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# MINIMUM AREA OF A REGION BOUNDED BY A CLOSED POLYGON

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ABSTRACT. It is well-known that polygons that maximise area are cyclic. We study whether polygons that minimise area are also cyclic. We include crossed polygons.

## 1. INTRODUCTION AND MOTIVATIONS

In this paper we analyse planar polygons with prescribed side-lengths and their minimum area configurations. We arrived at the problem after studying maximum area configurations of polygons depicting plots of land in a precolumbian Mexican codex. [10]
In 1966 Mathematics Magazine published a paper where Huseyin Demir proved that the configuration with maximum area of a closed polygon with given side-lengths was the cyclic one ([6]).
He mentioned that minima of area should also be cyclic but did not elaborate further.
In 2008 Hoehn ([9]) gave another proof that maxima of area are cyclic and for quadrilaterals gave a formula for the radius of the circumcircle but minima were not mentioned.
Christoffer Bradley in [4] mentioned many properties of cyclic quadrilaterals but did not include maxima or minima of area.

## 2. MAIN RESULTS

Are minima also cyclic? It depends on the definition of polygon and the definition of area.
The definition used by Demir gives signed area and therefore the minimum area is just minus the maximum area and corresponds to the same shape but reflected with respect to the $x$-axis. With this definition, minima are trivially cyclic.
Demir did not mention polygons with self-intersections, also called crossed polygons.
What happens if we use the traditional, absolute value of the Shoelace Formula ([12]) for the area?

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2.1. **Quadrilaterals.** Let us look at quadrilaterals first. The Shoelace formula applies to normal as well as crossed quadrilaterals, the ones with self-intersections. Let us analyze the possible values of the area.
Figure 1 shows the construction of a quadrilateral with fixed side-lengths $(a, b, c, d)$ units, in clockwise order (such that no length is larger than the sum of the other three) and prescribed area $A_c$ square units following ([8]). One vertex is placed at the origin, another on the $x-$axis, with coordinates $(a, 0)$, another at the point(s) of intersection of a circle $\mathcal{C}$ with radius $b$ and center at the origin (continuous grey line) and the line $\mathcal{L}$ with

$$\mathcal{L} : Pu + Qv = R, \quad \mathcal{C} : u^2 + v^2 = b^2 \tag{2.1}$$

where

$$\begin{aligned} P &= 2a(a^2 - M - d^2) = a(a^2 + b^2 - c^2 - d^2),\ Q = 4\mathcal{A}_c a, \\ R &= (M - a^2)^2 + a^2 b^2 + 4\mathcal{A}_c^2 - d^2(a^2 + b^2), \\ 2M &= a^2 + c^2 - b^2 - d^2. \end{aligned}$$

Here $u$ is the horizontal coordinate and $v$ the vertical coordinate. The intersections between $\mathcal{L}$ and $\mathcal{C}$ are called $(u_1, v_1)$ and $(u_2, v_2)$. They coincide in the case of tangency.
The last vertex is placed so that the side-lengths are the given ones. It is labeled $(w_j, z_j)$, $j = 1, 2$. Again, in the case of tangency, $(w_1, z_1) = (w_2, z_2)$.
The position and slope of the line $\mathcal{L}$ depend on the side-lengths and on the area.
The area of a quadrilateral is given by Bretschneider's formula:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{\alpha+\gamma}{2}\right)},$$

where $s$ is the semi-perimeter and $\alpha, \gamma$ are opposite angles. This coincides with the absolute value of the Shoelace formula.
From this formula we can see that the maximum possible area is given by

$$A_{max} = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

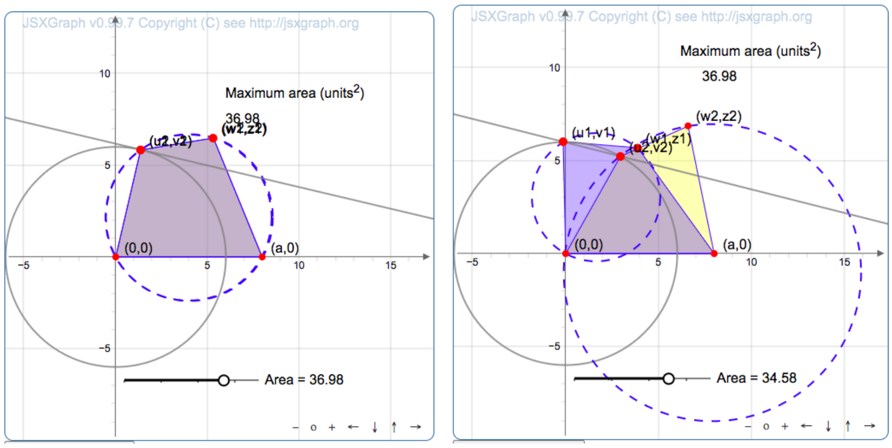


**Figure 1.** Configuration with maximum area for quadrilateral with side-lengths 8,6,4,7 units and the two configurations for a slightly smaller area.

If the prescribed area is larger than the maximum possible area then there is no intersection between the line and the circle. When $A_c = A_{max}$ the line is tangent to the circle and we obtain the unique configuration with maximum area. This configuration is cyclic because $\cos((\alpha+\gamma)/2) = 0$ therefore $\alpha + \gamma = \pi$ radians. In the maximum area configuration the dotted circle is the circumcircle.
When there are two intersections there are two possible configurations with the given area for the quadrilateral.
Four side-lengths define a quadrilateral only if no side is larger that the sum of the other three. In terms of this construction what happens is that in the impossible case, if we prescribe a large area and then decrease it to zero, the line and circle never intersect.
It is not trivial to see what happens when we look for the minimum possible area by decreasing the prescribed area towards zero. The slope of the line becomes infinite when the area is zero (except for rectangles) but this can occur before or after $\mathcal{L}$ leaves $\mathcal{C}$ . This is what we are going to study now.
We recommend playing with the programs in Irregular polygons to better understand the problem. The programs were created with JSXGraph.
If self-intersections are not allowed then minima of area consist of triangles and line segments ([3]). These minima are not cyclic in general since at least three vertices will be colinear.
For example, for a quadrilateral with (clockwise) side-lengths 9,5,7,12 units, Figure 2 shows the configuration with maximum area and two possible minima without self-intersections.
For a quadrilateral there is a maximum of eight ways of deforming to a triangle or a triangle plus a segment since each of the four angles could possibly be deformed to be zero or $\pi$ radians although in general, depending on the side-lengths, not all of those configurations will be possible. The minimum area is the minimum between the areas of those configurations that are possible. Usually configurations with triangle plus segment will give a smaller area. These minima are in general not cyclic. Since at least three points are colinear, the (degenerate) circumcircle would be that line.
If self-intersections are allowed then the minimum will be cyclic if the line $\mathcal{L}$ leaves the circle $\mathcal{C}$ before becoming vertical (area 0).
Consider for example the quadrilateral with side-lengths 8,7,5,12 units. The configuration with minimum area is not cyclic (see Figure (3).
For the quadrilateral with side-lengths 8,6,4,7 the minimum area configuration is a cyclic crossed quadrilateral. One vertex is given by the point of tangency between $\mathcal{L}$ and $\mathcal{C}$. For crossed quadrilaterals the area is the absolute value of the *difference* between the areas of the two triangular regions ([7]).
Alternatively, from Bretschneider´s Formula,

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2((\alpha+\gamma)/2)}$$

therefore the cyclic minimum area configuration will exist if

$$(s-a)(s-b)(s-c)(s-d) \geq abcd.$$

In this case $\cos((\alpha+\gamma)/2) = -1$ therefore $\alpha + \gamma = 2\pi$ radians and the quadrilateral is cyclic.

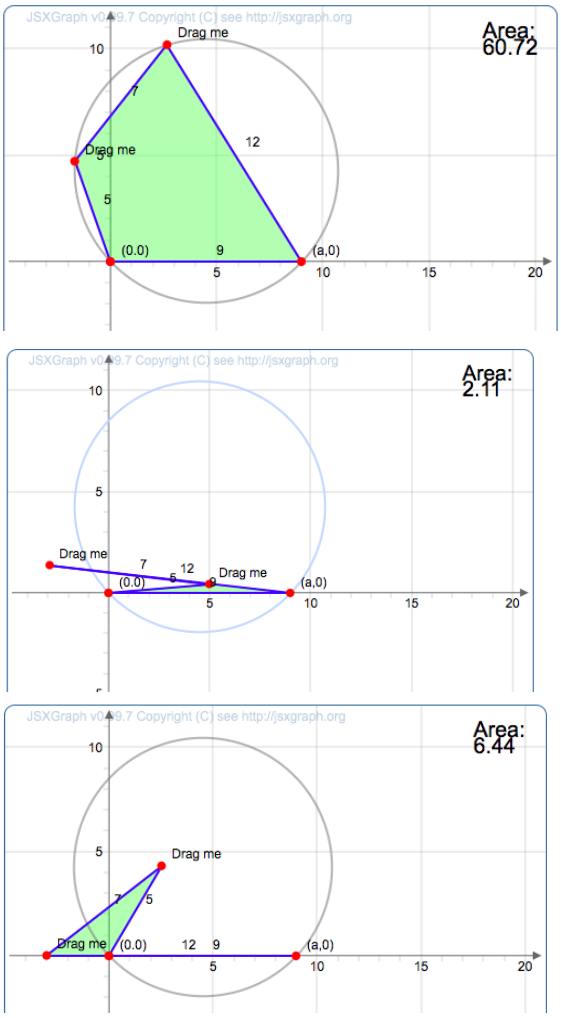


**Figure 2.** Configuration with maximum and two possible minima of area, without self-intersections, for the quadrilateral with side-lengths 9,5,7,12 units.

**Lemma** A crossed quadrilateral is cyclic if and only if opposite angles add up to $2\pi$ radians.
To prove this fact look at Figure 5.
The angles are $\alpha, \beta, \gamma, \delta$. If the quadrilateral is cyclic then $\beta = 2\pi - \delta$ radians because both angles are subtended by the same arc. Equally, $\alpha$ and $2\pi$ minus $\gamma$ are equal because they are subtended by the same arc. Therefore $\alpha$ plus its opposite angle, $\gamma$, add up to $2\pi$ radians and $\beta + \delta = 2\pi$ radians.
Inversely, if opposite angles add up to $2\pi$ then $\alpha + \gamma = 2\pi$ therefore the internal angle at $C$ is equal to $\alpha$. In the same way, angle $\beta$ and the internal angle at $D$ are also equal. If we call $E$ the point where the sides cross then triangles $AEB$ and $CED$ are similar and hence

$$\frac{AE}{ED} = \frac{BE}{CE}$$

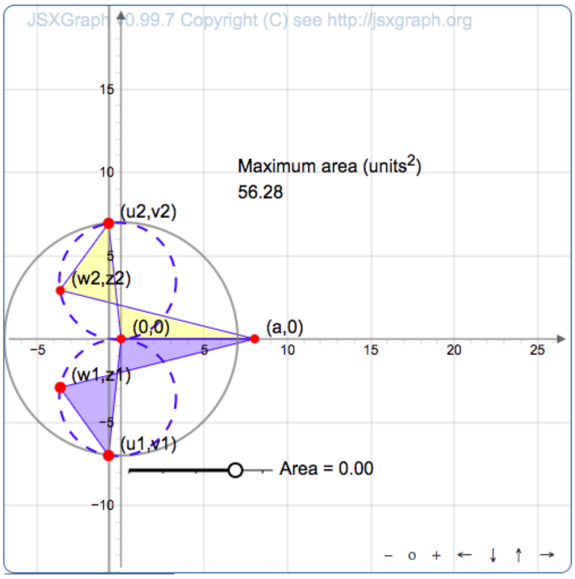


**Figure 3.** Side-lengths 8, 7, 5, 12 units. Two configurations with minimum area. They are not cyclic.

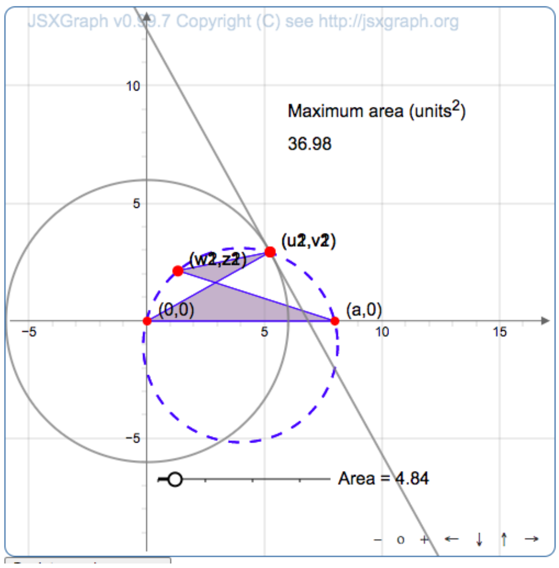


**Figure 4.** Side-lengths 8, 6, 4, 7 units. The area minimizer is cyclic.

therefore

$$AE \cdot EC = BE \cdot ED$$

and hence by [5] (cited in [13]) the quadrilateral is cyclic.
When the minimum is cyclic, what is the radius of the circumcircle?

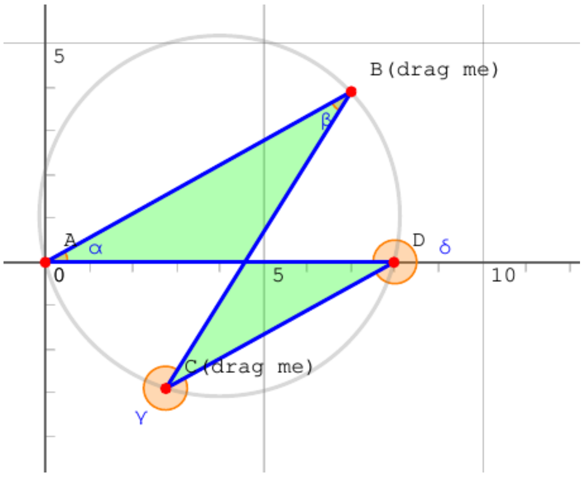


**Figure 5.** Cyclic crossed quadrilateral

It is known (see [9],[1]) that the radius of the circumcircle of the configuration with maximum area for the quadrilateral with side-lengths $a, b, c, d$ is

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}, \tag{2.2}$$

where $s$ is the semi-perimeter, $(a+b+c+d)/2$.
The important thing to observe is that this circumradius coincides with the circumcircle for the configuration with minimum area for a quadrilateral (when it is cyclic) with side-lengths $a, p, c, q$, where $p$ and $q$ are the diagonals of the quadrilateral. Therefore if we can write $b$ and $d$ in terms of $a, p, c, q$ we can write the circumradius of the minimum area configuration.
From Ptolemy´s Theorem ([11])

$$pq = ac + bd$$

and

$$\frac{p}{q} = \frac{ad+bc}{ab+cd}$$

and from these two expressions

$$b = \frac{pq - ac}{d} \tag{2.3}$$

and

$$d = \sqrt{\frac{(pq-ac)(cq-pa)}{pc-qa}}. \tag{2.4}$$

As an example consider Figure 4. To compute the radius of the circumcircle of this minimum area configuration we view the sides that cross as diagonals $p, q$ and compute the missing sides $b$ and $d$ from (2.3) and (2.4). We start from $a = 8, p = 6, c = 4, q = 7$ units. We compute $d = 5/2$ and $b = 4$. We now use equation (2.2) to calculate the circumradius of the quadrilateral with sides $a, b, c, d$. We obtain $16/\sqrt{15}$ or approximately 4.13 units in this case. It is the circumradius for the maximum area configuration of the quadrilateral with sides $a$, $b$, $c$, $d$ and diagonals $p$ and $q$ and for the (crossed) quadrilateral with sides $a$,$p$, $c$, $q$ and minimum area.

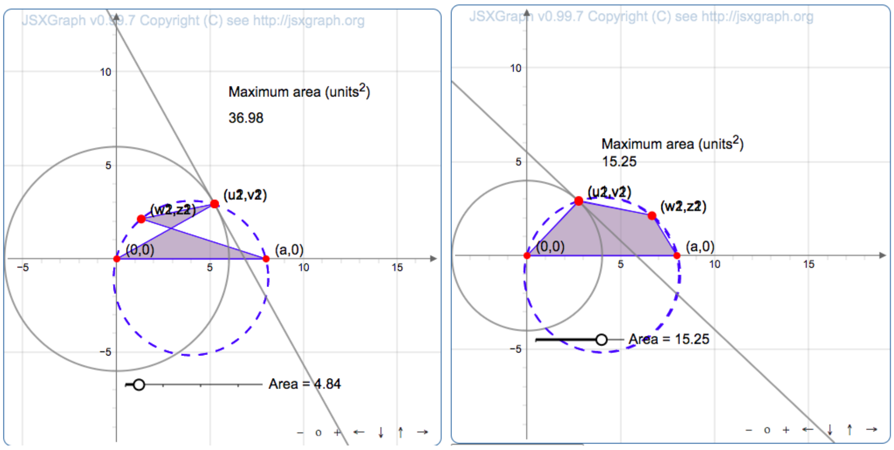


**Figure 6.** Minimum area configuration for side-lengths 8,6,4,7 units and maximum area configuration for side-lengths 8, 4, 4, 2.5 units. The circumradius is the same

For the non-cyclic case of Figure 3 the sidelengths are 8, 7, 5, 12 units. If there was a cyclic minimum area configuration it would have $a = 8, p = 7, c = 5, q = 12$ and equation (2.4) would give $d = \sqrt{176/(-57)}$, an imaginary number, and therefore the cyclic minimum area configuration is impossible.
Equivalently we can verify that

$$(s-a)(s-b)(s-c)(s-d) = 3,168 < abcd = 3,360$$

so we again conclude that the minimum is not cyclic.

2.2. **Polygons with more tha four sides.** For polygons with more than four sides the problem is considerably more complicated. Minima when only non-crossed configurations are permitted were studied in ([3]). For example for a pentagon, a triangle plus some line segments can be obtained by combinations of two angles being either zero or $\pi$ radians. These are the minimum area configurations in this context. They are not cyclic in general. See Figure (7) for the maximum area configuration of a pentagon with sides 8, 5, 9, 6, 10 units and Figure (8) for three possible non-crossed minima.

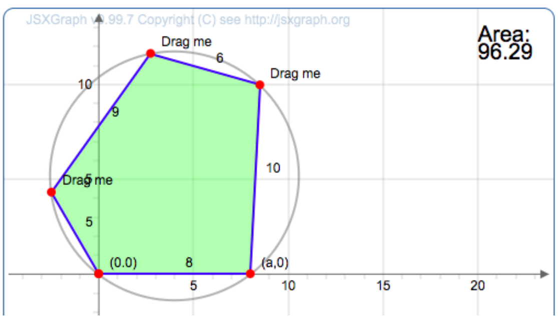


**Figure 7.** Side-lengths 8, 5, 9, 6, 10 units. Maximum area configuration.

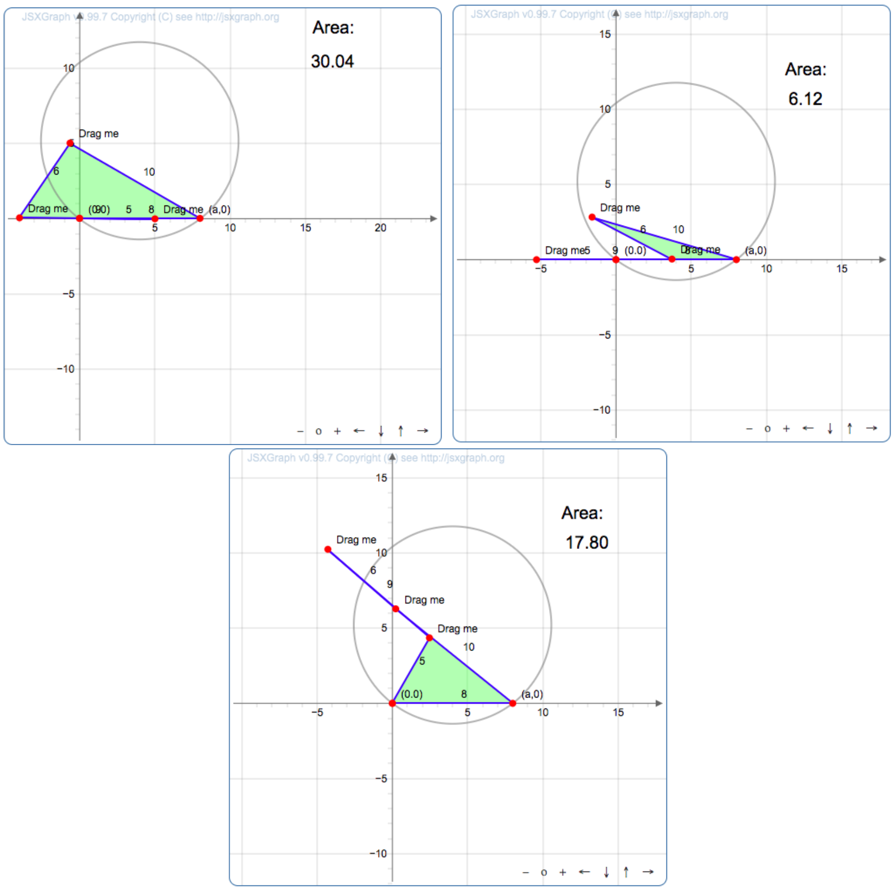


**Figure 8.** Side-lengths 8, 5, 9, 6, 10 units. Some non-crossed minimum configurations.

Figure (9) shows a couple of crossed configurations that appear to be minima. They are not cyclic.
We know of no formula to actually compute the minimum area so these figures are numerical approximations.
We cannot say more in the general case.

If the side-lengths are equal, that is, for regular polygons of side-length $a$, if the number of sides is odd then one possible minimum area configuration is a triangle with side $a$, which is seen in Figure (10). Another possible minimum is given by the star configuration as seen in Figure 11. The area of the star is smaller than that of the triangle. Both are cyclic. The triangle is numerically seen to be a saddle point for the area. Dragging the top of the triangle downwards decreases the area until we reach the star and then increases again.
Dragging the top vertex of the triangle upwards gives the true area minimum, which is zero. The configuration is not cyclic (see Figure (12)).
For a regular heptagon, a minimum area configuration is shown in Figure (13).
We conjecture that all regular polygons with an odd number of sides (greater than 3) have many similar configurations with area zero. Figure(14) shows two such configurations for a nonagon.

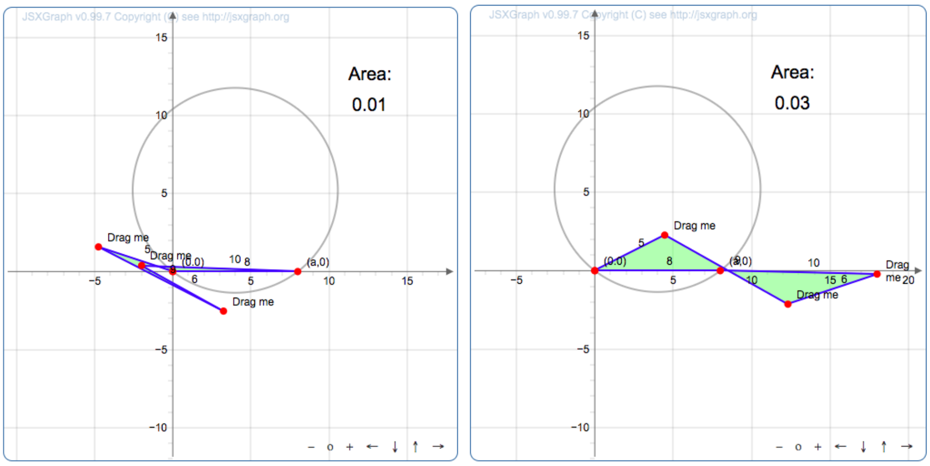


**Figure 9.** Side-lengths 8, 5, 9, 6, 10 units. Crossed (seemingly) minimum area configurations.

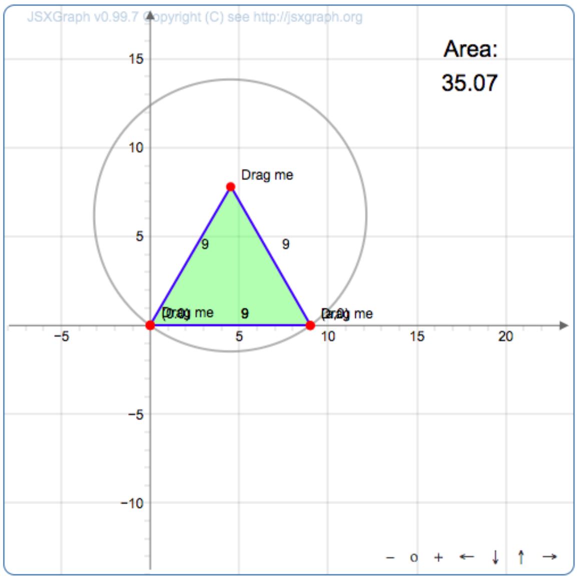


**Figure 10.** Regular pentagon with side-length 9 units deformed to a triangle

If the side-lengths are equal and there is an even number of sides then the figure can be deformed to a line, or several line segments, the minimum area is zero and it can be achieved in many ways so the minimum is not cyclic and not unique.

**Conclusions** We considered mainly crossed polygons that minimize area given by the absolute value of the Shoelace formula. For four sides we showed that some minima are cyclic and some are not. We gave three ways (the circle-line construction, Bretschneider's Formula and equation (2.4)) to tell which ones are cyclic and a way to compute the radius of the circumcircle when it exists.

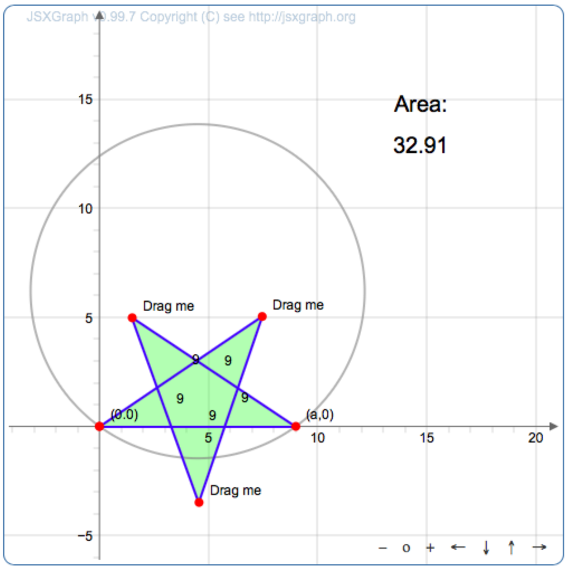


**Figure 11.** Regular pentagon with side-length 9 units deformed to a star shape.

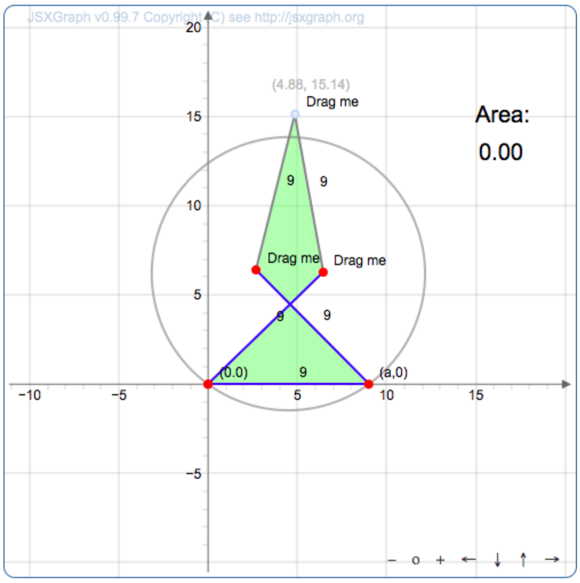


**Figure 12.** Regular pentagon with side-length 9 units deformed to a figure with area zero square units.

We analysed regular polygons with more than four sides and found configurations with area zero that are not cyclic.

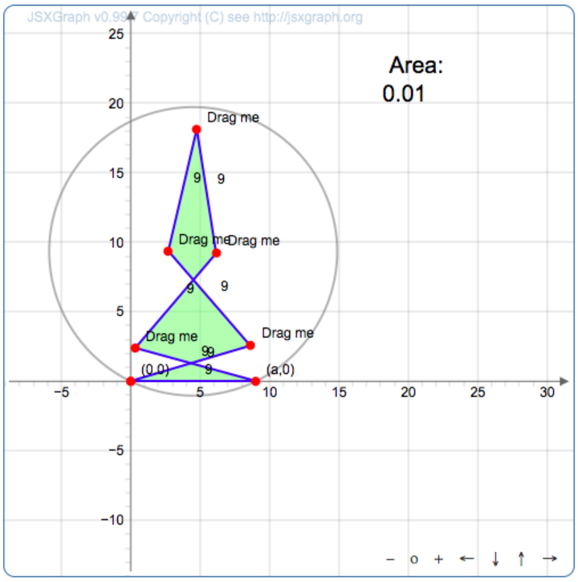


**Figure 13.** Regular heptagon with side-length 9 units deformed to a figure with area zero.

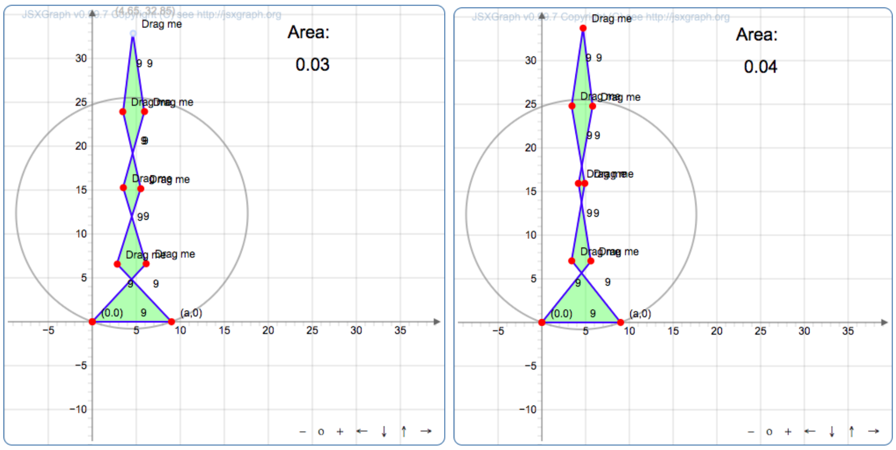


**Figure 14.** Regular nonagon with side-length 9 units deformed to a figure with area zero square units.

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