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A DIRECT PROOF OF LESTER'S THEOREM AND A RELATED TOPIC

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Abstract. In this paper is given an elementary proof of Lester's theorem using the formulas for the distances between the points O, N, F^+, F^-. Also, using the properties of the orthocentroidal circle and cosymmedian triangles, other quadruples of concyclic points are indicated.

In 1997, *June Lester* has discovered that the Fermat points, the circumcenter, and the nine-point center are concyclic [9]. The circle on which these points are located is called *Lester's circle*. This topic has interested many researchers. Lester's theorem has been demonstrated with various methods. *R. Shail* provides a Cartesian proof [12], *J. Rigby* uses the complex numbers [10] , *J.A. Scott* gives two proofs using baricentric coordinates [11], and *M. Duff* a projective proof [6] of this theorem. *N.I. Beluhov* gives an ingenious synthetic demonstration in [2], [3]. Etc., etc.
This work is closely related to [4] and [5]. We aim to give a simple and elementary demonstration to Lester's theorem using the formulas for the distances between the points O, N, F^+, F^- set out in [5].

1. A proof of Lester's theorem

Consider a triangle ABC with the sidelenghts a, b, c and area Δ. The notations O, H, G, N, K are standard. Also, F^+, F^-denote the first and second Fermat (isogonic) points, and J^+, J^- denote the first and second isodynamic points, respectively (see [8], [13]). The Euler line OH, the Fermat axis F^+F^-, and the Brocard axis OK are denoted by e, f, b, respectively. Let M be the midpoint of HG (Fig. 1). It is known that $M \in e \cap f$, $K \in f \cap b$, and $O \in e \cap b$.

Lemma 1.1. *The quadrilateral ONF^+F^- is convex.*

Proof. It is known that the order of points O, N, M on the Euler line is $O - N - M$, and of the points F^+, F^-, M on the Fermat axis is $F^- - F^+ - M$. These two properties imply our claim. □

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Let's recall the formulas we will need below (for demonstration, see [5]). We have:

$$F^+F^- = \frac{1}{\sqrt{3}}\frac{\sqrt{f}}{l_+l_-}, \tag{1.1}$$

$$F^+O^2 = \frac{1}{144\Delta^2 l_+^2}\left[32\Delta^2 l_+^2 l_-^2 + \left(2l_+^2 - l_-^2\right) f\right], \tag{1.2}$$

$$F^-O^2 = \frac{1}{144\Delta^2 l_-^2}\left[32\Delta^2 l_+^2 l_-^2 + \left(2l_-^2 - l_+^2\right) f\right], \tag{1.3}$$

$$F^+N^2 = \frac{1}{576\Delta^2 l_+^2}\left[32\Delta^2 l_+^2 l_-^2 + \left(2l_-^2 - l_+^2\right) f\right], \tag{1.4}$$

$$F^-N^2 = \frac{1}{576\Delta^2 l_-^2}\left[32\Delta^2 l_+^2 l_-^2 + \left(2l_+^2 - l_-^2\right) f\right], \tag{1.5}$$

where $f(a,b,c) = a^6 + b^6 + c^6 + 3a^2b^2c^2 - a^4b^2 - a^2b^4 - a^4c^2 - a^2c^4 - b^4c^2 - b^2c^4$ and

$$l_+^2 = \frac{1}{2}\left(a^2 + b^2 + c^2 + 4\sqrt{3}\Delta\right),\ l_-^2 = \frac{1}{2}\left(a^2 + b^2 + c^2 - 4\sqrt{3}\Delta\right).$$

It is known that $ON^2 = \frac{1}{4}OH^2 = \frac{1}{4}\left[9R^2 - \left(a^2 + b^2 + c^2\right)\right]$. Using the formula $4R\Delta = abc$, after a simple calculation, we get $ON^2 = \frac{1}{4}\cdot\frac{f}{16\Delta^2}$, i.e.

$$ON = \frac{\sqrt{f}}{8\Delta}. \tag{1.6}$$

It is easy to see that

$$F^+O = 2\frac{l_-}{l_+}F^-N \text{ and } F^-O = 2\frac{l_+}{l_-}F^+N. \tag{1.7}$$

Proposition 1.1. (Lester's theorem, 1997) *The points F^+, F^-, O, N are concyclic.*

Proof. These points are concyclic if and only if Ptolemy's equation for the convex quadrilateral ONF^+F^- is verified, i.e. the equation

$$F^+O\cdot F^-N = F^+F^-\cdot ON + F^+N\cdot F^-O$$

holds. According to (1.7), we can write this equation in the form

$$2l_-^2\cdot F^-N^2 - 2l_+^2\cdot F^+N^2 = l_+l_-\cdot F^+F^-\cdot ON.$$

Taking into account (1.5), (1.4), (1.1), (1.6) and performing a simple calculation, we obtain

$$\frac{2}{576\Delta^2}\left[\left(2l_+^2 - l_-^2\right) f - \left(2l_-^2 - l_+^2\right) f\right] = \frac{f}{8\sqrt{3}\Delta},$$

what is checked immediately. So, the proof is concluded. □

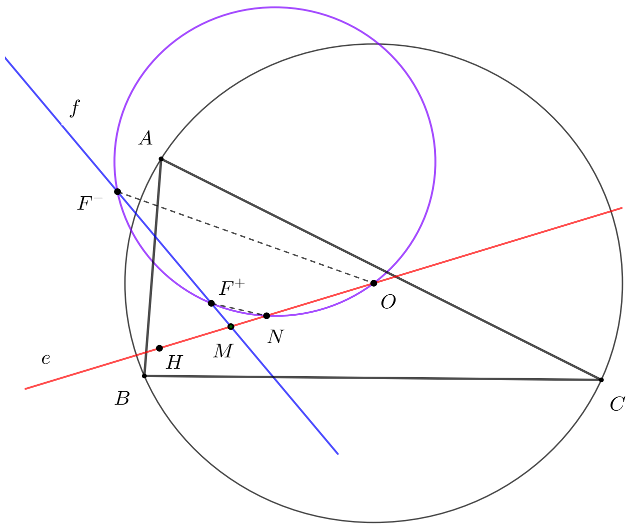

Figure 1

Remark 1.1. If ABC is an isosceles triangle, the points O and H, as well as F^+ and F^-, are on the axis of the triangle (i.e. Euler line and Fermat axis coincide with this axis). Then, the points F^+, F^-, O, N are collinear. This is a limiting case: Lester's circle degenerates into the axis of the isosceles triangle.

2. Some quadruples of concyclic points

The *cosymmedian triangle* associated with ABC is the triangle $A'B'C'$ with vertex A' defined by $A' = AK \cap (ABC)$, $A' \neq A$, and B', C' defined similarly ((XYZ) denotes the circle determined by the points X, Y, Z). Both triangles ABC and $A'B'C'$ are inscribed in the circle $\mathcal{C}_0 \equiv (ABC)$. The *orthocentroidal circle* of triangle ABC is the circle on HG as diameter, denoted here $\mathcal{C}_1$. Obviously, his center is M. It is known that the triangle $A_1B_1C_1$ determined by the projections of G on the altitudes of ABC and the triangle $A_1'B_1'C_1'$ determined by the projections of H on the medians of ABC are cosymmedian and inscribed in $\mathcal{C}_1$. Recall that $A_1B_1C_1$ is called the *orthocentroidal triangle* associated with ABC. We agree to denote with $H', G', N', M', F'^+, F'^-$ etc. and $H_1, G_1, N_1, M_1, F_1^+, F_1^-$ etc. the notable points corresponding to the triangle $A'B'C'$ and $A_1B_1C_1$, respectively. We assume that the basic properties of cosymmedian triangles and orthocentroidal circles are known (see [1], [4], [7] and others). Thus, we will use below that the triangles ABC and $A'B'C'$ have the same isodynamic points (i.e., $J'^+ = J^+$ and $J'^- = J^-$) and the fact that ABC and $A_1B_1C_1$ are inversely similar and $J_1^+ = F^+, J_1^- = F^-$.
The certain connection between the triangles $ABC, A'B'C'$ and $A_1B_1C_1$ is emphasized by the following theorem.

Proposition 2.1. *The following properties are true* (Fig. 2):

1) $GG' \parallel HH' \parallel NN' \parallel MM'$,
2) $F^+J^+ \parallel F^-J^- \parallel e$ ([8],Table 5.3, p.139) *and* $F'^+J^+ \parallel F'^-J^- \parallel e'$,
3) $F^+F'^+ \parallel F^-F'^- \parallel MM'$,
4) $F_1^+F^+ \parallel F_1^-F^- \parallel e_1$.

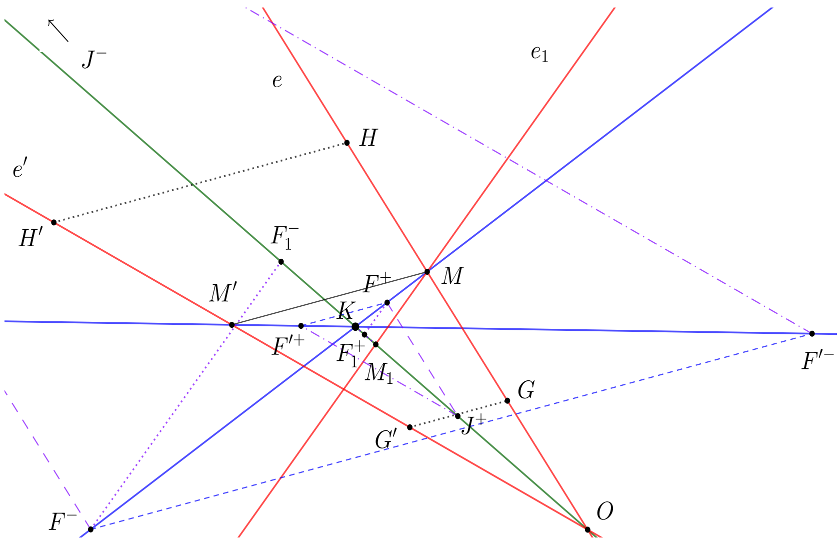

Figure 2

Proof. 1) The relative positions of points O, H, G, N, M and O', H', G', N', M' on Euler lines e and e', respectively, are determined by the same constant distance-ratios. Then we apply Thales' theorem.

2) We'll just show that $F^+J^+ \parallel e$. In the same way it is shown that $F^-J^- \parallel e$. For this it is enough to check that the equation

$$\frac{KF^+}{F^+M} = \frac{KJ^+}{J^+O}. \tag{2.1}$$

holds. Indeed, in [5] it was established the formulae:

$$KF^+ = \frac{\sqrt{f}}{\sqrt{3}\,(a^2+b^2+c^2)}\frac{l_-}{l_+} \quad \text{and} \quad F^+M = \frac{\sqrt{f}}{12\Delta}\cdot\frac{l_-}{l_+}; \tag{2.2}$$

$$KJ^+ = \frac{\sqrt{3}abc}{a^2+b^2+c^2}\cdot\frac{l_-}{l_+} \quad \text{and} \quad J^+O = \frac{abc}{4\Delta}\cdot\frac{l_-}{l_+}. \tag{2.3}$$

By substitution, we immediately verify that (2.1) is true.

3) Let's prove that $F^+F'^+ \parallel MM'$. It is enough to show that

$$\frac{KF^+}{F^+M} = \frac{KF'^+}{F'^+M'}. \tag{2.4}$$

The terms KF^+ and F^+M are given by (2.2), while KF'^+ and F'^+M' have expressions of the same shape in the sidelengths of the of the triangle $A'B'C'$. Then, (2.4) is equivalent to

$$\frac{\Delta}{a^2+b^2+c^2} = \frac{\Delta'}{(a')^2+(b')^2+(c')^2}$$

or

$$\frac{abc}{a^2+b^2+c^2} = \frac{a'b'c'}{(a')^2+(b')^2+(c')^2}$$

(because $4R\Delta = abc$ and $4R\Delta' = a'b'c'$). Since the triangles ABC and $A'B'C'$ are cosymmedians, the sidelenghts of the one are proportional to the medians of the other

([7], p.283; [4], p.3); more specifically, we have $a' = k \cdot m_a, b' = k \cdot m_b, c' = k \cdot m_c$ with $k = \frac{3}{4}\frac{abc}{m_a m_b m_c}$. Then the previous equation is written in the form

$$\frac{abc}{a^2+b^2+c^2} = \frac{k \cdot m_a m_b m_c}{(m_a)^2+(m_b)^2+(m_c)^2}.$$

Finally, taking into account the fomula $(m_a)^2+(m_b)^2+(m_c)^2 = \frac{3}{4}\left(a^2+b^2+c^2\right)$ and the expression of k, we get $1 = 1$.

4) It is the first property in 2) written for the triangle orthocentroidal.

The proof is complete. □

Corollary 2.1. *The triangles $KF^+F_1^+, KJ^+F^+, KF^-F_1^-$, and KJ^-F^- are similar, i.e. we have:*

$$KF^+F_1^+ \sim KJ^+F^+ \sim KF^-F_1^- \sim KJ^-F^-. \tag{2.5}$$

Proof. By Proposition 2.1, $F^+J^+ \parallel F^-J^-$ and $F_1^+F^+ \parallel F_1^-F^-$. Hence, $KJ^+F^+ \sim KJ^-F^-$ and $KF^+F_1^+ \sim KF^-F_1^-$.

On the other hand, $\frac{KF_1^+}{KF^+} = \frac{KF_1^+}{KJ_1^+}$ (since $J_1^+ = F^+$) and $\frac{KF_1^+}{KJ_1^+} = \frac{KF^+}{KJ^+}$ (since $A_1B_1C_1 \sim ABC$). Hence, $\frac{KF_1^+}{KF^+} = \frac{KF^+}{KJ^+}$. This equality and $\widehat{F^+KF_1^+} = \widehat{J^+KF^+}$ (same angle) imply that $KF^+F_1^+ \sim KJ^+F^+$ (Fig. 2). So, (2.5) is proven. □

Proposition 2.2. *The points in quadruples $\left(F^+, F^-, J^+, F_1^-\right)$ and $\left(F^+, F^-, J^-, F_1^+\right)$ are concyclic. The radical axis of the determined circles is the Fermat axis F^+F^-* (Fig. 3).

Proof. From (2.5) we have $KJ^+F^+ \sim KF^-F_1^-$. So $\widehat{KF^+J^+} = \widehat{KF_1^-F^-}$ or $\widehat{F^-F^+J^+} = \widehat{J^+F_1^-F^-}$, from which it follows that F^+, F^-, J^+, F_1^- are concyclic points.

In the same way it is shown that F^+, F^-, J^-, F_1^+ are concyclic points. The circles intersect in F^+ and F^-, so F^+F^- is their radical axis.

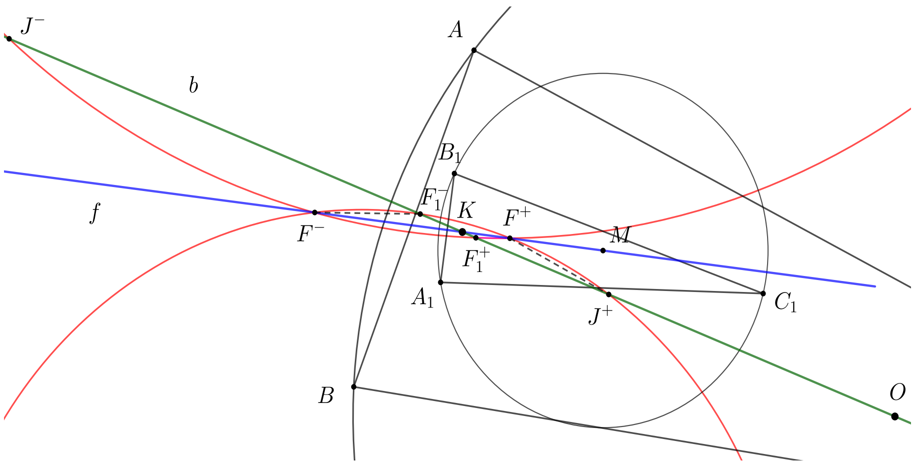

Figure 3

Second proof. To show that F^+, F^-, J^+, F_1^- are concyclic points it is enough to check that

$$KF^+ \cdot KF^- = KJ^+ \cdot KF_1^-.$$

Because $ABC \sim A_1B_1C_1$, we have $KF_1^- = \dfrac{HG}{2R} \cdot KF^-$. Then, the previous equation takes the form

$$2R \cdot KF^+ = KJ^+ \cdot HG. \tag{2.6}$$

This is checked immediately using the formulae (2.2), (2.3), and $HG = \dfrac{\sqrt{f}}{6\Delta}$ and making a simple calculation. Similarly for the points F^+, F^-, J^-, F_1^+.
All statements of the proposition are proven. $\square$

Remark 2.1. The circles $\left(F^+F^-J^+F_1^-\right)$, and $\left(F^+F^-J^-F_1^+\right)$ belong to the intersecting coaxal system determined by Fermat points F^+ and F^-.

In the same manner we will show the following result.

Proposition 2.3. *The points in quadruples* (F^+, J^+, M, M_1) *and* (F^-, J^-, M, M_1) *are concyclic. The radical axis of the determined circles is the Euler line* e_1 *of the orthocentroidal triangle* $A_1B_1C_1$ (Fig. 4).

Proof. We only show that F^+, J^+, M, M_1 are concyclic points. The remaining statements are shown by the same reasoning.
By (2.5), we have: $KJ^+F^+ \sim KF^+F_1^+$. Since $F^+F_1^+ \| MM_1$, we infer that $KF^+F_1^+ \sim KMM_1$. Hence, $KJ^+F^+ \sim KMM_1$ and we obtain: $\widehat{KJ^+F^+} = \widehat{KMM_1}$ or $\widehat{M_1J^+F^+} = \widehat{F^+MM_1}$. Therefore, the points F^+, J^+, M, M_1 are concyclic.

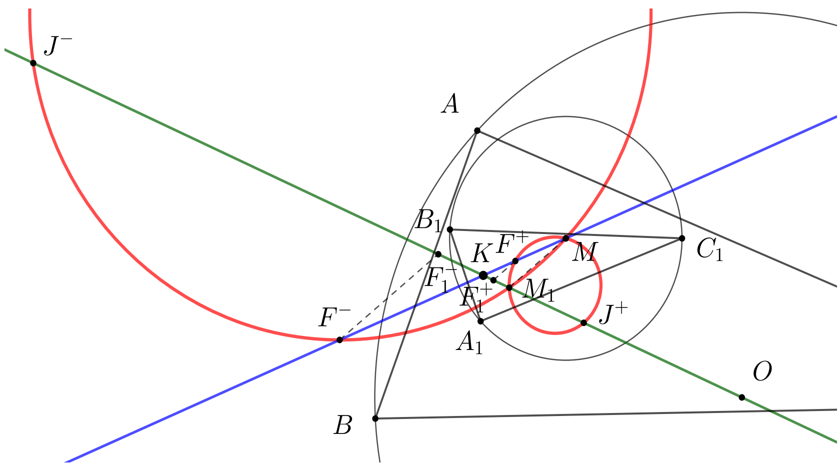

Figure 4

Second proof. Let's check that

$$KF^+ \cdot KM = KM_1 \cdot KJ^+,$$

is true. But $KM_1 = \dfrac{HG}{2R} \cdot KM$. Then, we have to verify that

$$2R \cdot KF^+ = HG \cdot KJ^+,$$

what was done above. So, F^+, J^+, M, M_1 are concyclic points.
The proposition is now completely proven. □

The results of Propositions 2.2 and 2.3 can be easily extended.
Starting from the triangle $A_1B_1C_1$ and doing the same as with ABC at the beginning of the previous section, we get the triangles $A_2B_2C_2$ and $A_2'B_2'C_2'$. Repeating this method we finally get two sequences of triangles $(A_nB_nC_n)_{n\geq 0}$ and $(A_n'B_n'C_n')_{n\geq 0}$ (with $A_0B_0C_0 = ABC$ and $A_0'B_0'C_0' = A'B'C'$).
Our attention is directed to the sequence $(A_nB_nC_n)_{n\geq 0}$. This is the sequence of orthocentroidal triangles starting from the triangle ABC. In [4], Section 3, many properties of this sequence were highlighted. The notations O_n, H_n, G_n, F_n^+, F_n^-, J_n^+, J_n^- relates to the triangles $A_nB_nC_n$ and they are easy to understand (with $O_0 \equiv O, O_1 \equiv M, O_2 \equiv M_1$).

Corollary 2.2. *For any $n \geq 0$, the points in the quadruples $(F_n^+,\ F_n^-, J_n^+,\ F_{n+1}^-)$, $(F_n^+, F_n^-, J_n^-, F_{n+1}^+)$, $(F_n^+, J_n^+, O_{n+1}, O_{n+2})$ and $(F_n^-, J_n^-, O_{n+1}, O_{n+2})$ are concyclic.*

Proof. By Propositions 2.2 and 2.3, the statement is true for $n = 0$. Since any triangle is similar to its orthocentroidal triangle, we have: $A_nB_nC_n \sim ABC$, for any $n \geq 1$. From these, the claims requested result immediately. □

Remark 2.2. *J. Lester* also conjectured in [9] the existence of a circle through the symmedian point, the Feuerbach point, the Lemoine-Clawson point, and the homothetic center of the orthic and the intangent triangles. This statement was validated in [14]. Since $O_n, F_n^+, F_n^-, J_n^+, J_n^-$, $n \geq 0$, are centers of the triangle ABC [4], Propositions 17 and 18, the four quadruple sequences of concyclic points above are inscribed in the same order of things.

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