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A GENERALIZATION OF IONESCU-WEITZENBÖCK'S INEQUALITY, USING HUYGENS-STEINER'S THEOREM FROM MECHANICS

MARTIN CELLI

ABSTRACT. The purpose of this note is to give a new proof of a known generalization of Ionescu-Weitzenböck's inequality, using Huygens-Steiner's theorem from mechanics. This will show an unexpected link between rigid body dynamics and triangle geometry.

1. HUYGENS-STEINER'S THEOREM FROM MECHANICS

Huygens-Steiner's theorem ([6]) expresses the moment of inertia as a quadratic function.
Huygens-Steiner's theorem. *Let $A_1, \ldots, A_N$ be N points and $m_1, \ldots, m_N$ be N numbers. Let us define the moment of inertia function about a point M, in the following way:*

$$F(M) = \sum_{k=1}^{N} m_k A_k M^2 \cdot$$

Then we have, for all Ω, M,

$$F(M) = F(\Omega) - 2\left(\sum_{k=1}^{N} m_k \overrightarrow{\Omega A_k}\right) \cdot \overrightarrow{\Omega M} + \left(\sum_{k=1}^{N} m_k\right) \Omega M^2 \cdot$$

This theorem just follows from the identity

$$A_k M^2 = ||\overrightarrow{A_k \Omega} + \overrightarrow{\Omega M}||^2 = A_k \Omega^2 + 2\overrightarrow{A_k \Omega} \cdot \overrightarrow{\Omega M} + \Omega M^2 \cdot$$

In mechanics, the m_k are positive masses. So their sum is non-zero. Thus, we can choose the center of inertia as Ω:

$$\Omega = O + \frac{1}{\sum_{k=1}^{N} m_k} \sum_{k=1}^{N} m_k \overrightarrow{OA_k} \cdot$$

This definition does not depend on the origin O. Taking $O = \Omega$, we obtain

$$\sum_{k=1}^{N} m_k \overrightarrow{\Omega A_k} = \vec{0} \cdot$$

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This gives us a simpler form of Huygens-Steiner's theorem:

$$F(M) = F(\Omega) + \left(\sum_{k=1}^{N} m_k\right) \Omega M^2 \cdot$$

In dimension 3, we can obtain a variant of this theorem called the parallel axis theorem, by replacing the center of inertia Ω with a line passing through Ω, the point M with a line parallel to Ω, the distance $A_k\Omega$ with the distance between the point A_k and the line Ω, the distance A_kM with the distance between the point A_k and the line M, the distance ΩM with the distance between the line Ω and the line M.

2. A GENERALIZATION OF IONESCU-WEITZENBÖCK'S INEQUALITY

Here, we will take fictitious positive and negative masses m_k whose sum is 0. In this case, the vector of inertia

$$\vec{U} = \sum_{k=1}^{N} m_k \overrightarrow{OA_k}$$

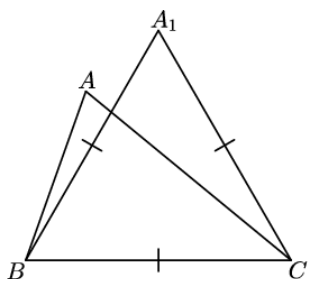
does not depend on the origin O, and we obtain another simpler form of Huygens-Steiner's theorem:

$$F(M) = F(\Omega) - 2\vec{U} \cdot \overrightarrow{\Omega M} \cdot$$

More precisely, let us take

$$A_2 = B, A_3 = C, m_1 = 2, m_2 = m_3 = -1, \Omega = B, M = A \cdot$$

As A_1 and O, we can choose the point such that the triangle A_1BC is equilateral, situated on the same side of BC as A (see the figure).



We obtain

$$2A_1A^2 - BA^2 - CA^2 = 2A_1B^2 - BC^2 + 2(\overrightarrow{A_1B} + \overrightarrow{A_1C}) \cdot \overrightarrow{BA},$$

$$2A_1A^2 = BA^2 + CA^2 + BC^2 - 2R\left(2H\frac{\overrightarrow{BC}}{BC}\right) \cdot \overrightarrow{BA},$$

where H is the altitude of the equilateral triangle A_1BC and R is the rotation of angle 90 degrees in the direction $(\overrightarrow{BC}, \overrightarrow{BA})$. Thus,

$$BC^2 + CA^2 + AB^2 = 2\sqrt{3}R(\overrightarrow{BC}) \cdot \overrightarrow{BA} + 2A_1A^2 = 4\sqrt{3}\mathcal{A} + 2A_1A^2,$$

where $\mathcal{A}$ is the area of the triangle ABC. This gives us the following theorem:
Theorem. *Let ABC be a triangle with area $\mathcal{A}$. Let A_1 be the point such that the triangle A_1BC is equilateral, situated on the same side of BC as A. Then we have*

$$BC^2 + CA^2 + AB^2 = 4\sqrt{3}\mathcal{A} + 2A_1A^2\cdot$$

This theorem states that for any constant points B, C, the level curves of $CA^2 + AB^2 - 4\sqrt{3}\mathcal{A}$, seen as a function of A, are the circles with center A_1. An immediate consequence of this theorem is Ionescu-Weitzenböck's classic inequality:

$$BC^2 + CA^2 + AB^2 \geq 4\sqrt{3}\mathcal{A}\cdot$$

Another proof of this theorem can be found in [4], proofs of other generalizations of Ionescu-Weitzenböck's inequality can be found in [1], [2], [3], [5], and [7] (where another vector argument is used).

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DEPARTMENT OF MATHEMATICS, UNIVERSIDAD AUTÓNOMA METROPOLITANA
AV. SAN RAFAEL ATLIXCO, 186, C.P. 09340, COL. VICENTINA
ALC. IZTAPALAPA, MEXICO CITY, MEXICO.
Email address: `cell@xanum.uam.mx`