Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.13, (2024), Issue 1, pp.11-26

CONFORMAL RICCI COLLINEATIONS ON THREE-DIMENSIONAL LORENTZIAN LIE GROUPS

YANLI WANG

ABSTRACT. In this paper, we determine all conformal Ricci collineations associated to the Levi-Civita connection on three-dimensional Lorentzian Lie groups.

1. INTRODUCTION AND MOTIVATIONS

Symmetry is an important part of the study of spacetime, and collineations are the symmetry properties of spacetime. Collineations are the symmetry of geometric quantities, which can be regarded as vector fields that preserve geometric quantities. Preserving geometric quantities can be understood as the Lie derivative of the geometric quantity in the direction of the vector field is 0. In 1969, Katzin first mentioned the concept of Ricci collineations in [2], that is

Definition 1.1. *Let X is a vector field, Ric is the Ricci tensor. If X satisfies*

$$L_X Ric = 0,$$

then X is the Ricci collineation.

If only consider Ricci tensor Ric preserves its conformal class, this case is the classic conformal Ricci collineations given by Duggal in [3], i.e. the vector field X satisfies $L_X Ric = 2\lambda Ric$. In [4], Kühnel considered metric g and Ricci tensor Ric preserve conformal class and defined a more general conformal Ricci collineation by the combination of $L_X g = 2\sigma g$ and $L_X Ric = 2\lambda Ric$. So

Definition 1.2. *Let X is a vector field, Ric is the Ricci tensor. If X satisfies*

$$L_X Ric = 2\lambda g,$$

then X is the conformal Ricci collineation, where λ is a scalar function, up to a constant.

2010 *Mathematics Subject Classification.* Primary 53C40; 53C42.
Key words and phrases. Conformal Ricci collineations; the Levi-Civita connection; three-dimensional Lorentzian Lie groups.

When $\lambda = 0$, conformal Ricci collineation is reduced to Ricci collineation. Ricci Collineations and conformal Ricci collineations have been discussed and determined in more different spacetimes and for various other ricci tensors in [5, 6, 7, 8].
In [12, 14], Calvaruso and Cordero divided three-dimensional Lorentzian Lie groups into $\{G_i\}_{i=1,\cdots,7}$, and geted corresponding Lie algebra $\{\mathfrak{g}\}_{i=1,\cdots,7}$. By this classification, Batat studied algebraic Ricci solitons on three-dimensional Lorentzian Lie groups in [10]. In 2022, Yong Wang computed canonical connections and Kobayashi-Nomizu connections and their Ricci tensor in [11]. In this paper, similar to [11], we compute the Ricci tensor of $\{G_i\}_{i=1,\cdots,7}$ associated to the Levi-Civita connection ∇^L, and then determine all conformal Ricci collineations associated to the Levi-Civita connection on $\{G_i\}_{i=1,\cdots,7}$. Where $\{G_i\}_{i=1,\cdots,4}$ are three-dimensional unimodular Lorentzian Lie groups; $\{G_i\}_{i=5,\cdots,7}$ are three-dimensional non-unimodular Lorentzian Lie groups.
In section 2, we first recall the definition of the Ricci collineation, and generalize to conformal Ricci collineation, then we determine conformal Ricci collineations associated to the Levi-Civita connection on three-dimensional Lorentzian Lie groups.

2. MAIN RESULTS

Calvaruso and Cordero classified three-dimensional Lorentzian Lie groups into $\{G_i\}_{i=1,\cdots,7}$ equipped with a left-invariant Lorentzian metric g in [12, 14](see Theorem 2.1 and Theorem 2.2 in [10]), and geted corresponding Lie algebra $\{\mathfrak{g}\}_{i=1,\cdots,7}$. In [10], Batat studied algebraic Ricci solitons of three-dimensional Lorentzian Lie groups, similar to his calculations, we compute the Ricci tensor of $\{G_i\}_{i=1,\cdots,7}$ associated to the Levi-Civita connection ∇^L. We define the curvature tensor of the Levi-Civita connection ∇^L:

$$R^L(X,Y)Z = \nabla^L_X\nabla^L_Y Z - \nabla^L_Y\nabla^L_X Z - \nabla^L_{[X,Y]}Z. \tag{2.1}$$

The Ricci tensor of (G_i, g) associated to the Levi-Civita connection ∇^L is defined by

$$\rho^L(X,Y) = -g(R^L(X,e_1)Y,e_1) - g(R^L(X,e_2)Y,e_2) + g(R^L(X,e_3)Y,e_3). \tag{2.2}$$

where $g(e_i,e_j) = 0, i \neq j;\ g(e_i,e_i) = 1, i = 1,2;\ g(e_3,e_3) = -1$. Let

$$Ric^L(X,Y) = \frac{\rho^L(X,Y) + \rho^L(Y,X)}{2}. \tag{2.3}$$

we get the symmetric Ricci tensor.
We know that for vector X, Y, V, the Lie derivative of the Ricci tensor Ric^L associated to V can be defined:

$$(L_V Ric^L)(X,Y) := V[Ric^L(X,Y)] - Ric^L([V,X],Y) - Ric^L(X,[V,Y]) \tag{2.4}$$

(G_i, g) admits left-invariant Ricci collineations associated to the Levi-Civita connection ∇^L if and only if it satisfies $(L_V Ric^L)(X,Y) = 0$. Further, conformal Ricci collineations can be discribed:

Theorem 2.1. *(G_i, g) admits conformal Ricci collineations associated to the Levi-Civita connection ∇^L if and only if it satisfies*

$$(L_V Ric^L) = 2\lambda g, \tag{2.5}$$

where $V = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ is a left-invariant vector field and $\lambda_1, \lambda_2, \lambda_3$ are real numbers.

2.1 Conformal Ricci collineations of G_1

By [10], we have the following Lie algebra of G_1 satisfies

$$[e_1,e_2]=\alpha e_1-\beta e_3,\ \ [e_1,e_3]=-\alpha e_1-\beta e_2,\ \ [e_2,e_3]=\beta e_1+\alpha e_2+\alpha e_3,\ \ \alpha\neq 0. \tag{2.6}$$

where e_1, e_2, e_3 is a pseudo-orthonormal basis, with e_3 timelike.

Lemma 2.1. *The Ricci tensor of (G_1, g) associated to the Levi-Civita connection ∇^L is determined by*

$$\begin{aligned}
&Ric^L(e_1,e_1)=-\frac{\beta^2}{2},\ \ Ric^L(e_1,e_2)=-\alpha\beta,\ \ Ric^L(e_1,e_3)=\alpha\beta,\\
&Ric^L(e_2,e_2)=-2\alpha^2-\frac{\beta^2}{2},\ \ Ric^L(e_2,e_3)=2\alpha^2,\ \ Ric^L(e_3,e_3)=-2\alpha^2+\frac{\beta^2}{2}.
\end{aligned} \tag{2.7}$$

Let $V=\lambda_1e_1+\lambda_2e_2+\lambda_3e_3$ is a left-invariant vector field, we can get the Lie derivative of the Ricci tensor Ric^L associated to V.

Lemma 2.2.

$$\begin{aligned}
(L_VRic^L)(e_1,e_1)&=-3\alpha\beta^2\lambda_2+3\alpha\beta^2\lambda_3,\\
(L_VRic^L)(e_1,e_2)&=\frac{3}{2}\alpha\beta^2\lambda_1-3\alpha^2\beta\lambda_2+3\alpha^2\beta\lambda_3,\\
(L_VRic^L)(e_1,e_3)&=-\frac{3}{2}\alpha\beta^2\lambda_1+3\alpha^2\beta\lambda_2+\alpha^2\beta\lambda_3,\\
(L_VRic^L)(e_2,e_2)&=6\alpha^2\beta\lambda_1-3\alpha\beta^2\lambda_3,\\
(L_VRic^L)(e_2,e_3)&=-6\alpha^2\beta\lambda_1+\frac{3}{2}\alpha\beta^2\lambda_2+\frac{3}{2}\alpha\beta^2\lambda_3,\\
(L_VRic^L)(e_3,e_3)&=6\alpha^2\beta\lambda_1-3\alpha\beta^2\lambda_2.
\end{aligned} \tag{2.8}$$

Then, if V is a conformal Ricci collineation associated to the Levi-Civita connection, by Lemma 2.2 and Theorem 2.1, we have the following equations:

$$\begin{cases}
-3\alpha\beta^2\lambda_2+3\alpha\beta^2\lambda_3=2\lambda\\
\frac{3}{2}\alpha\beta^2\lambda_1-3\alpha^2\beta\lambda_2+3\alpha^2\beta\lambda_3=0\\
-\frac{3}{2}\alpha\beta^2\lambda_1+3\alpha^2\beta\lambda_2+\alpha^2\beta\lambda_3=0\\
6\alpha^2\beta\lambda_1-3\alpha\beta^2\lambda_3=2\lambda\\
-6\alpha^2\beta\lambda_1+\frac{3}{2}\alpha\beta^2\lambda_2+\frac{3}{2}\alpha\beta^2\lambda_3=0\\
6\alpha^2\beta\lambda_1-3\alpha\beta^2\lambda_2=-2\lambda
\end{cases} \tag{2.9}$$

By solving (2.9) , we get

Theorem 2.2. *(G_1, g, V) admits conformal Ricci collineations associated to the Levi-Civita connection if and only if: $\alpha\neq 0, \beta=0, \lambda=0$, $\mathscr{V}_{\mathscr{R}C}=<e_1,e_2,e_3>$, where $\mathscr{V}_{\mathscr{R}C}$ is the vector space of conformal Ricci collineations on (G_1, g, V).*

Proof. *We know that* $\alpha \neq 0$. *By the second and third equations, we get* $\alpha^2\beta\lambda_3 = 0$, *then* $\beta\lambda_3 = 0$, *so*
case 1) *If* $\lambda_3 = 0$, *by* (2.9),

$$\begin{cases} -3\alpha\beta^2\lambda_2 = 2\lambda \\ \frac{3}{2}\alpha\beta^2\lambda_1 - 3\alpha^2\beta\lambda_2 = 0 \\ 6\alpha^2\beta\lambda_1 = 2\lambda \\ -6\alpha^2\beta\lambda_1 + \frac{3}{2}\alpha\beta^2\lambda_2 = 0 \\ 6\alpha^2\beta\lambda_1 - 3\alpha\beta^2\lambda_2 = -2\lambda \end{cases} \tag{2.10}$$

then we get $\lambda = \beta = 0$.
case 2) *If* $\lambda_3 \neq 0$, *then we get* $\beta = 0$. (2.9) *holds.*
So we have $\alpha \neq 0, \beta = 0, \lambda = 0$.

2.2 Conformal Ricci collineations of G_2

By [10], we have the following Lie algebra of G_2 satisfies

$$[e_1, e_2] = \gamma e_2 - \beta e_3, \ \ [e_1, e_3] = -\beta e_2 - \gamma e_3, \ \ [e_2, e_3] = \alpha e_1, \ \ \gamma \neq 0. \tag{2.11}$$

where e_1, e_2, e_3 is a pseudo-orthonormal basis, with e_3 timelike.

Lemma 2.3. *The Ricci tensor of* (G_2, g) *associated to the Levi-Civita connection* ∇^L *is determined by*

$$Ric^L(e_1, e_1) = -\frac{\alpha^2}{2} - 2\gamma^2, \quad Ric^L(e_1, e_2) = 0, \quad Ric^L(e_1, e_3) = 0, \tag{2.12}$$
$$Ric^L(e_2, e_2) = \frac{\alpha^2}{2} - \alpha\beta, \quad Ric^L(e_2, e_3) = -\alpha\gamma + 2\beta\gamma, \quad Ric^L(e_3, e_3) = -\frac{\alpha^2}{2} + \alpha\beta.$$

Similar to the previous lemma, we can get the Lie derivative of the Ricci tensor Ric^L associated to V.

Lemma 2.4.

$$(L_V Ric^L)(e_1, e_1) = 0, \tag{2.13}$$
$$(L_V Ric^L)(e_1, e_2) = (\frac{1}{2}\alpha^2\gamma - 2\beta^2\gamma)\lambda_2 + (-\frac{1}{2}\alpha^3 - \alpha\gamma^2 - 2\beta\gamma^2 + \alpha\beta^2 - \frac{1}{2}\alpha^2\beta)\lambda_3,$$
$$(L_V Ric^L)(e_1, e_3) = (\frac{1}{2}\alpha^3 + \alpha\gamma^2 + 2\beta\gamma^2 - \alpha\beta^2 + \frac{1}{2}\alpha^2\beta)\lambda_2 + (\frac{1}{2}\alpha^2\gamma - 2\beta^2\gamma)\lambda_3,$$
$$(L_V Ric^L)(e_2, e_2) = (-\alpha^2\gamma + 4\beta^2\gamma)\lambda_1,$$
$$(L_V Ric^L)(e_2, e_3) = 0,$$
$$(L_V Ric^L)(e_3, e_3) = (-\alpha^2\gamma + 4\beta^2\gamma)\lambda_1.$$

Then, if V is a conformal Ricci collineation associated to the Levi-Civita connection, by Lemma 2.4 and Theorem 2.1, we have the following equations:

$$\begin{cases} 0 = 2\lambda \\ (\frac{1}{2}\alpha^2\gamma - 2\beta^2\gamma)\lambda_2 + (-\frac{1}{2}\alpha^3 - \alpha\gamma^2 - 2\beta\gamma^2 + \alpha\beta^2 - \frac{1}{2}\alpha^2\beta)\lambda_3 = 0 \\ (\frac{1}{2}\alpha^3 + \alpha\gamma^2 + 2\beta\gamma^2 - \alpha\beta^2 + \frac{1}{2}\alpha^2\beta)\lambda_2 + (\frac{1}{2}\alpha^2\gamma - 2\beta^2\gamma)\lambda_3 = 0 \\ (-\alpha^2\gamma + 4\beta^2\gamma)\lambda_1 = 2\lambda \end{cases} \tag{2.14}$$

By solving (2.14) , we get

Theorem 2.3. *(G_2, g, V) admits conformal Ricci collineations associated to the Levi-Civita connection if and only if one of the following holds:*
(1) $\lambda = 0, \alpha = 2\beta, \gamma \neq 0, \gamma \neq 0, \beta \neq 0$, $\mathscr{V}_{\mathscr{R}C} =< e_1 >$,
(2) $\lambda = 0, \alpha = -2\beta, \gamma \neq 0$, $\mathscr{V}_{\mathscr{R}C} =< e_1, e_2, e_3 >$,
where $\mathscr{V}_{\mathscr{R}C}$ is the vector space of conformal Ricci collineations on (G_2, g, V).

Proof. *We know that $\gamma \neq 0, \lambda = 0$,*
case 1) *If $\frac{1}{2}\alpha^2\gamma - 2\beta^2\gamma = 0$, i.e. $\alpha^2 = 4\beta^2$, then $(-\frac{1}{2}\alpha^3 - \alpha\gamma^2 - 2\beta\gamma^2 + \alpha\beta^2 - \frac{1}{2}\alpha^2\beta)\lambda_3 = 0$.*
case 1-1) *If $\alpha = 2\beta$, then we have $\beta(\beta^2 + \gamma^2)\lambda_3 = 0$, i.e. $\beta\lambda_3 = 0$. When $\beta \neq 0$, we have $\lambda_2 = \lambda_3 = 0$, we get* (1). *When $\beta = 0$, we have $\alpha = 0$,* (2.14) *holds. This situation falls into* (2).
case 1-2) *If $\alpha = -2\beta$,* (2.14) *holds. This situation falls into* (2).
case 2) *If $\frac{1}{2}\alpha^2\gamma - 2\beta^2\gamma \neq 0$, i.e. $\alpha^2 \neq 4\beta^2$, then $\lambda_2 = \frac{b}{a}\lambda_3$, by the third equation we have*

$$b\frac{b}{a}\lambda_3 + a\lambda_3 = 0,$$

i.e. $(b^2 + a^2)\lambda_3 = 0$, we have $a = b = 0$, where $b = \frac{1}{2}\alpha^3 + \alpha\gamma^2 + 2\beta\gamma^2 - \alpha\beta^2 + \frac{1}{2}\alpha^2\beta, a = \frac{1}{2}\alpha^2\gamma - 2\beta^2\gamma$. This is a contradiction.

2.3 Conformal Ricci collineations of G_3

By [10], we have the following Lie algebra of G_3 satisfies

$$[e_1, e_2] = -\gamma e_3, \ \ [e_1, e_3] = -\beta e_2, \ \ [e_2, e_3] = \alpha e_1. \tag{2.15}$$

where e_1, e_2, e_3 is a pseudo-orthonormal basis, with e_3 timelike.

Lemma 2.5. *The Ricci tensor of (G_3, g) associated to the Levi-Civita connection ∇^L is determined by*

$$Ric^L(e_1, e_1) = -\alpha a_1 - \beta a_2 - \gamma a_3, \ \ Ric^L(e_1, e_2) = 0, \ \ Ric^L(e_1, e_3) = 0, \tag{2.16}$$
$$Ric^L(e_2, e_2) = \alpha a_1 + \beta a_2 - \gamma a_3, \ \ Ric^L(e_2, e_3) = 0, \ \ Ric^L(e_3, e_3) = -\alpha a_1 + \beta a_2 - \gamma a_3.$$

Similar to the previous lemma, we can get the Lie derivative of the Ricci tensor Ric^L associated to V.

Lemma 2.6.

$$(L_V Ric^L)(e_1, e_1) = 0, \tag{2.17}$$
$$(L_V Ric^L)(e_1, e_2) = (\alpha - \beta)[\gamma^2 - (\alpha + \beta)^2]\lambda_3,$$
$$(L_V Ric^L)(e_1, e_3) = (\alpha - \gamma)[\beta^2 - (\alpha + \gamma)^2]\lambda_2,$$
$$(L_V Ric^L)(e_2, e_2) = 0,$$
$$(L_V Ric^L)(e_2, e_3) = (\beta - \gamma)[\alpha^2 - (\beta + \gamma)^2]\lambda_1,$$
$$(L_V Ric^L)(e_3, e_3) = 0.$$

Then, if V is a conformal Ricci collineation associated to the Levi-Civita connection, by Lemma 2.6 and Theorem 2.1, we have the following equations:

$$\begin{cases} 0 = 2\lambda \\ (\alpha - \beta)[\gamma^2 - (\alpha + \beta)^2]\lambda_3 = 0 \\ (\alpha - \gamma)[\beta^2 - (\alpha + \gamma)^2]\lambda_2 = 0 \\ (\beta - \gamma)[\alpha^2 - (\beta + \gamma)^2]\lambda_1 = 0 \end{cases} \tag{2.18}$$

By solving (2.18) , we get

Theorem 2.4. *(G_3, g, V) admits conformal Ricci collineations associated to the Levi-Civita connection if and only if one of the following holds:*
(1) $\alpha + \beta + \gamma = 0, \lambda = 0, \mathscr{V}_{\mathscr{R}C} =< e_1, e_2, e_3 >$,
(2) $\alpha + \beta + \gamma \neq 0, \alpha = \beta = \gamma \neq 0, \lambda = 0, \mathscr{V}_{\mathscr{R}C} =< e_1, e_2, e_3 >$,
(3) $\alpha = 0, \beta = \gamma \neq 0, \lambda = 0, \mathscr{V}_{\mathscr{R}C} =< e_1, e_2, e_3 >$,
(4) $\beta = 0, \alpha = \gamma \neq 0, \lambda = 0, \mathscr{V}_{\mathscr{R}C} =< e_1, e_2, e_3 >$,
(5) $\gamma = 0, \alpha = \beta \neq 0, \lambda = 0, \mathscr{V}_{\mathscr{R}C} =< e_1, e_2, e_3 >$,
(6) $\alpha + \beta + \gamma \neq 0, \alpha \neq 0, \beta = \gamma, \alpha \neq \gamma, \lambda = 0, \mathscr{V}_{\mathscr{R}C} =< e_1 >$,
(7) $\alpha + \beta + \gamma \neq 0, \beta \neq 0, \alpha = \gamma, \alpha \neq \beta, \lambda = 0, \mathscr{V}_{\mathscr{R}C} =< e_2 >$,
(8) $\alpha + \beta + \gamma \neq 0, \gamma \neq 0, \alpha = \beta, \beta \neq \gamma, \lambda = 0, \mathscr{V}_{\mathscr{R}C} =< e_3 >$,
(9) $\alpha - \beta - \gamma = 0, \alpha \neq 0, \beta \neq 0, \gamma \neq 0, \beta \neq \gamma, \lambda = 0, \mathscr{V}_{\mathscr{R}C} =< e_1 >$,
(10) $-\alpha + \beta - \gamma = 0, \alpha \neq 0, \beta \neq 0, \gamma \neq 0, \alpha \neq \gamma, \lambda = 0, \mathscr{V}_{\mathscr{R}C} =< e_2 >$,
(11) $\gamma - \alpha - \beta = 0, \alpha \neq 0, \beta \neq 0, \gamma \neq 0, \alpha \neq \beta, \lambda = 0, \mathscr{V}_{\mathscr{R}C} =< e_3 >$,

where $\mathscr{V}_{\mathscr{R}C}$ is the vector space of conformal Ricci collineations on (G_3, g, V).

Proof. *We know that $\lambda = 0$. By the fourth equation, we have $(\beta - \gamma)(\alpha - \beta - \gamma)(\alpha + \beta + \gamma)\lambda_1 = 0$.*
case 1) *If $\alpha + \beta + \gamma = 0$,* (2.18) *holds. We get* (1).
case 2) *If $\alpha + \beta + \gamma \neq 0$, $\beta - \gamma = 0$, by* (2.18),

$$\begin{cases} \alpha(\alpha - \gamma)\lambda_3 = 0 \\ \alpha(\alpha - \gamma)\lambda_2 = 0 \end{cases} \tag{2.19}$$

case 2-1) *If $\alpha = 0$, then $\beta = \gamma \neq 0$,* (2.19) *holds. We get* (3).
case 2-2) *If $\alpha \neq 0$, then $\alpha = \gamma$. We get* (2).
case 2-3) *If $\alpha \neq 0$, $\alpha \neq \gamma$, then $\lambda_2 = \lambda_3 = 0$. We get* (6).

case 3) *If* $\alpha+\beta+\gamma\neq 0$, $\beta-\gamma\neq 0$, *then* $\alpha-\beta-\gamma=0$. *By* (2.18),

$$\begin{cases}(\alpha-\beta)\beta\lambda_3=0\\ (\alpha-\gamma)\gamma\lambda_2=0\end{cases} \tag{2.20}$$

case 3-1) *If* $\beta=0$, *then* $\alpha=\gamma\neq 0$. (2.20) *holds. We get* (4).
case 3-2) *If* $\beta\neq 0$, $\alpha-\beta=0$, *then* $\gamma=0$. (2.20) *holds. We get* (5).
case 3-3) *If* $\beta\neq 0$, $\alpha-\beta\neq 0$, *we have* $\alpha\neq 0,\beta\neq 0,\gamma\neq 0$, *then* $\lambda_2=\lambda_3=0$. *We get* (9).
case 4) *If* $\alpha+\beta+\gamma\neq 0$, $\beta-\gamma\neq 0$, $\alpha-\beta-\gamma\neq 0$, *then* $\lambda_1=0$. *By* (2.18),

$$\begin{cases}(\alpha-\beta)(\gamma-\alpha-\beta)\lambda_3=0\\ (\alpha-\gamma)(-\alpha+\beta-\gamma)\lambda_2=0\end{cases} \tag{2.21}$$

case 4-1) *If* $\alpha=\beta$, *then* $\gamma\neq 0$. *By* (2.21) *we have* $(\alpha-\gamma)\gamma\lambda_2=0$, *then* $\lambda_2=0$. *We get* (8).
case 4-2) *If* $\alpha\neq\beta$, $\gamma-\alpha-\beta=0$, *then* $\alpha\neq 0,\gamma\neq 0,\beta\neq 0$. *By* (2.) *we have* $(\alpha-\gamma)\beta\lambda_2=0$, *then* $\lambda_2=0$. *We get* (11).
case 4-3) *If* $\alpha\neq\beta$, $\gamma-\alpha-\beta\neq 0$, *then* $\lambda_3=0$. *If* $\alpha=\gamma$, *then* $\beta\neq 0$. (2.21) *holds. We get* (7). *If* $\alpha\neq\gamma$, *then* $-\alpha+\beta-\gamma=0$, *we have* $\alpha\neq 0,\beta\neq 0,\gamma\neq 0$. *We get* (10).

2.4 Conformal Ricci collineations of G_4

By [10], we have the following Lie algebra of G_4 satisfies

$$[e_1,e_2]=-e_2+(2\eta-\beta)e_3,\ \ \eta=1\text{ or }-1,\ \ [e_1,e_3]=-\beta e_2+e_3,\ \ [e_2,e_3]=\alpha e_1. \tag{2.22}$$

where e_1,e_2,e_3 is a pseudo-orthonormal basis, with e_3 timelike.

Lemma 2.7. *The Ricci tensor of* (G_4,g) *associated to the Levi-Civita connection* ∇^L *is determined by*

$$\begin{aligned}&Ric^L(e_1,e_1)=-\frac{\alpha^2}{2},\quad Ric^L(e_1,e_2)=0,\quad Ric^L(e_1,e_3)=0,\\ &Ric^L(e_2,e_2)=\frac{\alpha^2}{2}+2\eta(\alpha-\beta)-\alpha\beta+2,\quad Ric^L(e_2,e_3)=\alpha+2\eta-\beta,\\ &Ric^L(e_3,e_3)=-\frac{\alpha^2}{2}-2\beta\eta+\alpha\beta+2.\end{aligned} \tag{2.23}$$

Similar to the previous lemma, we can get the Lie derivative of the Ricci tensor Ric^L associated to V.

Lemma 2.8.

$$
\begin{aligned}
(L_V Ric^L)(e_1,e_1) &= 0, \qquad (2.24)\\
(L_V Ric^L)(e_1,e_2) &= (-\frac{\alpha^2}{2}+\beta^2-2\beta\eta+2)\lambda_2\\
&\quad +(-\frac{\alpha^3}{2}-\frac{\alpha^2\beta}{2}+\alpha\beta^2-2\alpha\beta\eta+2\beta^2\eta+\alpha-3\beta+2\eta)\lambda_3,\\
(L_V Ric^L)(e_1,e_3) &= (\frac{\alpha^3}{2}+\frac{\alpha^2\beta}{2}-\alpha\beta^2+2\alpha\beta\eta+2\beta^2\eta-\alpha^2\eta-\alpha-5\beta+2\eta)\lambda_2\\
&\quad +(-\frac{\alpha^2}{2}+\beta^2-4\beta\eta+2)\lambda_3,\\
(L_V Ric^L)(e_2,e_2) &= (\alpha^2-2\beta^2+4\beta\eta+4)\lambda_1,\\
(L_V Ric^L)(e_2,e_3) &= \eta(\alpha-2\beta+2\eta)(\alpha+2\beta-2\eta)\lambda_1,\\
(L_V Ric^L)(e_3,e_3) &= (\alpha^2-2\beta^2+8\beta\eta-4)\lambda_1.
\end{aligned}
$$

Then, if V is a conformal Ricci collineation associated to the Levi-Civita connection, by Lemma 2.8 and Theorem 2.1, we have the following equations:

$$
\begin{cases}
0=2\lambda\\
(-\frac{\alpha^2}{2}+\beta^2-2\beta\eta+2)\lambda_2\\
\qquad +(-\frac{\alpha^3}{2}-\frac{\alpha^2\beta}{2}+\alpha\beta^2-2\alpha\beta\eta+2\beta^2\eta+\alpha-3\beta+2\eta)\lambda_3=0\\
(\frac{\alpha^3}{2}+\frac{\alpha^2\beta}{2}-\alpha\beta^2+2\alpha\beta\eta+2\beta^2\eta-\alpha^2\eta-\alpha-5\beta+2\eta)\lambda_2\\
\qquad +(-\frac{\alpha^2}{2}+\beta^2-4\beta\eta+2)\lambda_3=0\\
(\alpha^2-2\beta^2+4\beta\eta+4)\lambda_1=2\lambda\\
\eta(\alpha-2\beta+2\eta)(\alpha+2\beta-2\eta)\lambda_1=0\\
(\alpha^2-2\beta^2+8\beta\eta-4)\lambda_1=-2\lambda
\end{cases} \qquad (2.25)
$$

By solving (2.25) , we get

Theorem 2.5. *(G_4,g,V) admits conformal Ricci collineations associated to the Levi-Civita connection if and only if one of the following holds:*
(1) $\eta=1$ or -1, $\lambda=0$, $\alpha=2\eta$, $\beta=0$, $\mathscr{V}_{\mathscr{R}C}=<e_2,e_3>$,
(2) $\eta=1$ or -1, $\lambda=0$, $m=0$, $n=0$, $\alpha^2-4\neq 0$, $\mathscr{V}_{\mathscr{R}C}=<\frac{2\beta\eta}{\alpha^2-4}e_2+e_3>$,
(3) $\eta=1$ or -1, $\lambda=0$, $m=0$, $n\neq 0$, $\alpha^2-4-n=0$, $\mathscr{V}_{\mathscr{R}C}=<e_2>$.
(4) $\eta=1$ or -1, $\lambda=0$, $m\neq 0$, $m(m-2\beta)-n(4\beta^2-\alpha^2-8\beta+4-n)=0$,
$\mathscr{V}_{\mathscr{R}C}=<-\frac{n}{m}e_2+e_3>$.
where $m=-\frac{\alpha^2}{2}+\beta^2-2\beta\eta+2$, $n=-\frac{\alpha^3}{2}-\frac{\alpha^2\beta}{2}+\alpha\beta^2-2\alpha\beta\eta+2\beta^2\eta+\alpha-3\beta+2\eta$, $\mathscr{V}_{\mathscr{R}C}$ *is the vector space of conformal Ricci collineations on (G_4,g,V).*

Proof. *We know that* $\eta=1$ or -1. *By the fifth equation, we have* $(\alpha-2\beta+2\eta)(\alpha+2\beta-2\eta)\lambda_1=0$.

Then by the sixth equation, $2\beta^2\lambda_1 = 0$. *If* $\beta = 0$, *then* $(\alpha^2 + 4)\lambda_1 = 0$, *i.e.* $\lambda_1 = 0$; *otherwise* $\beta \neq 0$, *then* $\lambda_1 = 0$. *So* $\lambda = \lambda_1 = 0$, *let* $m = -\frac{\alpha^2}{2} + \beta^2 - 2\beta\eta + 2, n = -\frac{\alpha^3}{2} - \frac{\alpha^2\beta}{2} + \alpha\beta^2 - 2\alpha\beta\eta + 2\beta^2\eta + \alpha - 3\beta + 2\eta$, (2.25) *can be simplified to*

$$\begin{cases} m\lambda_2 + n\lambda_3 = 0 \\ (4\beta^2\eta - \alpha^2\eta - 8\beta + 4\eta - n)\lambda_2 + (m - 2\beta\eta)\lambda_3 = 0 \end{cases} \tag{2.26}$$

case 1) *If* $\eta = 1$, *let* $m_1 = m$, $n_1 = n$, *by* (2.) *we have*

$$\begin{cases} m_1\lambda_2 + n_1\lambda_3 = 0 \\ (4\beta^2 - \alpha^2 - 8\beta + 4 - n_1)\lambda_2 + (m_1 - 2\beta)\lambda_3 = 0 \end{cases} \tag{2.27}$$

case 1-1) *If* $m_1 = 0$, *then* $n_1\lambda_3 = 0$.
case 1-1-1) *If* $n_1 = 0$, *by* $m_1 = -\frac{\alpha^2}{2} + (\beta - 1)^2 + 1 = 0$, (2.27) *can be simplified to*

$$(-\alpha^2 + 4(\beta - 1)^2)\lambda_2 - 2\beta\lambda_3 = (\alpha^2 - 4)\lambda_2 - 2\beta\lambda_3 = 0.$$

If $\alpha^2 - 4 = 0$, *then* $\beta = 0$ or 2, $2\beta\lambda_3 = 0$. *When* $\beta = 0$, *by* $n_1 = 0$ *we have* $\alpha = 2$, *this situation falls into* (1); *otherwise* $\beta = 2$, *by* $n_1 = 0$, *we get a contradiction.*
If $\alpha^2 - 4 \neq 0$, *then* $\lambda_2 = \frac{2\beta}{\alpha^2 - 4}\lambda_3$. *This situation falls into* (2).
case 1-1-2) *If* $n_1 \neq 0$, *then* $\lambda_3 = 0$. *By* (2.27) *we have*

$$(\alpha^2 - 4 - n_1)\lambda_2 = 0,$$

then $\alpha^2 - 4 - n_1 = 0$. *This situation falls into* (3).
case 1-2) *If* $m_1 \neq 0$, *then* $\lambda_2 = -\frac{n_1}{m_1}\lambda_3$, $m_1(m_1 - 2\beta) - n_1(4\beta^2 - \alpha^2 - 8\beta + 4 - n_1) = 0$. *This situation falls into* (4).
case 2) *If* $\eta = -1$, *similar to* $\eta = 1$, $\lambda = \lambda_1 = 0$, *let* $m_2 = m$, $n_2 = n$, *we have*
case 2-1) $\alpha = -2$, $\beta = 0$, *this situation falls into* (1).
case 2-2) $m_2 = 0$, $n_2 = 0$, $\alpha^2 - 4 \neq 0$, $\lambda_2 = -\frac{2\beta}{\alpha^2 - 4}\lambda_3$. *This situation falls into* (2).
case 2-3) $m_2 = 0$, $n_2 \neq 0$, $\alpha^2 - 4 - n_2 = 0$, $\lambda_3 = 0$. *This situation falls into* (3).
case 2-4) $m_2 \neq 0$, $m_2(m_2 - 2\beta) - n_2(4\beta^2 - \alpha^2 - 8\beta + 4 - n_2) = 0$, $\lambda_2 = -\frac{n_2}{m_2}\lambda_3$. *This situation falls into* (4).

2.5 Conformal Ricci collineations of G_5

By [10], we have the following Lie algebra of G_5 satisfies

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \alpha e_1 + \beta e_2, \quad [e_2, e_3] = \gamma e_1 + \delta e_2, \quad \alpha + \delta \neq 0, \quad \alpha\gamma + \beta\delta = 0. \tag{2.28}$$

where e_1, e_2, e_3 is a pseudo-orthonormal basis, with e_3 timelike.

Lemma 2.9. *The Ricci tensor of (G_5, g) associated to the Levi-Civita connection ∇^L is determined by*

$$Ric^L(e_1,e_1) = \alpha^2 + \alpha\delta + \frac{\beta^2-\gamma^2}{2}, \quad Ric^L(e_1,e_2) = 0, \quad Ric^L(e_1,e_3) = 0, \tag{2.29}$$
$$Ric^L(e_2,e_2) = \delta^2 + \alpha\delta - \frac{\beta^2-\gamma^2}{2}, \quad Ric^L(e_2,e_3) = 0, \quad Ric^L(e_3,e_3) = -(\alpha^2+\delta^2+\frac{(\beta+\gamma)^2}{2}).$$

Similar to the previous lemma, we can get the Lie derivative of the Ricci tensor Ric^L associated to V.

Lemma 2.10.

$$\begin{aligned}
(L_V Ric^L)(e_1,e_1) &= 2\alpha(\alpha^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2})\lambda_3, \\
(L_V Ric^L)(e_1,e_2) &= [\beta(\delta^2+\alpha\delta-\frac{\beta^2-\gamma^2}{2}) + \gamma(\alpha^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2})]\lambda_3, \\
(L_V Ric^L)(e_1,e_3) &= -\alpha(\alpha^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2})\lambda_1 - \gamma(\alpha^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2})\lambda_2, \\
(L_V Ric^L)(e_2,e_2) &= 2\delta(\delta^2+\alpha\delta-\frac{\beta^2-\gamma^2}{2})\lambda_3, \\
(L_V Ric^L)(e_2,e_3) &= -\beta(\delta^2+\alpha\delta-\frac{\beta^2-\gamma^2}{2})\lambda_1 - \delta(\delta^2+\alpha\delta-\frac{\beta^2-\gamma^2}{2})\lambda_2, \\
(L_V Ric^L)(e_3,e_3) &= 0.
\end{aligned} \tag{2.30}$$

Then, if V is a conformal Ricci collineation associated to the Levi-Civita connection, by Lemma 2.10 and Theorem 2.1, we have the following equations:

$$\begin{cases}
2\alpha(\alpha^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2})\lambda_3 = 2\lambda \\
[\beta(\delta^2+\alpha\delta-\frac{\beta^2-\gamma^2}{2}) + \gamma(\alpha^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2})]\lambda_3 = 0 \\
(\alpha^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2})(\alpha\lambda_1+\gamma\lambda_2) = 0 \\
2\delta(\delta^2+\alpha\delta-\frac{\beta^2-\gamma^2}{2})\lambda_3 = 2\lambda \\
(\delta^2+\alpha\delta-\frac{\beta^2-\gamma^2}{2})(\beta\lambda_1+\delta\lambda_2) = 0 \\
0 = -2\lambda
\end{cases} \tag{2.31}$$

By solving (2.31) , we get

Theorem 2.6. *(G_5, g, V) admits conformal Ricci collineations associated to the Levi-Civita connection if and only if one of the following holds:*

(1) $\alpha^2+\alpha\delta-\frac{\gamma^2}{2} = 0, \beta = 0, \alpha+\delta \neq 0, \alpha\gamma = 0, \delta \neq 0, \lambda = 0, \ \mathscr{V}_{\mathscr{R}C} =< e_1 >,$

(2) $\alpha^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2} = 0, \beta \neq 0, \alpha+\delta \neq 0, \alpha\gamma+\beta\delta = 0, \lambda = 0, \ \mathscr{V}_{\mathscr{R}C} =< -\frac{\delta}{\beta}e_1 + e_2 >,$

(3) $\alpha = 0, \beta = 0, \delta \neq 0, \gamma \neq 0, \lambda = 0, \ \mathscr{V}_{\mathscr{R}C} =< e_1 >,$

(4) $\delta^2+\alpha\delta-\frac{\beta^2-\gamma^2}{2} = 0, \alpha \neq 0, \alpha+\delta \neq 0, \alpha\gamma+\beta\delta = 0, \lambda = 0, \ \mathscr{V}_{\mathscr{R}C} =< -\frac{\gamma}{\alpha}e_1 + e_2 >,$

(5) $\alpha^2 + \alpha\delta + \frac{\beta^2-\gamma^2}{2} \neq 0, \delta^2 + \alpha\delta - \frac{\beta^2-\gamma^2}{2} \neq 0, \alpha \neq 0, \alpha+\delta \neq 0, \alpha\gamma+\beta\delta = 0, \alpha\delta - \beta\gamma = 0, \lambda = 0,$ $\mathscr{V}_{\mathscr{R}C} = < -\frac{\gamma}{\alpha}e_1 + e_2 >$.
where $\mathscr{V}_{\mathscr{R}C}$ *is the vector space of conformal Ricci collineations on* (G_5, g, V).

Proof. *We know that* $\lambda = 0$, $\alpha+\delta \neq 0$, $\alpha\gamma+\beta\delta = 0$. *By the fourth equation, we have* $\delta(\delta^2+\alpha\delta-\frac{\beta^2-\gamma^2}{2})\lambda_3 = 0$.
case 1) *If* $\lambda_3 = 0$, (2.31) *can be simplified to*

$$\begin{cases} (\alpha^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2})(\alpha\lambda_1+\gamma\lambda_2) = 0 \\ (\delta^2+\alpha\delta-\frac{\beta^2-\gamma^2}{2})(\beta\lambda_1+\delta\lambda_2) = 0 \end{cases} \tag{2.32}$$

case 1-1) *If* $\alpha^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2} = 0$, (2.32) *can be simplified to*

$$(\alpha+\delta)^2(\beta\lambda_1+\delta\lambda_2) = 0,$$

i.e. $\beta\lambda_1+\delta\lambda_2 = 0$.
case 1-1-1) *If* $\beta = 0$, *then* $\delta\lambda_2 = 0$. *When* $\delta = 0$, *by* $\alpha+\delta = 0$, $\alpha\gamma+\beta\delta = 0$, *we have* $\alpha \neq 0$, $\gamma = 0$, $\alpha^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2} = \alpha^2 \neq 0$, *we get a contradiction; otherwise* $\delta \neq 0$, $\lambda_2 = 0$. *we get* (1).
case 1-1-2) *If* $\beta \neq 0$, *then* $\lambda_1 = -\frac{\delta}{\beta}\lambda_2$. *We get* (2).
case 1-2) *If* $\alpha^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2} \neq 0$, *then* $\alpha\lambda_1+\gamma\lambda_2 = 0$.
case 1-2-1) *If* $\alpha = 0$, *by* $\alpha+\delta = 0$, $\alpha\gamma+\beta\delta = 0$, *we have* $\delta \neq 0$, $\beta = 0$, $\gamma \neq 0$, *then* $\lambda_2 = 0$. *We get* (3).
case 1-2-2) *If* $\alpha \neq 0$, *then* $\lambda_1 = -\frac{\gamma}{\alpha}\lambda_2$. (2.32) *can be simplified to*

$$(\delta^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2})(-\beta\gamma+\alpha\delta)\lambda_2 = 0.$$

If $\delta^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2} = 0$, (2.32) *holds. we get* (4).
If $\delta^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2} \neq 0$, *then* $\alpha\delta-\beta\gamma = 0$. *We get* (5).
case 2) *If* $\lambda_3 \neq 0$, $\delta = 0$, *we have* $\alpha \neq 0$, $\gamma = 0$, *by* (2.31) *we get* $\beta = 0$, $\alpha = 0$. *This is a contradiction.*
So $\delta \neq 0$, $\delta^2+\alpha\delta-\frac{\beta^2-\gamma^2}{2} = 0$, *then* $\alpha^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2} \neq 0$. *By* (2.31) *we have* $\alpha = \gamma = 0$, *by* $\alpha\gamma-\beta\delta = 0$, *we have* $\beta = 0$, *so* $\delta^2+\alpha\delta+\frac{\beta^2-\gamma^2}{2} \neq 0$, *we get a contradiction.*

2.6 Conformal Ricci collineations of G_6

By [10], we have the following Lie algebra of G_6 satisfies

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \ \ [e_1, e_3] = \gamma e_2 + \delta e_3, \ \ [e_2, e_3] = 0, \ \ \alpha + \delta \neq 0, \ \ \alpha\gamma - \beta\delta = 0. \quad (2.33)$$

where e_1, e_2, e_3 is a pseudo-orthonormal basis, with e_3 timelike.

Lemma 2.11. *The Ricci tensor of (G_6, g) associated to the Levi-Civita connection ∇^L is determined by*

$$Ric^L(e_1, e_1) = -\alpha^2 - \delta^2 + \frac{(\beta-\gamma)^2}{2}, \ \ Ric^L(e_1, e_2) = 0, \ \ Ric^L(e_1, e_3) = 0, \quad (2.34)$$
$$Ric^L(e_2, e_2) = -\alpha^2 - \alpha\delta + \frac{\beta^2-\gamma^2}{2}, \ \ Ric^L(e_2, e_3) = 0, \ \ Ric^L(e_3, e_3) = \delta^2 + \alpha\delta + \frac{\beta^2-\gamma^2}{2}.$$

Similar to the previous lemma, we can get the Lie derivative of the Ricci tensor Ric^L associated to V.

Lemma 2.12.

$$(L_V Ric^L)(e_1, e_1) = 0, \quad (2.35)$$
$$(L_V Ric^L)(e_1, e_2) = \alpha(-\alpha^2 - \alpha\delta + \frac{\beta^2-\gamma^2}{2})\lambda_2 + \gamma(-\alpha^2 - \alpha\delta + \frac{\beta^2-\gamma^2}{2})\lambda_3,$$
$$(L_V Ric^L)(e_1, e_3) = \beta(\delta^2 + \alpha\delta + \frac{\beta^2-\gamma^2}{2})\lambda_2 + \delta(\delta^2 + \alpha\delta + \frac{\beta^2-\gamma^2}{2})\lambda_3,$$
$$(L_V Ric^L)(e_2, e_2) = -2\alpha(-\alpha^2 - \alpha\delta + \frac{\beta^2-\gamma^2}{2})\lambda_1,$$
$$(L_V Ric^L)(e_2, e_3) = [-\beta(\delta^2 + \alpha\delta + \frac{\beta^2-\gamma^2}{2}) - \gamma(-\alpha^2 - \alpha\delta + \frac{\beta^2-\gamma^2}{2})]\lambda_1,$$
$$(L_V Ric^L)(e_3, e_3) = -2\delta(\delta^2 + \alpha\delta + \frac{\beta^2-\gamma^2}{2})\lambda_1.$$

Then, if V is a conformal Ricci collineation associated to the Levi-Civita connection, by Lemma 2.12 and Theorem 2.1, we have the following equations:

$$\begin{cases} 0 = 2\lambda \\ (-\alpha^2 - \alpha\delta + \frac{\beta^2-\gamma^2}{2})(\alpha\lambda_2 + \gamma\lambda_3) = 0 \\ (\delta^2 + \alpha\delta + \frac{\beta^2-\gamma^2}{2})(\beta\lambda_2 + \delta\lambda_3) = 0 \\ -2\alpha(-\alpha^2 - \alpha\delta + \frac{\beta^2-\gamma^2}{2})\lambda_1 = 2\lambda \\ [-\beta(\delta^2 + \alpha\delta + \frac{\beta^2-\gamma^2}{2}) - \gamma(-\alpha^2 - \alpha\delta + \frac{\beta^2-\gamma^2}{2})]\lambda_1 = 0 \\ -2\delta(\delta^2 + \alpha\delta + \frac{\beta^2-\gamma^2}{2})\lambda_1 = -2\lambda \end{cases} \quad (2.36)$$

By solving (2.36) , we get

Theorem 2.7. *(G_6, g, V) admits conformal Ricci collineations associated to the Levi-Civita connection if and only if one of the following holds:*

(1) $\alpha^2 + \alpha\delta + \frac{\gamma^2}{2} = 0, \beta = 0, \alpha + \delta \neq 0, \alpha\gamma = 0, \delta \neq 0, \lambda = 0, \ \mathscr{V}_{\mathscr{RC}} = < e_2 >,$

(2) $\alpha^2 + \alpha\delta - \frac{\beta^2 - \gamma^2}{2} = 0, \beta \neq 0, \alpha + \delta \neq 0, \alpha\gamma - \beta\delta = 0, \lambda = 0,$ $\mathscr{V}_{\mathscr{R}C} =< -\frac{\delta}{\beta} e_2 + e_3 >,$
(3) $\alpha = 0, \beta = 0, \delta \neq 0, \gamma \neq 0, \lambda = 0,$ $\mathscr{V}_{\mathscr{R}C} =< e_2 >,$
(4) $\delta^2 + \alpha\delta + \frac{\beta^2 - \gamma^2}{2} = 0, \alpha \neq 0, \alpha + \delta \neq 0, \alpha\gamma - \beta\delta = 0, \lambda = 0,$ $\mathscr{V}_{\mathscr{R}C} =< -\frac{\gamma}{\alpha} e_2 + e_3 >,$
(5) $\alpha^2 + \alpha\delta - \frac{\beta^2 - \gamma^2}{2} \neq 0, \delta^2 + \alpha\delta + \frac{\beta^2 - \gamma^2}{2} \neq 0, \alpha \neq 0, \alpha + \delta \neq 0, \alpha\gamma - \beta\delta = 0, \alpha\delta - \beta\gamma = 0, \lambda = 0,$ $\mathscr{V}_{\mathscr{R}C} =< -\frac{\gamma}{\alpha} e_2 + e_3 >.$
where $\mathscr{V}_{\mathscr{R}C}$ *is the vector space of conformal Ricci collineations on* (G_6, g, V).

Proof. *We know that* $\lambda = 0$, $\alpha + \delta \neq 0$, $\alpha\gamma - \beta\delta = 0$. *By the fifth equation, we have* $\alpha(\alpha^2 + \alpha\delta - \frac{\beta^2 - \gamma^2}{2})\lambda_1 = 0$.
case 1) *If* $\lambda_1 = 0$, (2.36) *can be simplified to*

$$\begin{cases} (\alpha^2 + \alpha\delta - \frac{\beta^2 - \gamma^2}{2})(\alpha\lambda_2 + \gamma\lambda_3) = 0 \\ (\delta^2 + \alpha\delta + \frac{\beta^2 - \gamma^2}{2})(\beta\lambda_2 + \delta\lambda_3) = 0 \end{cases} \tag{2.37}$$

case 1-1) *If* $\alpha^2 + \alpha\delta - \frac{\beta^2 - \gamma^2}{2} = 0$, (2.37) *can be simplified to*

$$(\alpha + \delta)^2(\beta\lambda_2 + \delta\lambda_3) = 0,$$

i.e. $\beta\lambda_2 + \delta\lambda_3 = 0$.
case 1-1-1) *If* $\beta = 0$, *then* $\delta\lambda_3 = 0$. *When* $\delta = 0$, *by* $\alpha + \delta = 0$, $\alpha\gamma - \beta\delta = 0$, *we have* $\alpha \neq 0$, $\gamma = 0$, $\alpha^2 + \alpha\delta - \frac{\beta^2 - \gamma^2}{2} \neq 0$, *we get a contradiction; otherwise* $\delta \neq 0$, $\lambda_3 = 0$. *We get* (1).
case 1-1-2) *If* $\beta \neq 0$, *then* $\lambda_2 = -\frac{\delta}{\beta}\lambda_3$. *We get* (2).
case 1-2) *If* $\alpha^2 + \alpha\delta - \frac{\beta^2 - \gamma^2}{2} \neq 0$, *then* $\alpha\lambda_2 + \gamma\lambda_3 = 0$.
case 1-2-1) *If* $\alpha = 0$, *by* $\alpha + \delta \neq 0$, $\alpha\gamma - \beta\delta = 0$, *we have* $\delta \neq 0$, $\beta = 0$, $\gamma \neq 0$, *then* $\lambda_3 = 0$. *We get* (3).
case 1-2-2) *If* $\alpha \neq 0$, *then* $\lambda_2 = -\frac{\gamma}{\alpha}\lambda_3$. (2.37) *can be simplified to*

$$(\delta^2 + \alpha\delta + \frac{\beta^2 - \gamma^2}{2})(-\beta\gamma + \alpha\delta)\lambda_3 = 0.$$

If $\delta^2 + \alpha\delta + \frac{\beta^2 - \gamma^2}{2} = 0$, (2.37) *holds. We get* (4).
If $\delta^2 + \alpha\delta + \frac{\beta^2 - \gamma^2}{2} \neq 0$, *then* $\alpha\delta - \beta\gamma = 0$. *We get* (5).
case 2) *If* $\lambda_1 \neq 0$, $\alpha = 0$, *we have* $\delta \neq 0$, $\beta = 0$, *by* (2.36) *we get* $\gamma = 0$, $\delta = 0$. *This is a contradiction.*
So $\alpha \neq 0$, $\alpha^2 + \alpha\delta - \frac{\beta^2 - \gamma^2}{2} = 0$, *then* $\delta^2 + \alpha\delta + \frac{\beta^2 - \gamma^2}{2} \neq 0$. *By* (2.36) *we have* $\beta = \delta = 0$, *by* $\alpha\gamma - \beta\delta = 0$, *we have* $\gamma = 0$, *so*$\alpha^2 + \alpha\delta - \frac{\beta^2 - \gamma^2}{2} = \alpha^2 \neq 0$,, *we get a contradiction.*

2.7 Conformal Ricci collineations of G_7

By [10], we have the following Lie algebra of G_7 satisfies

$[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3,\ [e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3,\ [e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3,\ \alpha + \delta \neq 0,\ \alpha\gamma = 0.$

where e_1, e_2, e_3 is a pseudo-orthonormal basis, with e_3 timelike.

Lemma 2.13. *The Ricci tensor of (G_7, g) associated to the Levi-Civita connection ∇^L is determined by*

$$Ric^L(e_1, e_1) = -\frac{\gamma^2}{2}, \quad Ric^L(e_1, e_2) = 0, \quad Ric^L(e_1, e_3) = 0, \tag{2.38}$$
$$Ric^L(e_2, e_2) = -\alpha^2 - \frac{\gamma^2}{2} + \alpha\delta - \beta\gamma, \quad Ric^L(e_2, e_3) = \alpha^2 - \alpha\delta + \beta\gamma,$$
$$Ric^L(e_3, e_3) = -\alpha^2 - \frac{\gamma^2}{2} + \alpha\delta - \beta\gamma.$$

Similar to the previous lemma, we can get the Lie derivative of the Ricci tensor Ric^L associated to V.

Lemma 2.14.

$$(L_V Ric^L)(e_1, e_1) = 0, \tag{2.39}$$
$$(L_V Ric^L)(e_1, e_2) = -\frac{\beta\gamma^2}{2}\lambda_2 + (\frac{\beta\gamma^2}{2} - \frac{\gamma^3}{2})\lambda_3,$$
$$(L_V Ric^L)(e_1, e_3) = (\frac{\beta\gamma^2}{2} + \frac{\gamma^3}{2})\lambda_2 - \frac{\beta\gamma^2}{2}\lambda_3,$$
$$(L_V Ric^L)(e_2, e_2) = \beta\gamma^2\lambda_1 + \delta\gamma^2\lambda_3,$$
$$(L_V Ric^L)(e_2, e_3) = -\beta\gamma^2\lambda_1 - \frac{\delta\gamma^2}{2}\lambda_2 - \frac{\delta\gamma^2}{2}\lambda_3,$$
$$(L_V Ric^L)(e_3, e_3) = \beta\gamma^2\lambda_1 + \delta\gamma^2\lambda_2.$$

Then, if V is a conformal Ricci collineation associated to the Levi-Civita connection, by Lemma 2.14 and Theorem 2.1, we have the following equations:

$$\begin{cases} 0 = 2\lambda \\ -\frac{\beta\gamma^2}{2}\lambda_2 + (\frac{\beta\gamma^2}{2} - \frac{\gamma^3}{2})\lambda_3 = 0 \\ (\frac{\beta\gamma^2}{2} + \frac{\gamma^3}{2})\lambda_2 - \frac{\beta\gamma^2}{2}\lambda_3 = 0 \\ \beta\gamma^2\lambda_1 + \delta\gamma^2\lambda_3 = 2\lambda \\ -\beta\gamma^2\lambda_1 - \frac{\delta\gamma^2}{2}\lambda_2 - \frac{\delta\gamma^2}{2}\lambda_3 = 0 \\ \beta\gamma^2\lambda_1 + \delta\gamma^2\lambda_2 = -2\lambda \end{cases} \tag{2.40}$$

By solving (2.40) , we get

Theorem 2.8. *(G_7, g, V) admits conformal Ricci collineations associated to the Levi-Civita connection if and only if one of the following holds:*
(1) $\gamma = 0, \alpha + \delta \neq 0, \lambda = 0,\ \mathscr{V}_{\mathscr{R}C} =< e_1, e_2, e_3 >$,
(2) $\alpha = 0, \beta = 0, \delta \neq 0, \gamma \neq 0, \lambda = 0,\ \mathscr{V}_{\mathscr{R}C} =< e_1 >$.
where $\mathscr{V}_{\mathscr{R}C}$ is the vector space of conformal Ricci collineations on (G_7, g, V).

Proof. *We know that $\lambda = 0,\ \alpha + \delta \neq 0,\ \alpha\gamma = 0$. By (2.40), we have $\delta\gamma^2(\lambda_2 - \lambda_3) = 0$.*
case 1) If $\delta = 0$, by $\alpha + \delta \neq 0,\ \alpha\gamma = 0$, we have $\alpha \neq 0,\ \gamma = 0$, (2.40) holds. This situation falls into (1).
case 2) If $\delta \neq 0,\ \gamma = 0,\ \alpha + \delta \neq 0$, (2.40) holds. This situation falls into (1).
case 3) If $\delta \neq 0,\ \gamma \neq 0$, then $\lambda_2 = \lambda_3$, by $\alpha\gamma = 0$, we have $\alpha = 0$. (2.40) can be simplified to

$$\begin{cases} \frac{\gamma^3}{2}\lambda_2 = 0 \\ \beta\gamma^2\lambda_1 + \delta\gamma^2\lambda_2 = 0 \end{cases} \tag{2.41}$$

We get $\lambda_2 = \lambda_3 = 0$, $\beta = 0$. We get (2).

Acknowledgments. The author was supported in part by NSFC No.11771070.

References

[1] (EXAMPLE) Abd-Ellah, H. N. *Motion of tubular surfaces*. Italian J. Pure and Appl. Math., N. 22 (2007): 161-176.
[2] G. H. Katzin, J. Levine, W. R. Davis, *Curvature collineations: A fundamental symmetry property of the space-times of general relativity defined by the vanishing Lie derivative of the Riemann curvature tensor*. J. Mathematical Phys., 10 (1969): 617-629.
[3] K. L. Duggal, *Curvature inheritance symmetry in Riemannian spaces with applications to fluid space times*. J. Math. Phys., 33 (1992): 2989-2997.
[4] W. Kühnel, H. B. Rademacher, *Conformal Ricci collineations of space-times*. Gen. Relativity Gravitation, 33 (2001): 1905-1914.
[5] G. Contreras, L. A. Núñez, U. Percoco, *Ricci collineations for non-degenerate, diagonal and spherically symmetric Ricci tensors*. Gen. Relativity Gravitation, 32 (2000): 285-294.
[6] M. Akbar, Cai, Rong-Gen, *Ricci collineations of static space times with maximal symmetric transverse spaces*. Commun. Theor. Phys. (Beijing), 45 (2006): 95-104.
[7] U. Camci, A. Qadir, K. Saifullah, *Conformal Ricci collineations of static spherically symmetric spacetimes*. Commun. Theor. Phys. (Beijing). 49 (2008): 1527-1532.
[8] M. Afzal, U. Camci, K. Saifullah, *Integrability conditions for conformal Ricci collineation equations*. Mathematical physics, 103-115, World Sci. Publ., Hackensack, NJ, 2013.
[9] J. Lauret, *Ricci soliton homogeneous nilmanifolds*. Math. Ann., 319 (2001): 715-733.
[10] W. Batat, K. Onda, *Algebraic Ricci solitons of three-dimensional Lorentzian Lie groups*. J. Geom. Phys., 114 (2017): 138-152.
[11] Yong Wang,*Canonical Connections and Algebraic Ricci Solitons of Three-dimensional Lorentzian Lie Groups*. J. Chin. Ann. Math. Ser. B., 43 (2022): 443-458.
[12] G. Calvaruso, *Homogeneous structures on three-dimensional Lorentzian manifolds*. J. Geom. Phys., 57 (2007): 1279-1291.
[13] G. Calvaruso, *Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds*. Geom. Dedicata, 127 (2007): 99-119.
[14] L.A. Cordero, P.E. Parker, *Left-invariant Lorentzian metrics on 3-dimensional Lie groups*. Rend. Mat. Appl., 17 (1997): 129-155.

School of Mathematics and Statistics
Northeast Normal University
Changchun 130024, China.
Email address: wangyl553@nenu.edu.cn