi W)\\ &\quad+\alpha^2[g(X,Z)g(\varphi Y,W)-g(Y,Z)g(\varphi X,W)+g(Y,\varphi Z)g(X,W)\\ &\quad-g(X,\varphi Z)g(Y,W)]+2\alpha\beta[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\\ &\quad+g(Y,\varphi Z)g(\varphi X,W)-g(X,\varphi Z)g(\varphi Y,W)]+\beta^2[g(Y,Z)g(\varphi X,W)\\ &\quad-g(X,Z)g(\varphi Y,W)+g(\varphi Y,Z)g(X,W)-g(\varphi X,Z)g(Y,W)].\end{aligned}$$

Replacing Y by φX, W by Y, Z by Y

$$R(X,\varphi X,Y,\varphi Y)=R(X,Y,\varphi X,\varphi Y)+R(X,\varphi Y,Y,\varphi X).$$

By (3), we have

$$\begin{aligned}R(X,\varphi X,Y,\varphi Y) &= R(X,\varphi Y,Y,\varphi X)+R(X,Y,X,Y)\\ &\quad+\alpha^2[g(X,X)g(Y,Y)-g(X,Y)^2-g(X,\varphi Y)^2]\\ &\quad+\beta^2[g(X,Y)^2-g(X,X)g(Y,Y)-g(X,\varphi Y)^2].\end{aligned}$$

(5) Taking inner product of (2.1) with W and replacing Y by φY, W by X and Z by Y, we obtain

$$\begin{aligned}&R(X,\varphi Y,X,\varphi Y)-R(X,\varphi Y,Y,\varphi X)\\ &\quad=\alpha^2[g(X,Y)^2-g(Y,Y)g(X,X)+g(\varphi Y,X)^2]\\ &\quad\quad+\beta^2[g(X,X)g(Y,Y)-g(X,Y)^2-g(\varphi Y,X)^2].\end{aligned}$$

(6) Taking inner product of (2.1) with W and replacing X by φX, W by Y and Z by X, we obtain

$$\begin{aligned}&R(Y,\varphi X,Y,\varphi X)-R(X,\varphi Y,Y,\varphi X)\\ &\quad=\alpha^2[g(X,Y)^2+g(\varphi X,Y)^2-g(X,X)g(Y,Y)]\\ &\quad\quad+\beta^2[g(X,X)g(Y,Y)-g(X,Y)^2-g(\varphi X,Y)^2].\end{aligned}$$

□

Theorem 2.1. *Let $(M,\varphi,\xi,\eta,g,\alpha,\beta,c)$ be a trans-Sasakian space form, where α,β constants and c constant φ-sectional curvature. Then the curvature tensor R of M is*

$$\begin{aligned}4R(X,Y)Z &= [3(\alpha^2-\beta^2)+c][g(Y,Z)X-g(X,Z)Y]-(\alpha^2-\beta^2-c)\{\eta(Z)[\eta(X)Y\\ &\quad-\eta(Y)X]+[\eta(Y)g(X,Z)-\eta(X)g(Y,Z)]\xi+[g(X,\varphi Z)\varphi Y-g(Y,\varphi Z)\varphi X\\ &\quad+2g(X,\varphi Y)\varphi Z]\}-8\alpha\beta\{[\eta(Y)g(\varphi Z,X)-\eta(X)g(\varphi Z,Y)]\xi\\ &\quad+\eta(Z)[\eta(X)\varphi Y-\eta(Y)\varphi X]\}. \qquad (2.9)\end{aligned}$$

Proof. Let the vector fields U,V,W be orthogonal to ξ, then we have

$$R(U,\varphi U,U,\varphi U)=-cg(U,U)^2. \qquad (2.10)$$

Replacing U by $U+V$ in (2.10), we have

$$\begin{aligned}&2R(V,\varphi U,U,\varphi U)+2R(U,\varphi U,U,\varphi V)+2R(V,\varphi V,U,\varphi U)+2R(V,\varphi V,V,\varphi U)\\ &+2R(U,\varphi V,V,\varphi U)+2R(U,\varphi V,V,\varphi V)+R(U,\varphi V,U,\varphi V)+R(V,\varphi U,V,\varphi U)\\ &=-c[4g(U,V)^2+4g(U,U)g(U,V)+2g(U,U)g(V,V)+4g(U,V)g(V,V)]. \qquad (2.11)\end{aligned}$$

Replacing U by $U-V$ in (2.10), we get

$$\begin{aligned}&-2R(U,\varphi U,U,\varphi V)-2R(U,\varphi U,V,\varphi U)+2R(U,\varphi U,V,\varphi V)-2R(V,\varphi U,V,\varphi V)\\&+2R(U,\varphi V,V,\varphi U)-2R(U,\varphi V,V,\varphi V)+R(V,\varphi U,V,\varphi U)+R(U,\varphi V,U,\varphi V)\\&=-c[4g(U,V)^2-4g(U,U)g(U,V)-4g(U,V)g(V,V)+2g(U,U)g(V,V)].\end{aligned}\tag{2.12}$$

Adding (2.11) and (2.12), we have

$$\begin{aligned}2R(U,\varphi U,V,\varphi V)+2R(U,\varphi V,V,\varphi U)+R(U,\varphi V,U,\varphi V)+R(V,\varphi U,V,\varphi U)\\=-2c[2g(U,V)^2+g(U,U)g(V,V)].\end{aligned}\tag{2.13}$$

Using (2.4), (2.5), (2.6) in (2.13), we get

$$3R(U,\varphi V,V,\varphi U)+R(U,V,U,V)=-c[2g(U,V)^2+g(U,U)g(V,V)].\tag{2.14}$$

Replacing V by φV in (2.14), we get

$$3R(U,V,\varphi U,\varphi V)+R(U,\varphi V,U,\varphi V)=-c[2g(U,\varphi V)^2+g(U,U)g(V,V)].$$

Using (2.5) and (2.3) in preceding equation, we get

$$\begin{aligned}3R(U,V,U,V)+R(U,\varphi V,V,\varphi U)+2\alpha^2[g(U,U)g(V,V)\\-g(U,V)^2-g(U,\varphi V)^2]+2\beta^2[g(U,V)^2-g(U,U)g(V,V)\\+g(U,\varphi V)^2]=-c[2g(U,\varphi V)^2+g(U,U)g(V,V)].\end{aligned}\tag{2.15}$$

Multiplying (2.15) by 3 and then subtract (2.14), we obtain

$$\begin{aligned}4R(U,V,U,V)=3(\alpha^2-\beta^2-c)g(U,\varphi V)^2\\+[3(\alpha^2-\beta^2)+c]g(U,V)^2-[3(\alpha^2-\beta^2)+c]g(U,U)g(V,V).\end{aligned}\tag{2.16}$$

Replacing U by $U+W$ in (2.16), we get

$$\begin{aligned}4R(U,V,U,V)+4R(W,V,W,V)+8R(U,V,W,V)=3(\alpha^2-\beta^2-c)g(U+W,\varphi V)^2\\+[3(\alpha^2-\beta^2)+c]g(U+W,V)^2-[3(\alpha^2-\beta^2)+c]g(U+W,U+W)g(V,V).\end{aligned}$$

Using (2.16) in preceding equation, we have

$$4R(U,V)V=[3(\alpha^2-\beta^2)+c][g(V,V)U-g(U,V)V]-3(\alpha^2-\beta^2-c)g(U,\varphi V)\varphi V.\tag{2.17}$$

Replacing V by $V+W$, in (2.17), we get

$$\begin{aligned}4R(U,W)V+4R(U,V)W=[3(\alpha^2-\beta^2)+c][2g(V,W)U-g(U,V)W-g(U,W)V]\\-3(\alpha^2-\beta^2-c)[g(U,\varphi W)\varphi V+g(U,\varphi V)\varphi W].\end{aligned}\tag{2.18}$$

Replacing U by V and V by $-U$, we get

$$\begin{aligned}4\{R(U,V)W+R(W,V)U\}=[3(\alpha^2-\beta^2)+c][g(U,V)W+g(V,W)U-2g(U,W)V]\\+3(\alpha^2-\beta^2-c)[g(V,\varphi W)\varphi U+g(V,\varphi U)\varphi W].\end{aligned}\tag{2.19}$$

Adding (2.18) and (2.19), we get

$$\begin{aligned}8R(U,V)W+4\{R(U,W)V+R(W,V)U\}=3[3(\alpha^2-\beta^2)+c][g(V,W)U-g(U,W)V]\\-3(\alpha^2-\beta^2-c)[g(U,\varphi W)\varphi V+2g(U,\varphi V)\varphi W-g(V,\varphi W)\varphi U].\end{aligned}$$

Using Bianchi's identity

$$4R(U,V)W=[3(\alpha^2-\beta^2)+c][g(V,W)U-g(U,W)V]$$

$$-(\alpha^2-\beta^2-c)[g(U,\varphi W)\varphi V+2g(U,\varphi V)\varphi W-g(V,\varphi W)\varphi U]. \qquad (2.20)$$

Now, let X, Y, Z be arbitrary vectors fields. Then we can write

$$X = U+\eta(X)\xi, Y = V+\eta(Y)\xi \text{ and } Z = W+\eta(Z)\xi,$$

where U, V and W are orthogonal to ξ. Then from (2.20), we have

$$\begin{aligned}4R(X,Y)Z = {} & [3(\alpha^2-\beta^2)+c][g(Y,Z)X-g(X,Z)Y]-(\alpha^2-\beta^2-c)\{[\eta(X)\eta(Z)Y \\ & -\eta(Y)\eta(Z)X]+[\eta(Y)g(X,Z)\xi-\eta(X)g(Y,Z)\xi]+[g(X,\varphi Z)\varphi Y \\ & -g(Y,\varphi Z)\varphi X+2g(X,\varphi Y)\varphi Z]\}-8\alpha\beta[\eta(Y)g(\varphi Z,X)\xi \\ & -\eta(X)g(\varphi Z,Y)\xi+\eta(X)\eta(Z)\varphi Y-\eta(Y)\eta(Z)\varphi X].\end{aligned}$$

□

Corollary 2.1. *Let $(M,\varphi,\xi,\eta,g,\alpha,\beta,c)$ be a trans-Sasakian space form, where α,β constants and c constant φ-sectional curvature. Then the Ricci curvature tensor S of M is*

$$\begin{aligned}S(Y,Z) = {} & \frac{(3n-1)(\alpha^2-\beta^2)+(n+1)c}{2}g(Y,Z) \\ & +\frac{(n+1)(\alpha^2-\beta^2-c)}{2}\eta(Y)\eta(Z)+4\alpha\beta g(\varphi Z,Y)\end{aligned} \qquad (2.21)$$

Proof. Taking inner product of (2.9) with W, we get

$$\begin{aligned}4R(X,Y,Z,W) = {} & [3(\alpha^2-\beta^2)+c][g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \\ & -(\alpha^2-\beta^2-c)\{[\eta(X)\eta(Z)g(Y,W)-\eta(Y)\eta(Z)g(X,W)] \\ & +[\eta(Y)g(X,Z)g(\xi,W)-\eta(X)g(Y,Z)g(\xi,W)]+[g(X,\varphi Z)g(\varphi Y,W) \\ & -g(Y,\varphi Z)g(\varphi X,W)+2g(X,\varphi Y)g(\varphi Z,W)]\}-8\alpha\beta[\eta(Y)g(\varphi Z,X)g(\xi,W) \\ & -\eta(X)g(\varphi Z,Y)g(\xi,W)+\eta(X)\eta(Z)g(\varphi Y,W)-\eta(Y)\eta(Z)g(\varphi X,W)].\end{aligned}$$

Contracting X and W, we have

$$\begin{aligned}S(Y,Z) = {} & \frac{(3n-1)(\alpha^2-\beta^2)+(n+1)c}{2}g(Y,Z) \\ & +\frac{(n+1)(\alpha^2-\beta^2-c)}{2}\eta(Y)\eta(Z)+4\alpha\beta g(\varphi Z,Y).\end{aligned}$$

□

Using (1.4), (1.6) and (1.7) in cor 2.1, we have the following properties

$$S(X,\xi) = 2n(\alpha^2-\beta^2)\eta(X). \qquad (2.22)$$

$$S(\xi,\xi) = 2n(\alpha^2-\beta^2). \qquad (2.23)$$

$$QY = \frac{(3n-1)(\alpha^2-\beta^2)+(n+1)c}{2}Y+\frac{(n+1)(\alpha^2-\beta^2-c)}{2}\eta(Y)\xi-4\alpha\beta\varphi Y. \qquad (2.24)$$

Using (1.13), we have

$$\begin{aligned}(\nabla_Z S)(X,Y) = {} & \frac{(n+1)(\alpha^2-\beta^2-c)}{2}[\alpha\{\eta(Y)g(Z,\varphi X)+\eta(X)g(Z,\varphi Y)\} \\ & +\beta\{\eta(Y)g(\varphi Z,\varphi X)+\eta(X)g(\varphi Z,\varphi Y)\}].\end{aligned} \qquad (2.25)$$

Contracting Y and Z in (2.21), we have

$$\tau = n(3n+1)(\alpha^2-\beta^2)+n(n+1)c. \qquad (2.26)$$

For any differentiable function f on M, we have

$$div(h\xi) = \xi h + 2n\beta h. \tag{2.27}$$

2.2. **Riemann and almost Riemann soliton.** Using Kulkarni-Nomizu product formulas, rewrite the equation (1.2) in the following form:

$$2R(X,Y,U,W) + 2\lambda[g(X,W)g(Y,U) - g(X,U)g(Y,W)] + g(X,W)(\mathcal{L}_V g)(Y,U)$$

$$+g(Y,U)(\mathcal{L}_V g)(X,W) - g(X,U)(\mathcal{L}_V g)(Y,W) - g(Y,W)(\mathcal{L}_V g)(X,U) = 0$$

Contracting with respect to X and W, we obtain

$$2S(Y,U) + 4n\lambda g(Y,U) + (2n-1)(\mathcal{L}_V g)(Y,U) + 2(divV)g(Y,U) = 0. \tag{2.28}$$

Theorem 2.2. *If a trans-Sasakian space form $(M, g, \varphi, \eta, \xi, c, \alpha, \beta)$ with α, β constants admits Riemann soliton, then $c = \frac{(n+1)(\alpha^2-\beta^2)-2(2n-1)\beta}{n+1}$ and the soliton is shrinking, steady, or expanding according to $\alpha^2 > \beta^2 - \beta$, $\alpha^2 = \beta^2 - \beta$ or $\alpha^2 < \beta^2 - \beta$, respectively.*

Proof. Replacing V by ξ in the equation (2.28), we get

$$2S(Y,U) + 4n\lambda g(Y,U) + (2n-1)(\mathcal{L}_\xi g)(Y,U) + 2(div\ \xi)g(Y,U) = 0.$$

Using (1.12), (1.13) and (2.27) in the preceding equation, we have

$$S(Y,U) + 2n(\lambda + \beta)g(Y,U) + (2n-1)\beta g(\varphi Y, \varphi U) = 0. \tag{2.29}$$

Replacing Y, U by ξ in the equation (2.29), we have

$$(\alpha^2 - \beta^2) + (\lambda + \beta) = 0.$$

This implies

$$\lambda = -(\alpha^2 - \beta^2) - \beta. \tag{2.30}$$

Contracting the equation (2.29), we have

$$\tau + 2n(2n+1)(\lambda + \beta) + 2n(2n-1)\beta = 0.$$

Substitute the value of τ and using (2.30) in the preceding equation, we obtain

$$n(3n+1)(\alpha^2 - \beta^2) + n(n+1)c - 2n(2n+1)(\alpha^2 - \beta^2) + 2n(2n-1)\beta = 0.$$

This implies

$$c = \frac{(n+1)(\alpha^2 - \beta^2) - 2(2n-1)\beta}{n+1}.$$

From (2.30), we have $\lambda < 0$, $= 0$, > 0 when $\alpha^2 > \beta^2 - \beta$, $\alpha^2 = \beta^2 - \beta$ $\alpha^2 < \beta^2 - \beta$ respectively.
Thus, the soliton is shrinking, steady, or expanding according to $\alpha^2 > \beta^2 - \beta$, $\alpha^2 = \beta^2 - \beta$, or $\alpha^2 < \beta^2 - \beta$, respectively. □

Theorem 2.3. *If a trans-Sasakian space form $(M, g, \varphi, \eta, \xi, c, \alpha, \beta)$ with α, β constants admits Riemann soliton and the potential vector field V is pointwise colinear with ξ, then V is a constant multiple of ξ.*

Proof. Let $V = h\xi$, where h is a function on M. Then the equation (2.28) reduce to

$$2S(Y,U) + 4n\lambda g(Y,U) + (2n-1)[g((Yh)\xi + h\nabla_Y\xi, U) + g(Y,(Uh)\xi + h\nabla_U\xi)] + 2(div\, h\xi)g(Y,U) = 0.$$

Using (1.12) and (2.27) in the preceding equation, we have

$$2S(Y,U) + 4n\lambda g(Y,U) + (2n-1)[(Yh)\eta(U) + (Uh)\eta(Y)] + 2(2n-1)h\beta g(\varphi Y, \varphi U) + 2(\xi h + 2nh\beta)g(Y,U) = 0.$$

Replacing U by ξ in the preceding equation and using (2.22), we get

$$\left[4n(\alpha^2-\beta^2) + 4n\lambda + (2n+1)(\xi h) + 4nh\beta\right]\eta(Y) + (2n-1)(Yh) = 0. \tag{2.31}$$

Replacing Y by ξ in (2.31), we get

$$\xi h = -(\alpha^2-\beta^2) - \lambda - h\beta.$$

Substitute this value of ξh in (2.31), we get

$$Yh = -[(\alpha^2-\beta^2) + \lambda + h\beta]\eta(Y).$$

This implies

$$dh = -[(\alpha^2-\beta^2) + \lambda + h\beta]\eta. \tag{2.32}$$

Applying d on (2.32), we get

$$0 = d^2h = -[(\alpha^2-\beta^2) + \lambda + h\beta]d\eta - [\beta dh]\eta \tag{2.33}$$

Again we apply d on (2.33), we get

$$[(\alpha^2-\beta^2) + \lambda + h\beta]d^2\eta + [\beta dh]d\eta + [\beta d^2h]\eta + [\beta dh]d\eta = 0$$

This implies

$$2[\beta dh]d\eta = 0$$

Since $d\eta \neq 0$ in a trans-Sasakian manifold and $\beta \neq 0$, therefore

$$dh = 0. \tag{2.34}$$

Thus h is constant. □

Theorem 2.4. *If a trans-Sasakian space form $(M, g, \varphi, \eta, \xi, c, \alpha, \beta)$ with α, β constants admits almost Riemann soliton and divergence of potential field is constant, then almost Riemann soliton reduces to Riemann soliton.*

Proof. Utilising cor 2.1 in (2.28), we have

$$(\mathcal{L}_V g)(Y,U) = -\frac{1}{2n-1}\{[(3n-1)(\alpha^2-\beta^2) + (n+1)c + 4n\lambda + 2(divV)]g(Y,U) + (n+1)(\alpha^2-\beta^2-c)\eta(Y)\eta(U)\}. \tag{2.35}$$

Taking covariant derivative of (2.35) along the arbitrary vector field Z and considered as $divV$ =constant, we have

$$(\nabla_Z\mathcal{L}_V g)(Y,U) = -\frac{1}{2n-1}\{4n(Z\lambda)g(Y,U) + (n+1)(\alpha^2-\beta^2-c)[\eta(U)(\nabla_Z\eta)Y + \eta(Y)(\nabla_Z\eta)U]\}.$$

Using (1.13), we get

$$(\nabla_U \mathcal{L}_V g)(Y,Z) = -\frac{1}{2n-1}\{4n(U\lambda)g(Y,Z) + (n+1)(\alpha^2-\beta^2-c)[\alpha\{\eta(Z)g(U,\varphi Y) + \eta(Y)g(U,\varphi Z)\} + \beta\{\eta(Z)g(\varphi U,\varphi Y) + \eta(Y)g(\varphi U,\varphi Z)\}]\}.$$

Utilizing this in (1.17), we obtain

$$\begin{aligned}
2g((\mathcal{L}_V\nabla)(Z,Y),U) &= (\nabla_Z \mathcal{L}_V g)(Y,U) + (\nabla_Y \mathcal{L}_V g)(Z,U) - (\nabla_U \mathcal{L}_V g)(Y,Z) \\
&= -\frac{1}{2n-1}\{4n(Z\lambda)g(Y,U) + (n+1)(\alpha^2-\beta^2-c)[\alpha\{\eta(U)g(Z,\varphi Y) \\
&\quad + \eta(Y)g(Z,\varphi U)\} + \beta\{\eta(U)g(\varphi Z,\varphi Y) + \eta(Y)g(\varphi Z,\varphi U)\}]\} \\
&\quad - \frac{1}{2n-1}\{4n(Y\lambda)g(Z,U) + (n+1)(\alpha^2-\beta^2-c)[\alpha\{\eta(U)g(Y,\varphi Z) \\
&\quad + \eta(Z)g(Y,\varphi U)\} + \beta\{\eta(U)g(\varphi Y,\varphi Z) + \eta(Z)g(\varphi Y,\varphi U)\}]\} \\
&\quad + \frac{1}{2n-1}\{4n(U\lambda)g(Y,Z) + (n+1)(\alpha^2-\beta^2-c)[\alpha\{\eta(Z)g(U,\varphi Y) \\
&\quad + \eta(Y)g(U,\varphi Z)\} + \beta\{\eta(Z)g(\varphi U,\varphi Y) + \eta(Y)g(\varphi U,\varphi Z)\}]\}
\end{aligned}$$

Replacing $Z = Y = \xi$,

$$2g((\mathcal{L}_V\nabla)(\xi,\xi),U) = -\frac{1}{2n-1}\{4n(\xi\lambda)\eta(U) + 4n(\xi\lambda)\eta(U) - 4n(U\lambda)\}$$

Inserting $U = \xi$, we get

$$2g((\mathcal{L}_V\nabla)(\xi,\xi),\xi) = -\frac{1}{2n-1}\{4n(\xi\lambda) + 4n(\xi\lambda) - 4n(\xi\lambda)\}$$

This implies

$$(\mathcal{L}_V\nabla)(\xi,\xi) = -\frac{2n}{2n-3}D\lambda. \tag{2.36}$$

Now, taking covariant derivative of (2.28) along the vector field Z, we have

$$(\nabla_Z \mathcal{L}_V g)(Y,U) = -\frac{2}{2n-1}(\nabla_Z S)(Y,U),$$

where we applied V has a constant divergence and inserting it into (1.17), we have

$$g((\mathcal{L}_V\nabla)(Z,Y),U) = -\frac{1}{2n-1}[(\nabla_U S)(Z,Y) - (\nabla_Z S)(Y,U) - (\nabla_Y S)(Z,U)].$$

Using (2.25), we get

$$g((\mathcal{L}_V\nabla)(Z,Y),U) = -\frac{(n+1)(\alpha^2-\beta^2-c)\alpha}{2n-1}[\eta(Y)g(U,\varphi Z) + \eta(Z)g(U,\varphi Y)]$$

Replacing Z, Y and U by ξ, we have

$$(\mathcal{L}_V\nabla)(\xi,\xi) = 0. \tag{2.37}$$

By (2.36) and (2.37),

$$D\lambda = 0.$$

This implies that λ is constant and hence the almost Riemann soliton reduces to Riemann soliton. □

Lemma 2.2. *In a trans-Sasakian manifold $(M, g, \varphi, \eta, \xi, c, \alpha, \beta)$ with α, β constants, if a function $f \in C^\infty(M)$ is such that $Df = k\xi(f)\xi$ for some non-zero constant k, then f is constant.*

Proof. Taking covariant derivative on both side of $Df = k\xi(f)\xi$ with respect to X,

$$\nabla_X Df = k[X(\xi(f))\xi + \xi(f)\nabla_X \xi]$$

Using (1.12), we get

$$\nabla_X Df = k[X(\xi(f))\xi - \xi(f)[\alpha\varphi X + \beta\varphi^2 X]]$$

Taking inner product of preceding equation with Y, we get

$$g(\nabla_X Df, Y) = k[X(\xi(f))\eta(Y) - \xi(f)[\alpha g(\varphi X, Y) + \beta g(\varphi^2 X, Y)]] \tag{2.38}$$

Similarly,

$$g(\nabla_Y Df, X) = k[Y(\xi(f))\eta(X) - \xi(f)[\alpha g(X, \varphi Y) + \beta g(X, \varphi^2 Y)]] \tag{2.39}$$

We know that $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$, so by (2.38) and (2.39), we have

$$X(\xi(f))\eta(Y) - Y(\xi(f))\eta(X) + 2\alpha\xi(f)g(\varphi X, Y) = 0$$

Replacing X by φX and Y by φY, we have

$$\xi(f)g(\varphi X, Y) = 0.$$

Since $trace(\varphi^2) = 2n$, the preceding equation implies that

$$\xi(f) = 0.$$

Consequently $Df = 0$ and hence f is constant. □

Definition 2.1. *A vector field V on a trans-Sasakian manifold is said to be a contact vector field if it satisfies*

$$\mathcal{L}_V \eta = \varpi \eta \tag{2.40}$$

for some smooth function ϖ on the manifold.

Theorem 2.5. *If the potential vector field V of a Riemann soliton on a trans-Sasakian space form $(M, g, \varphi, \eta, \xi, c, \alpha, \beta)$ is a contact vector field, then it leaves the contact form η invariant, up to scaling.*

Proof. Now by (1.12), we have

$$d\eta(X, Y) = \alpha g(X, \varphi Y). \tag{2.41}$$

Taking Lie derivative in both sides along V, we get

$$\begin{aligned}(\mathcal{L}_V d\eta)(X, Y) &= \alpha(\mathcal{L}_V g)(X, \varphi Y) \\ &= -\frac{\alpha}{(2n-1)}[2S(X, \varphi Y) + 4n\lambda g(X, \varphi Y) + 2(divV)g(X, \varphi Y)].\end{aligned}$$

Replacing $Y = \xi$ in the preceding equation, we get

$$(\mathcal{L}_V d\eta)(X, \xi) = 0. \tag{2.42}$$

Applying the exterior derivative of operator d on (2.40), we get

$$(\mathcal{L}_V d\eta)(X, Y) = \varpi d\eta(X, Y) + \frac{1}{2}[d\varpi(X)\eta(Y) - d\varpi(Y)\eta(X)]$$

By (2.41), we have

$$(\mathcal{L}_V d\eta)(X,Y) = \alpha\varpi g(X,\varphi Y) + \frac{1}{2}[d\varpi(X)\eta(Y) - d\varpi(Y)\eta(X)]$$

Replacing $Y = \xi$ in the preceding equation, we get

$$(\mathcal{L}_V d\eta)(X,\xi) = \frac{1}{2}[d\varpi(X) - d\varpi(\xi)\eta(X)] \tag{2.43}$$

Equations (2.42) and (2.43) yield

$$d\varpi(X) = d\varpi(\xi)\eta(X).$$

This implies

$$g(D\varpi, X) = g((\xi\varpi)\xi, X).$$

This implies that

$$D\varpi = (\xi\varpi)\xi. \tag{2.44}$$

By Lemma 2.2, ϖ is constant. □

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