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THE STEINBART THEOREM, POLAR TRIANGLES AND THE REVERSION MAP

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ABSTRACT. We show that the Steinbart Theorem has an highly symmetric projective generalization in the framework of polar triangles and reversion maps of conics. The generalization has the additional feature that it is self-dual. This sheds new light on the original Steinbart Theorem, in particular on its dual, and yields a new conjugation in triangles.

1. INTRODUCTION

The Steinbart Theorem is one of those geometric incidences that makes you wonder how it could have remained hidden for millennia. In fact, the theorem was only discovered in 2000 by the German high school student Oliver Funck at the Steinbart Gymnasium in Duisburg. The theorem states the following:

Theorem 1.1. *Let $A_1A_2A_3$ be a triangle and P a point in the plane. The points A_1', A_2', A_3' denote the points of tangency of the incircle C of the triangle (see Figure 1). The line A_iP intersects C in a second point A_i'', $i = 1, 2, 3$. Then, the straight lines A_iA_i'' are concurrent in a point Q.*

Several special cases of the theorem were already known, for example, when the point P is the centroid of the triangle and Q its Exeter point. Humenberger lists in [7] seven such special cases.

Already Funck observed in his original work [3] that the theorem has a natural interpretation in the context of projective geometry, and formulated the corresponding version and some consequences. Funck proved the theorem with the help of a computer algebra system. Grinberg noted in [4], that the concurrency of the lines $A_i'A_i''$ is *not* a necessary condition for the concurrency of the lines A_iA_i'', and proved an extended version of the Steinbart Theorem. The aim of this article is to show that the Steinbart Theorem has an highly symmetric projective generalization in the framework of polar triangles and reversion maps of conics. This generalization has the additional nice feature that it is self-dual.

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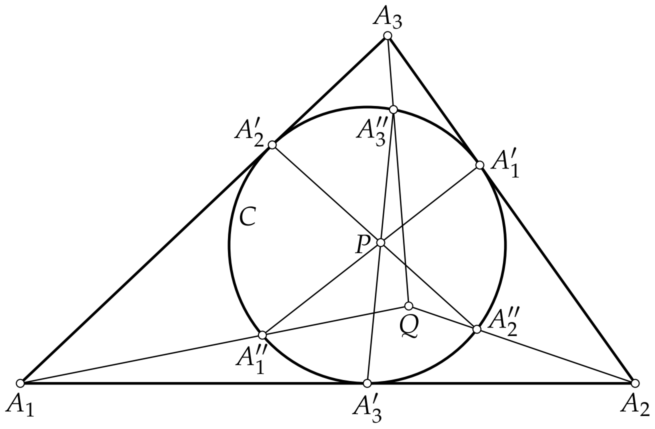


Figure 1. The Steinbart Theorem.

The article is organized as follows. In Section 2, we briefly describe the necessary theory of polar triangles and the reversion map. Section 3 contains the projective generalization of the Steinbart Theorem, and some additional features of the configuration. In Section 4 we describe some applications of the main theorem.

2. Polar triangles and the reversion map

The idea of the generalization is based on three pillars. First of all notice that the Steinbart Theorem remains valid, if we replace the incircle by an arbitrary non-degenerate conic C which touches the sides of the triangle. Secondly, observe that the situation in the Steinbart Theorem is just a special case of triangles $A_1A_2A_3$ and $A_1'A_2'A_3'$ which are conjugate with respect to C. And thirdly, notice that the map which associates the point A_i'' on C to the point A_i' on C has a projective extension to the whole projective plane, namely the reversion map.

Let us now explain the three pillars more explicitly and fix the notation at the same time.

2.1. **The projective plane.** We will work in the standard model of the real projective plane. The set of points $\mathbb{P}$ is given by $\mathbb{RP}^2 = \mathbb{R}^3 \setminus \{0\} / \sim$, where $X \sim Y \in \mathbb{R}^3 \setminus \{0\}$ are equivalent if $X = \lambda Y$ for some $\lambda \in \mathbb{R}$. Similarly, the set of lines $\mathbb{B}$ is also $\mathbb{R}^3 \setminus \{0\} / \sim$, where again $g \sim h \in \mathbb{R}^3 \setminus \{0\}$ are equivalent, if $g = \lambda h$ for some $\lambda \in \mathbb{R}$. A point $[X]$ and a line $[g]$ are incident if $\langle X, g \rangle = 0$, where we denoted equivalence classes by square brackets and the standard inner product in $\mathbb{R}^3$ by $\langle \cdot, \cdot \rangle$. Since we mostly work with representatives we will omit the square brackets in the notation of equivalence classes.

A non-degenerate conic in $\mathbb{RP}^2$ is given by the equation $\langle X, CX \rangle = 0$ where C is a regular, real, symmetric 3×3 matrix which has eigenvalues of both signs. By abuse of notation we will denote both, the conic and the matrix, with the same letter C. The polare line of a point P with respect to C is given by CP. Vice versa, the pole of a line p with respect to C is given by $C^{-1}p$. The intersection of two lines g and h can be computed by $g \times h$, where $\times$ is the cross product in $\mathbb{R}^3$. Similarly, the line passing through the points X and

Y is $X \times Y$. See, e.g., [9] for more information or a general introduction to projective geometry.

2.2. **Polar triangles.** Let A'_1, A'_2, A'_3 be three points, not on the same line, and C a non-degenerate conic. Then the polar line of A'_i is given by CA'_i, and the three polar lines CA'_i form the polar triangle $A_1A_2A_3$ of $A'_1A'_2A'_3$ with respect to C (see Figure 2). Vice versa, by de La Hire's Fundamental Theorem of poles and polars (see [8]), the triangle $A'_1A'_2A'_3$ is the polar triangle of $A_1A_2A_3$. So, because this relation is symmetric, it is justified to call the two triangles *conjugate* to each other with respect to C. Recall that by Chasles' Polar Triangle Theorem, conjugate triangles are centrally perspective, i.e., the lines $A_iA'_i$ are concurrent (see, e.g., [2, 5·61]). Moreover, conjugate triangles are also axially perspective by Desargues' Two-triangle Theorem (see, e.g., [2, 1·52]).

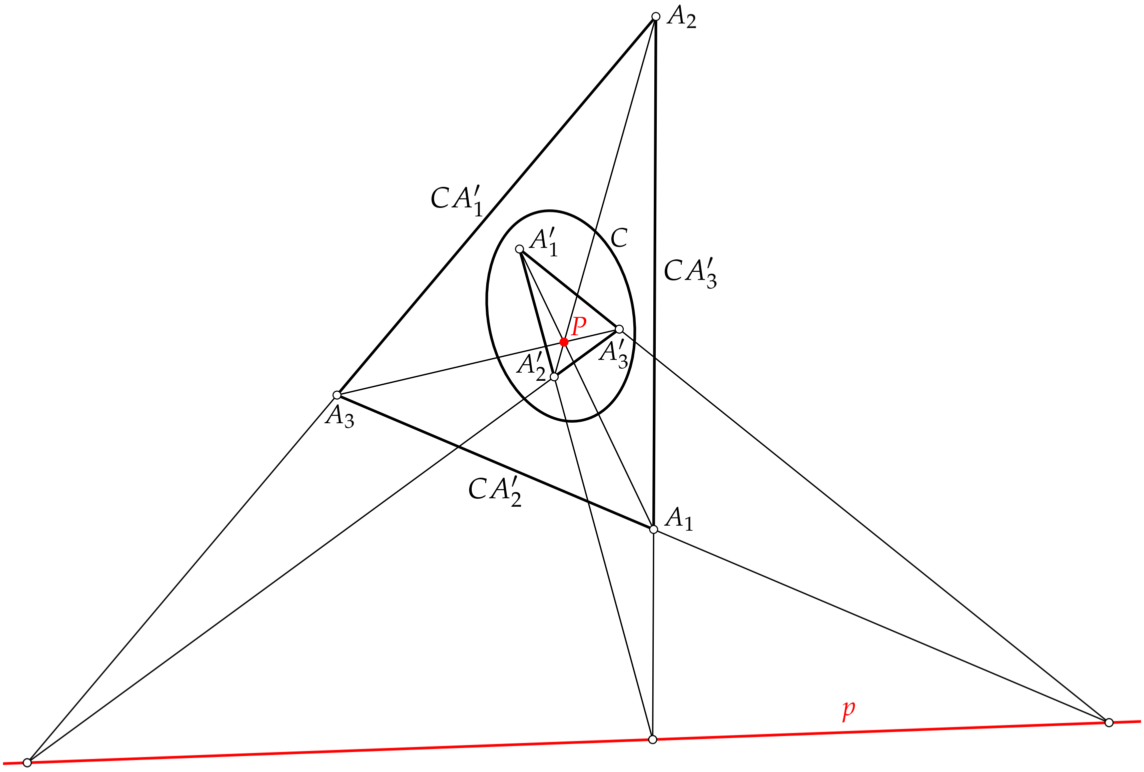


Figure 2. Chasles' Polar Triangle Theorem.

In view of the application below, we make an addition to Chasles' theorem.

Proposition 2.1. *If $A_1A_2A_3$ and $A'_1A'_2A'_3$ are conjugate triangles with respect to a conic C with perspectivity center P and perspectivity axis p, then P is the pole of p with respect to C.*

Proof. Observe that by de La Hire's Theorem the intersection of the line $A_{i-1}A_{i+1}$ and $A'_{i-1}A'_{i+1}$ is the pole of the line $A_iA'_i$ for all i. Here and below, indicies are read cyclically. This proves the claim. □

2.3. **The reversion map.** Let C be a non-degenerate conic and P a point not on C. The point reversion map

$$\varphi_P : C \to C, \;\; X \mapsto \varphi_P(X),$$

is defined by the requirement that the points $X, P, \varphi_P(X)$ are collinear, and that $X \neq \varphi_P(X)$ unless the line XP is a tangent of C (see Figure 3). Clearly, the point reversion map is an involution.

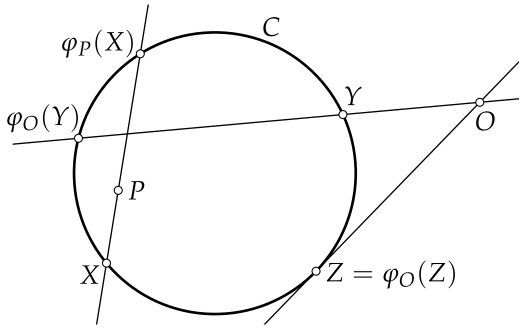


Figure 3. The point reversion map on the circle C.

This point reversion has a projective extension to the whole plane $\mathbb{RP}^2$, again denoted by φ_P, which can be written as

$$\varphi_P : \mathbb{P} \to \mathbb{P}, \quad X \mapsto \langle P, CP\rangle X - 2\langle X, CP\rangle P = MX. \tag{2.1}$$

Here M is the 3×3 matrix given by the expression in (2.1). Indeed, the three points $X, P, \varphi_P(X)$ are obviously collinear, and a short calculation shows that $\langle X, CX\rangle = 0$ implies $\langle \varphi_P(X), C\varphi_P(X)\rangle = 0$. Hence, φ_P maps points on C to points on C. Moreover, $X = \varphi_P(X)$ for X on C is equivalent to $\langle X, CP\rangle = 0$ and this is the case if and only if P lies on the tangent to C in X. Hence, the projective map given by (2.1) has the properties which define the point reversion. See [1] and [6] for more information about the point reversion map.

Geometrically the point $\varphi_P(X)$ can be constructed as shown in Figure 4: Let R be a point on C, not on the line XP. Then the line g through R and X intersects C in a second point S. The line through $\varphi_P(R)$ and $\varphi_P(S)$ intersects the line XP in the point $\varphi_P(X)$, and this point does not depend on the choice of R.

If $A'_1A'_2A'_3$ is a triangle, C a non-degenerate conic and P a point, we will call $A''_1A''_2A''_3$, where $A''_i = \varphi_P(A'_i)$, the point reversed triangle of $A'_1A'_2A'_3$ with respect to P. Vice versa, because the reversion map is an involution, the triangle $A'_1A'_2A'_3$ is the point reversed triangle of $A''_1A''_2A''_3$. It is therefore justified to call the two triangles *point reversed* to each other. Since point reversed triangles are centrally perspective by construction, they are also axially perspective. Similarly to Proposion 2.1 we have the following property:

Proposition 2.2. *If $A'_1A'_2A'_3$ and $A''_1A''_2A''_3$ are point reversed triangles with respect to a conic C, with perspectivity center P and perspectivity axis p, then P is the pole of p with respect to C (see Figure 5).*

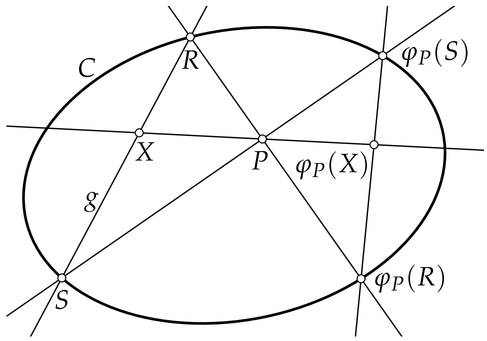


Figure 4. Construction of the reversion point $\varphi_P(X)$.

Proof. By applying a suitable projective map we may assume that C is a circle and either P its center, or P a point at infinity. In the first case, the map φ_P is just the reflection about the point P and hence the perspectivity axis is the ideal line, which proves the claim. In the other case, φ is the reflection about the polar line ℓ of P through the center of the circle. Thus, the perspectivity axis is ℓ and we are also done in this case. □

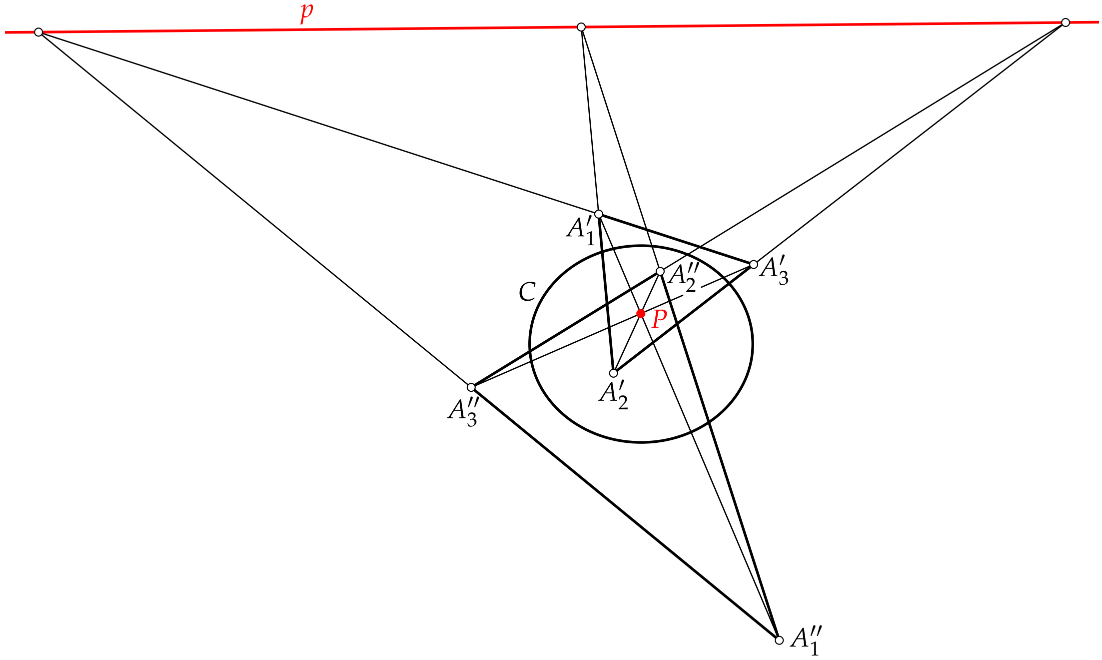


Figure 5. Point reversed triangles with respect to P are axially perspective with respect to the polar line p.

The point reversion φ_P maps points to points. Since it is a collineation it also induces a map $\bar{\varphi}_P$ which maps lines to lines, and which is given by the inverse transposed matrix M^{-t} from (2.1).

On the other hand, we can also define the dual of the point reversion map: This is the map φ_p for a given line p which maps the line x to the line $\varphi_p(x)$ with respect to the conic C. We call φ_p the line reversion map. It is given by the equation

$$\varphi_p : \mathbb{B} \to \mathbb{B}, \quad x \mapsto \langle p, C^{-1}p\rangle x - 2\langle x, C^{-1}p\rangle p.$$

It is instructive to interpret the line reversion geometrically by dualizing the construction in Figure 4 (see Figure 6): Select an exterior point E of C on the line x, and draw the tangents to C. These tangents intersect p in the points F and G. From F and G draw the second tangent to C. Their intersection is a point H. Then. $\varphi_p(x)$ is the line JH, where J is the intersection of x and p. Notice that this line does not depend on the initial choice of E.

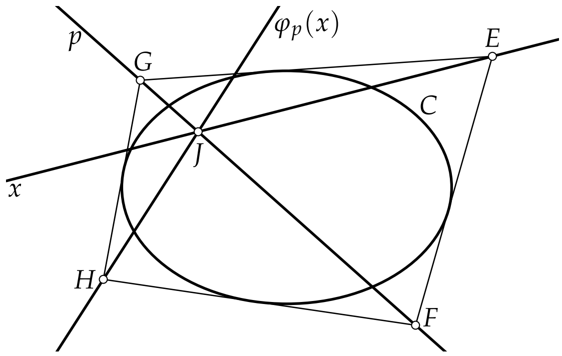


Figure 6. Construction of the line reverse $\varphi_p(x)$ of the line x with respect to the line p.

Observe that $\varphi_p : \mathbb{B} \to \mathbb{B}$ induces a map $\bar{\varphi}_p : \mathbb{P} \to \mathbb{P}$ by mapping the intersection of lines to the intersection of the image lines. The following proposition links the variants of the reversion maps:

Proposition 2.3. *Let P be the pole of the line p with respect to the conic C. Then the map $\varphi_P : \mathbb{P} \to \mathbb{P}$ agrees with the map $\bar{\varphi}_p : \mathbb{P} \to \mathbb{P}$, and vice versa, the map $\varphi_p : \mathbb{B} \to \mathbb{B}$ agrees with the map $\bar{\varphi}_P : \mathbb{B} \to \mathbb{B}$.*

Proof. Like in the proof of Proposition 2.2 it suffices to consider the case of a circle with either P its center, or P at infinity. In both situations, the claim is easily checked. □

Since φ_P is a projective map that leaves C invariant the following ist true: Let Q be a point and q its polar line with respect to the conic C. Then the line $\varphi_P(q)$ is the polar line of $\varphi_P(Q)$ with respect to C. This leads immediately to the following statement.

Proposition 2.4. *If $A_1A_2A_3$ is the polar triangle of $A'_1A'_2A'_3$, and $A''_1A''_2A''_3$ is the point reversed triangle of $A'_1A'_2A'_3$ with respect to P, then the polar triangle A'''_1, A'''_2, A'''_3 of $A''_1A''_2A''_3$ is the point reversed triangle of $A_1A_2A_3$ with respect to P.*

With this preparation we are now ready to formulate the projective generalization of the Steinbart Theorem.

3. The projective generalization of the Steinbart Theorem

We set the stage as follows.

Proposition 3.1. *Let C be non-degenerate conic and P a point not on C. Let $\triangle' = A_1'A_2'A_3'$ and $\triangle'' = A_1''A_2''A_3''$ be point reversed triangles with respect to P. Moreover, let $\triangle = A_1A_2A_3$ be the polar triangle of $\triangle'$, and $\triangle''' = A_1'''A_2'''A_3'''$ be the polar triangle of $\triangle''$. Then the following statements hold.*

(a) The triangles $\triangle$ and $\triangle'$ are point perspective with respect to a point Q and axially perspective with respect to its polar line q.
(b) The triangles $\triangle''$ and $\triangle'''$ are point perspective with respect to a point R and axially perspective with respect to its polar line r.
(c) $R = \varphi_P(Q)$ and $r = \varphi_p(q)$.
(d) The triangles $\triangle'$ and $\triangle''$ are point perspective with respect to P and axially perspective with respect to its polar line p.
(e) The triangles $\triangle$ and $\triangle'''$ are point perspective with respect to P and axially perspective with respect to its polar line p.
(f) The points $A_1, A_2, A_3, A_1''', A_2''', A_3'''$ lie on a conic.
(g) The points $A_1', A_2', A_3', A_1'', A_2'', A_3''$ lie on a conic.

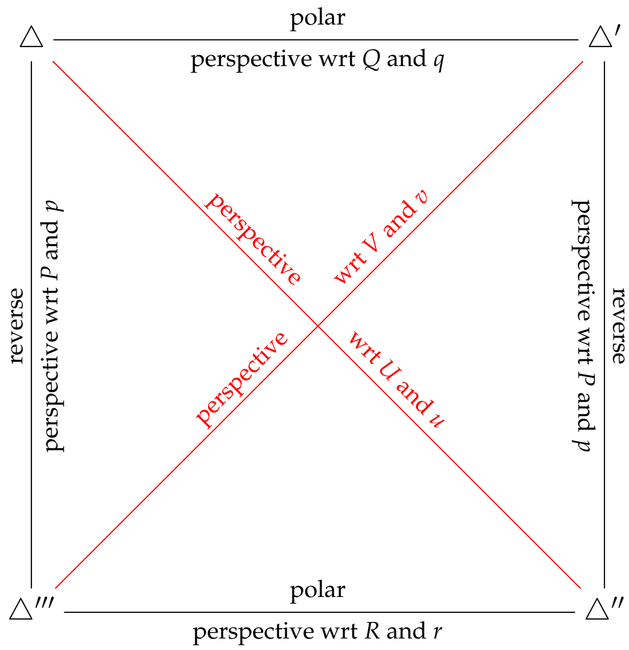
Proof. The statements (a) and (b) follow directly from Chasles' Theorem together with Proposition 2.1. (c) follows from the fact that φ_P maps $\triangle'$ to $\triangle''$, and, by Proposition 2.4, $\triangle$ to $\triangle'''$. The statements (d) and (e) follow from Proposition 2.1. For (f) we argue as follows: Since $\triangle'$ and $\triangle''$ are perspective, it follows from Biranchon's Theorem that the points $A_1', A_2'', A_3', A_1'', A_2', A_3''$ form a hexagon which is circumscribed around a conic D. Therefore the vertices of the polar hexagon $A_1, A_2''', A_3, A_1''', A_2, A_3'''$ are points on the conjugate conic E of D with respect to C (see [5]). The same argument applies when we start with the triangle $\triangle$ and $\triangle'''$, which gives (g). □

The miracle which happens now is that also $\triangle$ and $\triangle''$ are perspective, and the same is true for $\triangle'$ and $\triangle'''$.

Theorem 3.1 (Projective generalization of the Steinbart Theorem)**.** *Let C be non-degenerate conic and P a point not on C. Let $\triangle' = A_1'A_2'A_3'$ and $\triangle'' = A_1''A_2''A_3''$ be point reversed triangles with respect to the point P, or equivalently, line reversed with respect to its polar line p. Let $\triangle = A_1A_2A_3$ be the polar triangle of $\triangle'$, and let $A_1'''A_2'''A_3'''$ be the polar triangle of $\triangle''$. Then the following statements hold (see Figure 7).*

(a) The triangles $\triangle$ and $\triangle''$ are point perspective with respect to a point U, and axially perspective with respect to its polar line u.
(b) The triangles $\triangle'$ and $\triangle'''$ are point perspective with respect to a point V, and axially perspective with respect to its polar line v.
(c) $V = \varphi_P(U)$ and $v = \varphi_p(u)$.

The relations in the Propositions 3.1 and Theorem 3.1 are visualized in the following schema, the Steinbart relations of Theorem 3.1 are indicated in red:



Proof. We start with the points A'_i and denote their polar lines by $a'_i := CA'_i$. Then the points A_i are given by $A_i = a'_{i-1} \times a'_{i+1}$, where $\times$ is the cross product in $\mathbb{R}^3$. Here and below, indicies are read cyclically. For the points A''_i we can write $A''_i = MA'_i$, where M is the matrix given in (2.1). The three lines $A_iA''_i = A_i \times A''_i$, $i = 1, 2, 3$, are concurrent iff $d = \det(A_1 \times A''_1, A_2 \times A''_2, A_3 \times A''_3) = 0$. Inserting the expressions for A_i and A''_i we have

$$d = \det\big((a'_3 \times a'_2) \times MA'_1, (a'_1 \times a'_3) \times MA'_2, (a'_2 \times a'_1) \times MA'_3\big).$$

By expanding the triple cross product we find for the columns c_i in this matrix

$$c_i = a'_{i+1}\langle MA'_i, a'_{i-1}\rangle - a'_{i-1}\langle a'_{i+1}, MA'_i\rangle.$$

Now we use the linearity of the determinant to expand d. Observe that all terms which correspond to determinants having two columns which are parallel to the same vector a'_i are zero. We are left with

$$d = \det(a'_2\langle MA'_1, a'_3\rangle, a'_3\langle MA'_2, a'_1\rangle, a'_1\langle MA'_3, a'_2\rangle) \\ - \det(a'_3\langle MA'_1, a'_2\rangle, a_1\langle MA'_2, a'_3\rangle, a'_2\langle MA'_3, a'_1\rangle).$$

Therefore we have $d = 0$, if

$$\langle MA'_1, a'_3\rangle\langle MA'_2, a'_1\rangle\langle MA'_3, a'_2\rangle = \langle MA'_1, a'_2\rangle\langle MA'_2, a'_3\rangle\langle MA'_3, a'_1\rangle.$$

Indeed, we have

$$\langle MA'_i, a'_j\rangle = \langle MA'_j, a'_i\rangle,$$

since by (2.1)

$$\langle MA'_i, a'_j\rangle = \langle\langle P, CP\rangle A'_i - 2\langle A'_i, CP\rangle P, CA'_j\rangle = \langle P, CP\rangle\langle A'_i, CA'_j\rangle - 2\langle A'_i, CP\rangle\langle P, CA'_j\rangle$$

and

$$\langle MA'_j, a'_i \rangle = \langle \langle P, CP \rangle A'_j - 2\langle A'_j, CP \rangle P, CA'_i \rangle = \langle P, CP \rangle \langle A'_j, CA'_i \rangle - 2\langle A'_j, CP \rangle \langle P, CA'_i \rangle.$$

Both terms agree since C is symmetric. This shows statement (a). By exchanging the role of $\triangle'$ and $\triangle''$ we get in the same way the statement (b). Statement (c) is then evident. □

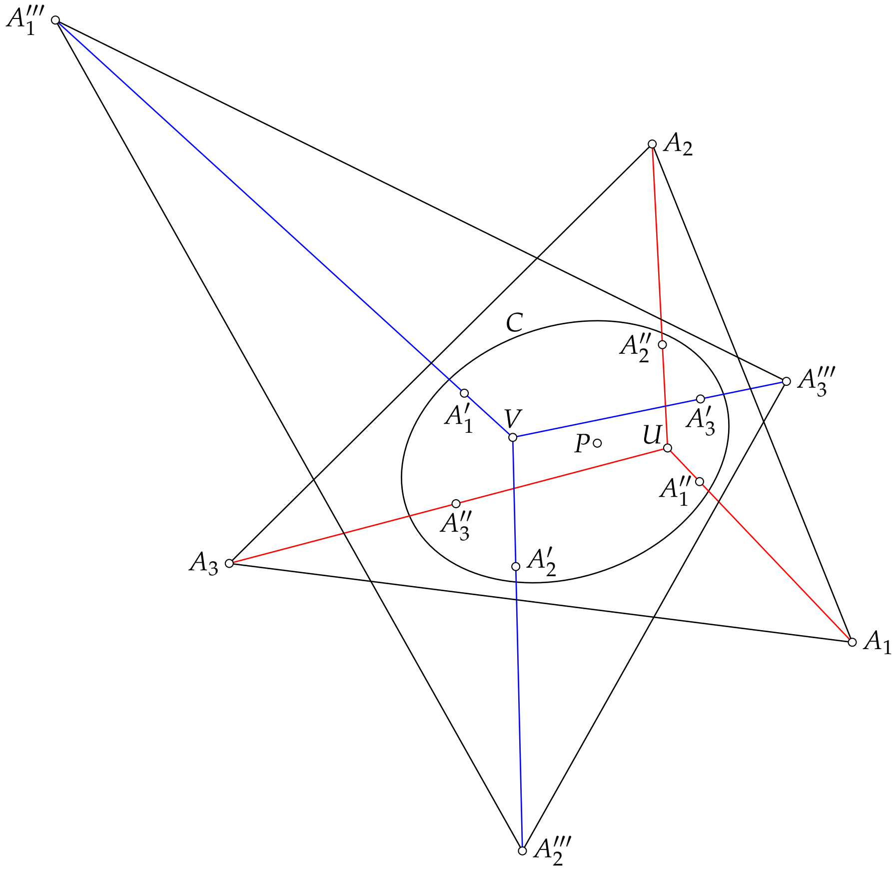


Figure 7. The projective generalization of the Steinbart Theorem.

4. Some consequences of the projective generalization of the Steinbart Theorem

Theorem 3.1 implies immediately that the Steinbart Theorem also holds, mutatis mutandis, for each excircle instead of the incircle.

Corollary 4.1 (The Steinbart Theorem for the excircle)**.** *Let $A_1A_2A_3$ be a triangle and P a point in the plane. The points A'_1, A'_2, A'_3 denote the points of tangency of the excircle C of the triangle (see Figure 8). The line A_iP intersects C in a second point A''_i, $i = 1, 2, 3$. Then, the straight lines $A_iA''_i$ are concurrent in a point Q.*

Incidence relations that are self-dual, like, e.g., Desargues' Theorem, are particularly appealing. The main Theorem 3.1 has this nice property.

Proposition 4.1. *The statement of Theorem 3.1 is self-dual.*

Proof. The statement that two triangles are conjugate with respect to a conic is a self-dual statement. Similarly, the statement that two triangles are point reverse is dual to the statement, that the two triangles are line reverse. Hence, when we dualize Theorem 3.1 we get again the same statement. □

In contrast, the original Theorem 1.1 is not self-dual. Its dual formulation reads as follows.

Corollary 4.2 (The dual form of the Steinbart Theorem 1.1)**.** *Let the lines $A_1A_2A_3$ form a triangle with circumcircle C, and let P be a line in the plane. The tangents in the vertices of the triangle $A_1A_2A_3$ are $A'_1A'_2A'_3$, numbered in the obvious way (see Figure 9). For each i, we draw the other tangent A''_i to C from the intersection of the line A'_i with the line P. Then the intersections of A_i and A''_i are collinear on a line Q.*

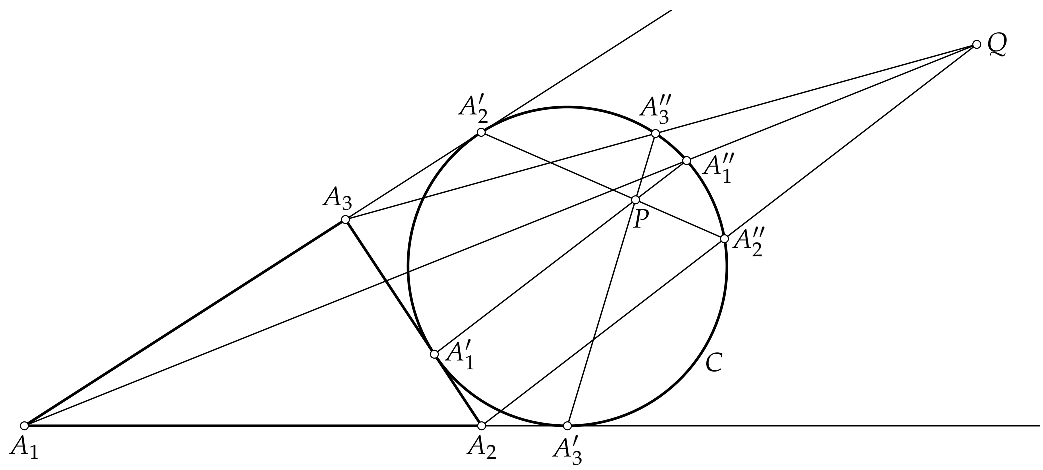


Figure 8. The Steinbart Theorem for the excircle.

However, in the light of Theorem 3.1, we can interpret the statement in Corollary 4.2 so that its true nature becomes apparent. Indeed, observe that the triangles with sides $A'_1A'_2A'_3$ and $A_1A_2A_3$ are polar triangles. The triangles $A'_1A'_2A'_3$ and $A''_1A''_2A''_3$ are line reverse with respect to the line P and therefore point reverse with respect to the pole of P (see the dotted lines in Figure 9). Hence, the triangles $A_1A_2A_3$ and $A''_1A''_2A''_3$ are axially perspective with respect to the line Q and therefore also point perspective (see the dashed lines in Figure 9).

As a last remark, we want to point out that, in Theorem 1.1, the map with associates the point Q to the point P is not an involution. However, in Theorem 3.1, the map which associates the point U to the point V is an involution (see Figure 7). Hence we may consider this map as a new conjugation in a triangle $A_1A_2A_3$. This conjugation has the nice property, that it leaves the conic C invariant.

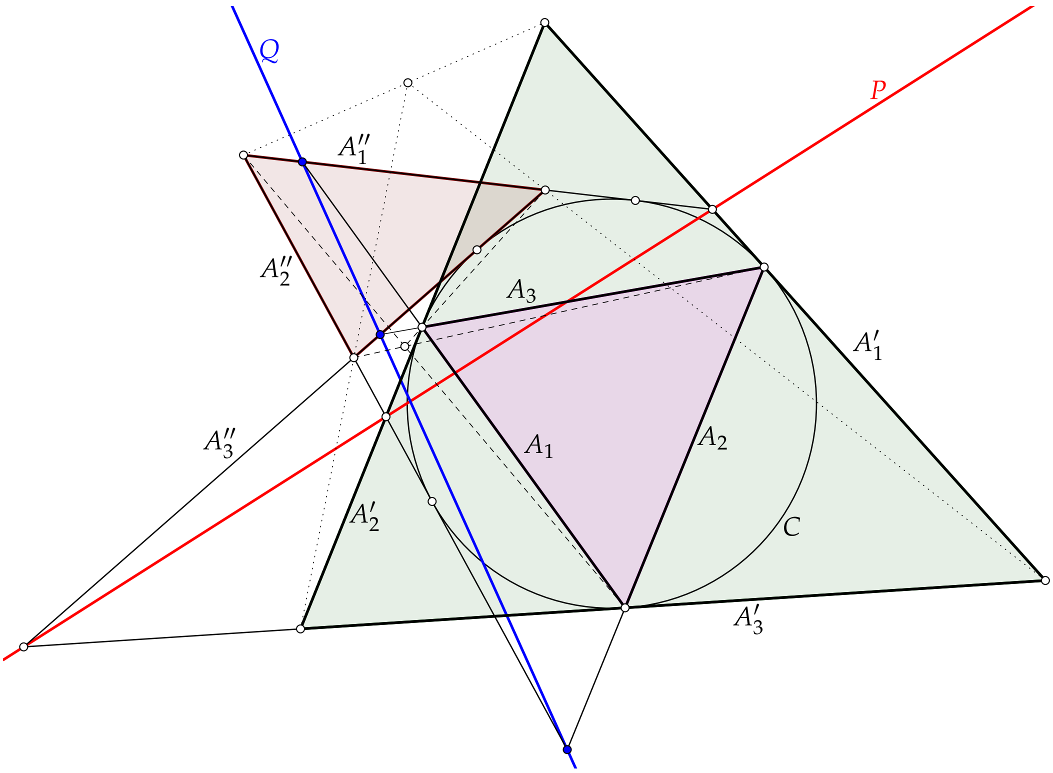


Figure 9. The dual of the Steinbart Theorem.

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