Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.13, (2024), Issue 1, pp.1-10

OBTUSE TRIANGLE, RECTANGULAR HYPERBOLA AND DE LONGCHAMPS' CIRCLE

PARIS PAMFILOS

ABSTRACT. In this article we study conics inscribed in a triangle, which appear always in a group of related four. We find conditions that such a group contains simultaneously a rectangular hyperbola and a parabola. These conditions involve the de Longchamps circle of the triangle of reference, which necessarily it is obtuse.

1. INTRODUCTION

This article grew out from an attempt to understand *"inconics"* of a triangle, i.e. conics inscribed in a triangle, which always appear in a group of four and which I call a *"four inconics group"*. In fact, given an inconic κ_D of the triangle ABC, its *"perspector"* D w.r.t. the triangle ([1, p. 105], [2]) is defined by the common point of the lines $\{AA', BB', CC'\}$ joining the vertices with the contacts $\{A', B', C'\}$ on the opposite sides (see Figure 1). Conversely, a point D, not lying on a side-line of the triangle, uniquely defines its *"traces"* $\{A', B', C'\}$ on the sides and through them an inconic κ_D tangent to the corresponding side-lines of the triangle at these three points. With $tr(X)$ we denote in the figure the *"trilinear polar"* or *"tripolar"* of the point X ([3, p.134], [4]). The inconic with perspector X is the envelope of the tripolars $tr(Y)$ of points $Y \in tr(X)$ ([2]).

Starting with an inconic κ_D and taking the three *"harmonic associates"* ([5, p.102]) of its perspector, i.e. the harmonic conjugates of the point D :

$$A^* = D(AA') \ , \ B^* = D(BB') \ , \ C^* = D(CC') \ ,$$

we define three other inconics $\{\kappa_A, \kappa_B, \kappa_C\}$ having the points $\{A^*, B^*, C^*\}$ as perspectors.

There is a kind of symmetry in this configuration, and the role of D can be interchanged with the role of any one of the points $\{A^*, B^*, C^*\}$, e.g. with B^*. Then, $\{A^*, C^*, D\}$ are again obtained as harmonic conjugates of the corresponding pairs defined by the traces $\{A'', C'', B'\}$ of B^* : $\{A^* = B^*(CC'') \ , \ C^* = B^*(AA'') \ , \ D = B^*(BB')\}$ and the conics $\{\kappa_A, \kappa_C, \kappa_D\}$ are defined through their perspectors $\{A^*, C^*, D\}$.

The four perspectors $\{D, A^*, B^*, C^*\}$ define a *"complete quadrilateral"* and ABC is its *"diagonal triangle"* ([6, p.7]). Also, each conic out of the four is tangent to the other three, with common tangent a different side of the triangle.

2020 *Mathematics Subject Classification.* 51-02; 51M15, 51N15, 51N20, 51N25.
Key words and phrases. Triangle, Triangle Geometry, Inconics, Rectangular hyperbola, de Longchamps.

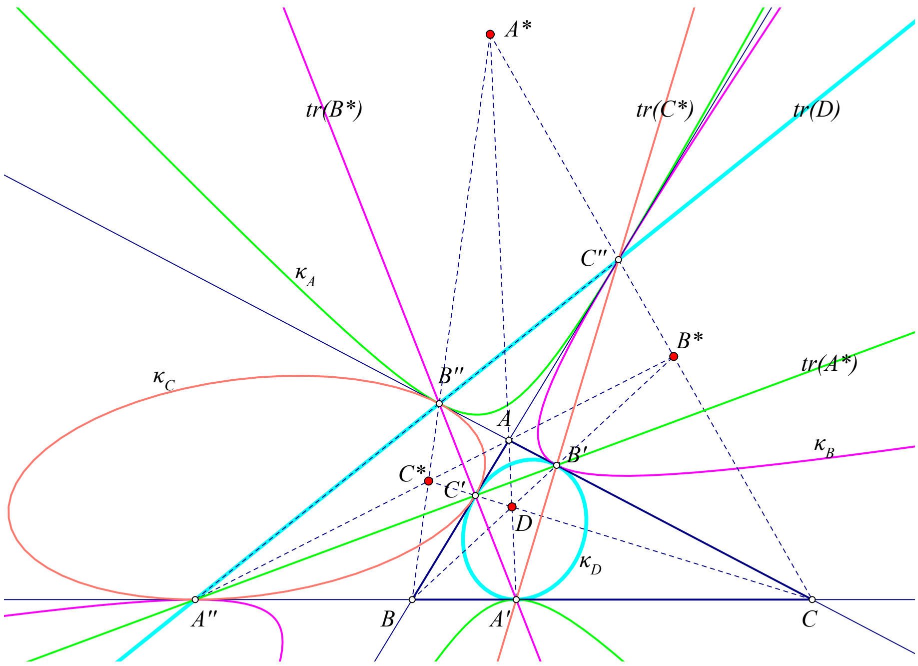

Figure 1. A *"four inconics group"* $\{\kappa_D, \kappa_A, \kappa_B, \kappa_C\}$ of the triangle ABC

A naturally arising question, representing also the core subject of this article, is the one about the kinds of the four inconics group. Which kinds appear in such a group? In the example seen in figure 1 appear two ellipses and two hyperbolas. It is also well known ([1, p.107], [5, p.127]), that the kind of the inscribed conic is determined by the location of its perspector relative to the *"Steiner ellipse"* σ circumscribing the triangle.

Inconics with perspectors lying inside/on/outside σ are correspondingly ellipses/ parabolas/hyperbolas. Figure 2 shows an example with two perspectors $\{D, B^*\}$ inside

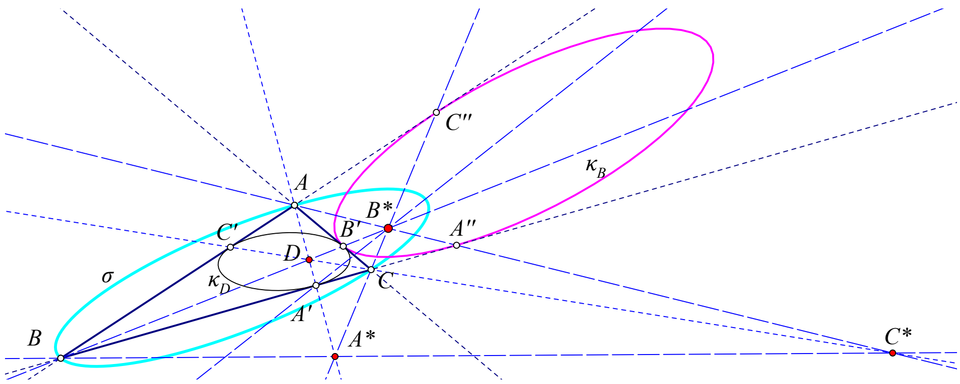

Figure 2. A four inconics group with two elliptical members

the Steiner ellipse σ. Certainly one, corresponding to the perspector D lying inside the triangle, hence also inside σ, defines an ellipse. Thus, one ellipse is mandatory and every four inconics group contains at least one ellipse. It is also easily seen that at most two ellipses can be contained in such a group, as in figure 2, in which besides κ_D we have the ellipse κ_B with perspector B^* lying also inside σ. One can then easily show that the polars of the other perspectors $\{A^*, C^*\}$ w.r.t. to the inner ellipse κ_D intersect this ellipse, consequently intersect also the Steiner ellipse σ containing κ_D, hence $\{A^*, C^*\}$ lie outside the ellipse σ, and the corresponding conics $\{\kappa_A, \kappa_C\}$ are hyperbolas.

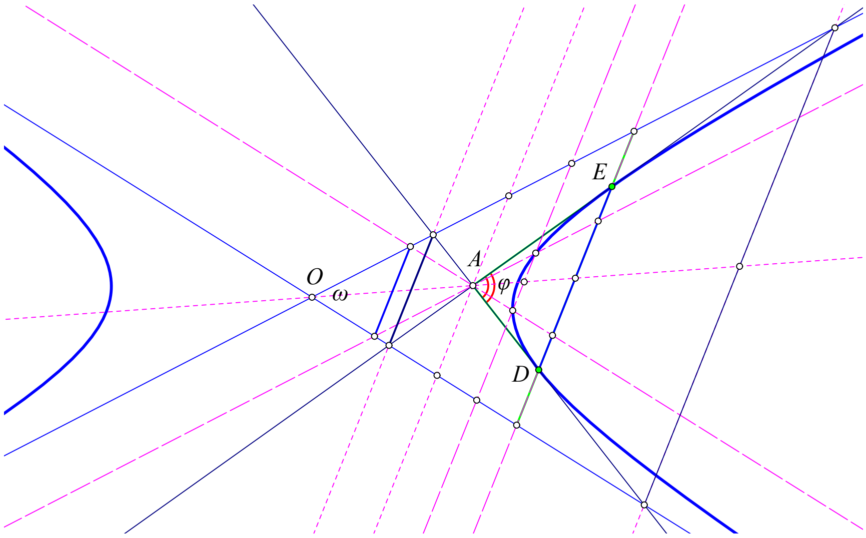

Figure 3. Angle $\phi = \widehat{DAE}$ *"viewing"* a branch is greater than ω

Next section handles the case of rectangular hyperbolas, locating the position of perspectors which deliver an inconic of this kind. Talking of hyperbolas inscribed in triangles we should notice that in such configurations there is always an angle, $\widehat{A}$ say, of the triangle with both sides tangent to the same branch of the hyperbola at two points $\{D, E\}$ of it (see Figure 3). Then, using general and elementary properties of the hyperbola ([7]), one can show that angle $\widehat{A}$ is greater than the angle ω of the asymptotes under which the center O of the hyperbola is *"viewing"* this branch. Thus, a triangle circumscribing such a hyperbola has one of its angles greater than ω. In particular, a triangle circumscribing a rectangular hyperbola has an angle greater than $90°$, i.e. it is obtuse and the rectangular hyperbola has a branch contained in the obtuse angle. This justifies the adopted below restriction to obtuse triangles, when considering the problem of inconics of rectangular hyperbola type. This implies also, that in a four inconics group there is at most one rectangular hyperbola member, which, if existing, has a branch contained in the obtuse angle.

2. THE ROLE OF DE LONGCHAMPS' CIRCLE

Here we work with barycentric coordinates or *"barycentrics"* ([5], [4]). The general form of an inconic described through these coordinates $\{(x : y : z)\}$ is

$$L^2x^2 + M^2y^2 + N^2z^2 - 2LMxy - 2MNyz - 2NLzx \ = \ 0, \qquad (2.1)$$

with $D\left(\frac{1}{L} : \frac{1}{M} : \frac{1}{N}\right)$ representing the perspector of the inconic [8, p.131]. Notice that $D^u(L : M : N)$ is the *"isotomic conjugate"* ([5, p.31]) of D. The *"de Longchamps circle"* of the triangle ABC is the circle centered at the *"de Longchamps point"* of the triangle, denoted by $X(20)$ in Kimberling's notation ([9]). This is the symmetric of the orthocenter w.r.t. to the circumcenter of the triangle. The de Longchamps circle λ is represented in barycentrics through equation ([4])

$$a^2x^2 + b^2y^2 + c^2z^2 + 2S_Cxy + 2S_Ayz + 2S_Bzx = 0 . \tag{2.2}$$

The constants appearing there are the side-lengths $\{a = |BC|, b = |CA|, c = |AB|\}$ and the *"Conway triangle symbols"*

$$S_A = \frac{b^2 + c^2 - a^2}{2} \quad , \quad S_B = \frac{c^2 + a^2 - b^2}{2} \quad , \quad S_C = \frac{a^2 + b^2 - c^2}{2} .$$

The de Longchamps circle is real for obtuse triangles and is orthogonal to the *"power circles"* i.e. the circles centered at the middles of the sides and passing through the opposite vertex of the triangle ([10]) (see Figure 4). Next theorem shows that the two notions are intimately connected.

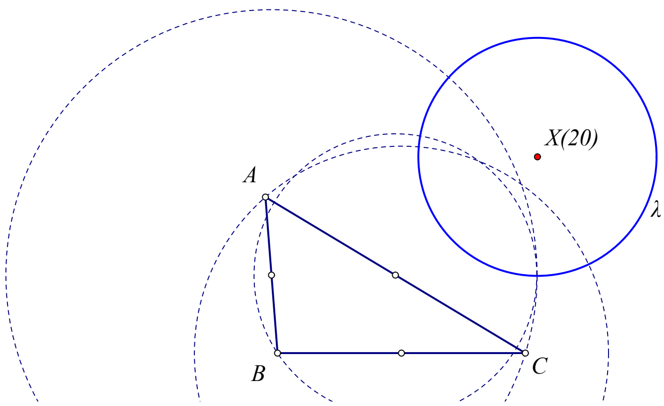

Figure 4. The de Longchamps circle of an obtuse triangle ABC

Theorem 2.1. *The inconic of the obtuse triangle ABC is a rectangular hyperbola, if and only if the isotomic $B^u(L : M : N)$ of its perspector $B^*(1/L : 1/M : 1/N)$ is a point of the de Longchamps circle λ of the triangle.*

Proof. Assuming the equation of the inconic in the form (2.1), its points at infinity are the intersections with the line at infinity represented in barycentrics through $x + y + z = 0$. Thus, setting $z = -x - y$ in equation (2.1) we arrive at the equation in (x, y) :

$$(N + L)^2x^2 + 2(N^2 + MN + LN - LM)xy + (N + M)^2y^2 = 0 \quad \text{which for} \quad t = \frac{x}{y} \quad \Rightarrow$$

$$(N + L)^2t^2 + 2(N^2 + MN + LN - LM)t + (N + M)^2 = 0 . \tag{2.3}$$

The roots $\{t_1, t_2\}$ of this equation determine the points $\{(u : v : w), u + v + w = 0\}$ of the inconic at infinity, representing also in barycentrics the directions of its asymptotes.

$$(u_1 : v_1 : w_1) = (t_1 : 1 : -(1 + t_1)) \quad , \quad (u_2 : v_2 : w_2) = (t_2 : 1 : -(1 + t_2)) \tag{2.4}$$

The condition of orthogonality, valid for the directions of the asymptotes of a rectangular hyperbola, leads to the equation satisfied by these two points ([4]):

$$\begin{aligned} S_A u_1 u_2 + S_B v_1 v_2 + S_C w_1 w_2 &= 0 \quad \Leftrightarrow \\ S_A t_1 t_2 + S_B + S_C(1+t_1)(1+t_2) &= 0 \quad \Leftrightarrow \\ (S_A+S_C)\frac{(N+M)^2}{(N+L)^2} - 2S_C\frac{N^2+MN+LN-LM}{(N+L)^2} + (S_C+S_B) &= 0 \quad \Leftrightarrow \\ a^2L^2 + b^2M^2 + c^2N^2 + 2S_C LM + 2S_A MN + 2S_B NL &= 0\,, \end{aligned} \tag{2.5}$$

which is the desired equation (2.2) of the de Longchamps circle. □

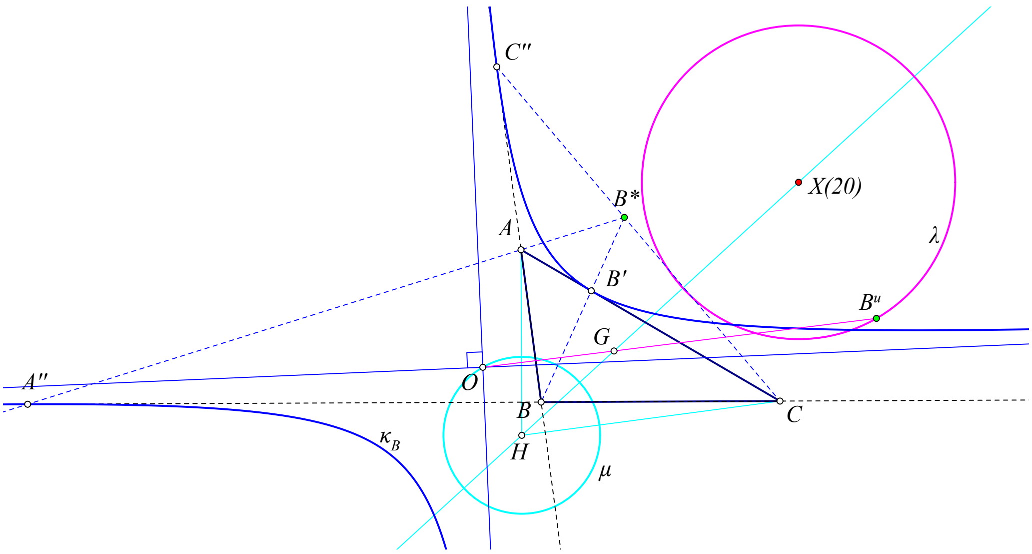

Figure 5. Points $\{B^u(L:M:N)\}$ defining inconics of rectangular hyperbola type

Next corollary is a refinement of a well known result for rectangular hyperbolas inscribed in a triangle ([11, p.338], [1, p.105], [5, p.128], [12]).

Corollary 2.1. *The center O of the inconic corresponding to the point $B^u(L:M:N)$ on the de Longchamps circle is homothetic w.r.t to the centroid G in ratio $(-1:2)$ to B^u and lies on the polar circle μ of the triangle ABC (see Figure 5).*

Proof. The center of the inconic defined by equation (2.1) through the point $B^u(L:M:N)$ of the de Longchamps circle λ is given by $O(M+N:N+L:L+M)$. The collinearity of the three points $\{B^u, G, O\}$ follows trivially from the vanishing of the determinant

$$\begin{vmatrix} M+N & N+L & L+M \\ 1 & 1 & 1 \\ L & M & N \end{vmatrix} = 0\,.$$

The proof for the homothety results from the equality

$$G = \frac{1}{2}\left(\frac{B^u}{L+M+N}\right) + \left(\frac{O}{2(L+M+N)}\right),$$

expressing the barycentric coordinates of G (up to a factor) as combination of the absolute barycentrics of the points $\{B^u, O\}$. The other claim relating the de Longchamps circle to the polar circle is a well known fact ([10]). □

Notice that the corollary implies an easy construction of all the rectangular hyperbolas inscribed in an obtuse triangle. To define such a conic select a point O on the polar circle μ and consider the symmetric w.r.t. to O of the side-lines of the triangle. The conic can then be drawn as one tangent to five given lines.

3. Combining a parabola with a rectangular hyperbola

Here we seek to find conditions for an obtuse triangle to accept a four inconics group having a parabola *and* a rectangular hyperbola member. As we noticed already, the perspector, C^* say, of the parabola lies on the Steiner ellipse σ of the triangle of reference ABC. Thus, we can start with such a point $C^* \in \sigma$ and see when the other perspectors of the group defined by C^* determine a rectangular hyperbola.

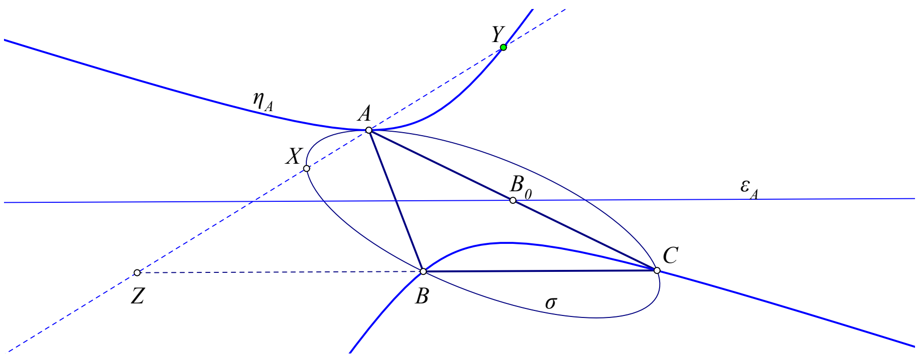

Figure 6. For $X \in \sigma$ point $Y = X(AZ)$ lies on the hyperbola η_A

Figure 6 shows a point X varying on the Steiner ellipse σ of the triangle ABC. Its harmonic associate Y varies then on a conic circumscribing the triangle of reference.

Theorem 3.1. *For a point X lying on the Steiner ellipse σ of the triangle ABC, the corresponding harmonic associate $Y = X(AZ)$ with $Z = AX \cap BC$ lies on a hyperbola η_A, whose isotomic image is the line ε_A parallel to BC through the middle B_0 of AC.*

Proof. We work in barycentrics ([5], [4]) w.r.t. the triangle ABC. The Steiner ellipse in these coordinates is represented through equation

$$\sigma \; : \; xy + yz + zx = 0 \, .$$

For a point $X(u:v:w) \in \sigma$ point $Z = (0:v:w)$ and writing $X = uA + Z$ we have $Y(u':v':w') = X(AZ) = uA - Z = (u:-v:-w)$ satisfying equation

$$u'v' - v'w' + w'u' \; = \; 0 \, .$$

This is a conic circumscribing the triangle and also is the isotomic image of the line $\varepsilon_A : -x + y + z = 0$ passing through the middles $\{B_0(1:0:1), C_0(1:1:0)\}$ of the sides $\{AC, AB\}$. The conic is obviously a hyperbola, since for X obtaining the position of the two intersections of $\sigma \cap \varepsilon_A$ the corresponding Y takes the position of two distinct points at infinity. □

A similar property can be proved for the harmonic associate $Y' = X(CZ')$ with point $Z' \in AB$, and we get a second hyperbola η_C containing all these $\{Y'\}$ and its isotomic image, which is a line ε_C parallel to AB through the middle B_0 of AC. Figure 7 shows the two hyperbolas, their isotomic lines $\{\varepsilon_A, \varepsilon_C\}$, and the de Longchamps circle λ, which play a fundamental role in next theorem.

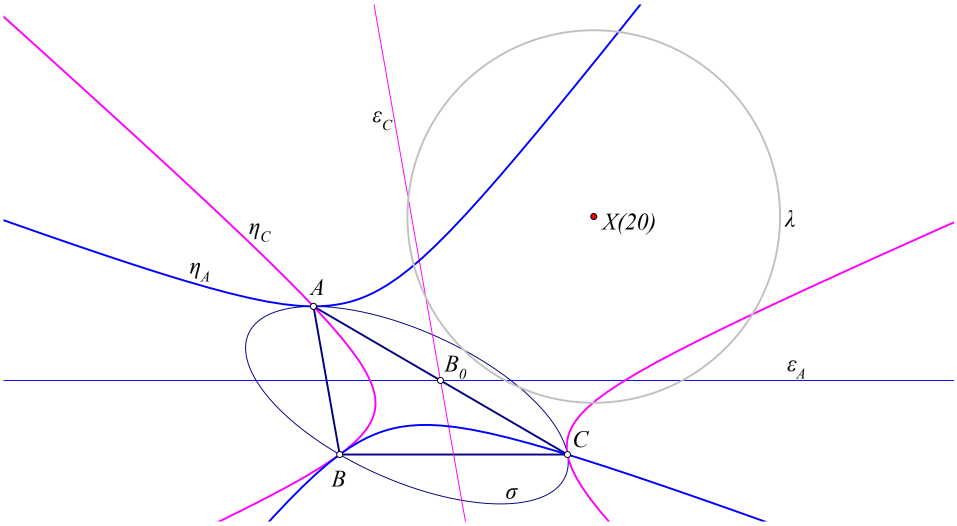

Figure 7. Lines $\{\varepsilon_A, \varepsilon_C\}$ isotomic of $\{\eta_A, \eta_C\}$ and the de Longchamps circle λ

Theorem 3.2. *With the preceding notation and conventions, the obtuse triangle ABC admits a four inconics group containing a parabola and a rectangular hyperbola, if and only if one of the lines $\{\varepsilon_A, \varepsilon_C\}$ intersects the de Longchamps circle λ. Depending on the number of intersections of this circle with the two lines, we may have 0 to 4 such inconics groups.*

Proof. In fact, assume that there is a four inconics group containing a parabola and a rectangular hyperbola. Then, as we noticed at the end of section 1, the hyperbola will have a branch contained in the obtuse angle, which we may assume to be $\widehat{B}$. Then the parabola will be contained to one of the other angles. The isotomic conjugate B^u of the perspector B^* of this hyperbola will be a point of the de Longchamps circle λ. Since the parabola and the rectangular hyperbola are members of the same four inconics group, their perspectors, $\{C^*, B^*\}$ say, will lie, C^* on the Steiner ellipse and B^*, in one of the hyperbolas, η_A say. It follows, that the isotomic B^u of B^* will be a point of the line ε_A and since B^*, by assumption, defines a rectangular hyperbola, B^u will be also a point of the de Longchamps circle λ. This shows the necessity of the condition. The argument though can be reversed and shows also the sufficiency. □

4. Some additional properties

Figure 8 shows the pair of a parabola and a rectangular hyperbola belonging to a four inconics group. By theorem 2.1, the perspector B^* of the hyperbola κ_B has its isotomic conjugate B^u at an intersection of the de Longchamps circle λ and the line ε_A, parallel to BC through the middle B_0 of AC. Point C^* is a harmonic associate of B^* and lies on the Steiner ellipse σ. It is also the perspector of a parabola κ_C of the four inconics group defined by the rectangular hyperbola κ_B.

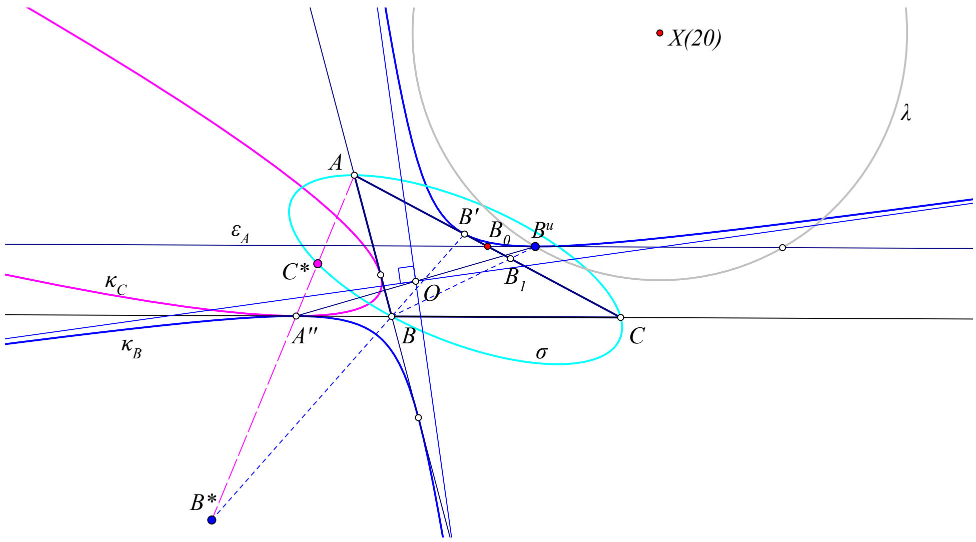

Figure 8. Pair of rectangular hyperbola and parabola defined by $B^u \in \lambda \cap \varepsilon_A$

Theorem 4.1. *With the preceding notation and conventions, an intersection point B^u of the de Longchamps circle λ and one of the lines $\{\varepsilon_A, \varepsilon_C\}$, ε_A say, lies on the rectangular hyperbola κ_B whose perspector is the isotomic conjugate B^* of B^u and its tangent there is the line ε_A.*

Proof. In fact, a parameterization of the line ε_A is given by $\{\ (1 : t : (1-t))\ ,\ t \in \mathbb{R}.\}$ The corresponding inconic given by equation (2.1) for $(L : M : N) = (1 : t : (1-t))$ is

$$f(x,y,z) \ = \ (z+y)^2t^2 - 2(z^2+yz-xz+xy)t + (z-x)^2 \ = \ 0\,.$$

It is readily seen that the equation is satisfied for $B^u = (1 : t : (1-t))$ and its tangent there is expressed through

$$2t(1-t)(-x+y+z) \ = \ 0\,,$$

coinciding with line ε_A. □

Corollary 4.1. *With the preceding notation and conventions, the center O of the hyperbola coincides with the middle of the segment $A''B^u$.*

Proof. Since ε_A and BC are parallel tangents to the hyperbola, its center lies on the middle of the segment joining the corresponding contact points. □

Notice that a four inconics group cannot contain more than one parabola. Since if it contained two parabolas, then the corresponding perspectors, $\{A^*, C^*\}$ say, would lie on the Steiner ellipse, which contains also the collinear to them vertex B of the triangle of reference ABC, i.e. the ellipse σ would intersect a line in three distinct points, which is impossible.

In figure 8 we notice also an other phenomenon. Since $\{B^u, B^*\}$ are isotomic conjugate, points $\{B', B_1\}$ are symmetric w.r.t. the middle B_0 of AC, point B' being the contact point of κ_B with AC and B_1 the intersection $B_1 = AC \cap BB''$.

Regarding the number of four inconics groups of a given obtuse triangle, we have to count the intersections of the lines $\{\varepsilon_A, \varepsilon_C\}$ with the de Longchamps circle λ. Replacing the parameterizations of the lines

$$\varepsilon_A \ : \ (1+t, 1, t) \quad \text{and} \quad \varepsilon_C \ : \ (1, t, 1+t) \, ,$$

in equation (2.2) representing the de Longchamps circle. This leads to two equations in t :

$$(2(a^2+c^2)-b^2)t^2+(3a^2+b^2+c^2)t+(2(a^2+b^2)-c^2) \ = \ 0\, , \qquad (4.1)$$

$$(2(b^2+c^2)-a^2)t^2+(a^2+b^2+3c^2)t+(2(a^2+c^2)-b^2) \ = \ 0\, , \qquad (4.2)$$

which determine the points on the de Longchamps circle defining the four inconics groups of the triangle. The discriminants of the quadratic polynomials are respectively

$$D_1 \ = \ 9b^4-2(a^2+9c^2)b^2+(c^2-a^2)(7a^2+9c^2)\, , \qquad (4.3)$$

$$D_2 \ = \ 9b^4-2(9a^2+c^2)b^2+(a^2-c^2)(9a^2+7c^2)\, , \qquad (4.4)$$

and their simultaneous vanishing leads to a special triangle. In fact their difference is

$$D_1 - D_2 \ = \ 16(c^2-a^2)(a^2+c^2-b^2)\, ,$$

and its vanishing, taking into account that our triangle ABC is obtuse, is only possible, if $a = c$, i.e. the triangle is isosceles. Replacing this into equation (4.3) we find the relations

$$b \ = \ \frac{2\sqrt{5}}{3}a \quad \text{and} \quad a = c \quad \Rightarrow \quad \cos\widehat{B} = -\frac{1}{9}\, .$$

This leads to the following theorem.

Theorem 4.2. *The only triangle for which both lines* $\{\varepsilon_A, \varepsilon_C\}$ *are tangent to the de Longchamps circle* λ *is the isosceles* ABC*, whose apex angle* $\widehat{B}$ *has* $\cos(\widehat{B}) = -\frac{1}{9}$.

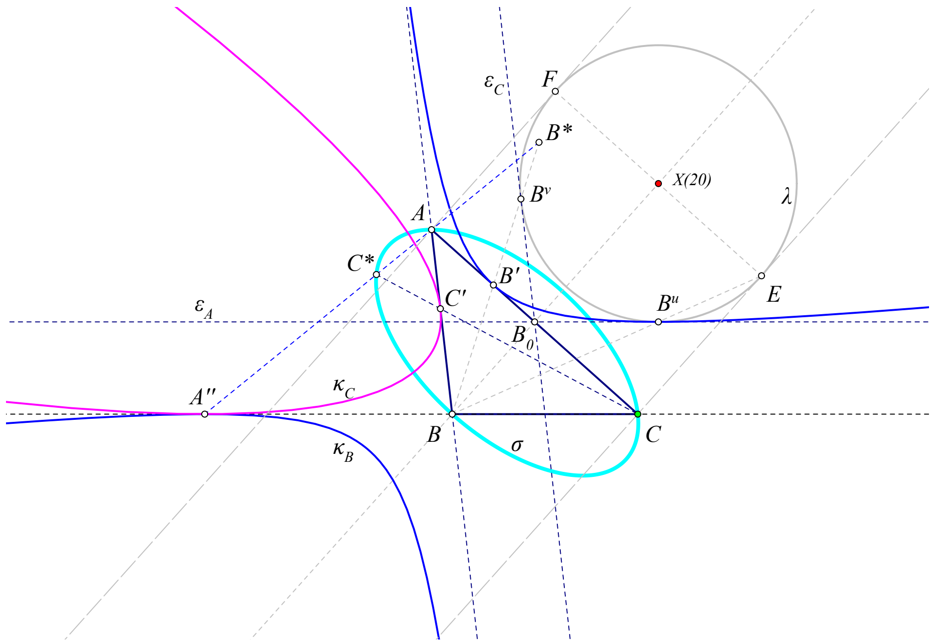

Figure 9. An exceptional isosceles with $\cos(\widehat{B}) = -\frac{1}{9}$

Figure 9 shows such an exceptional triangle and a rectangular hyperbola κ_B together with the associated parabola κ_C of the four inconics group determined by the point B^u of the de Longchamps circle, according to our discussion. The triangle possesses two such four inconics groups, the other one lying symmetrically to that suggested by the figure w.r.t. to the bisector of the obtuse angle $\widehat{B}$.

The rationality of $\cos(\widehat{B})$ implies simple relations between the distances of points and angles appearing in this configuration. Taking $|AB| = |BC| = 9$ we notice some of them, their proofs being easy exercises:

(1) $|AB| = |BC| = 9 \quad \Rightarrow \quad |AC| = 6\sqrt{5}$ and $|BB_0| = 6$.

(2) The radius of the polar circle is $\frac{3}{2}\sqrt{5}$ and its double, which is the radius of the de Longchamps circle is $3\sqrt{5}$.

(3) This implies that the de Longchamps circle λ has a diameter EF parallel and equal to AC and the tangents at its extremities pass respectively through $\{A, C\}$.

(4) The triangle BCE is isosceles and the four angles at B are equal.

(5) $\frac{|AB'|}{|B'B_0|} = \frac{3}{2}$, $\frac{|B'C|}{|CA|} = \frac{7}{10}$, $\frac{|AB|}{|BC'|} = \frac{7}{4}$, $\frac{|B^*B|}{|B^*B^v|} = \frac{24}{5}$.

REFERENCES

[1] Akopyan, A. , Zaslavsky, A. *Geometry of Conics*. American Mathematical Society, New York, (2012).

[2] Gibert, B. *Simmons Conics*. Forum Geometricorum, N. 6 (2006): 213-224.

[3] Eiden, J. *Geometrie Analytique Classique*. Calvage - Mounet, Paris, (2009).

[4] Pamfilos, P. *Barycentric coordinates*. `http://users.math.uoc.gr/~pamfilos/eGallery/problems/Barycentrics.pdf`, (2023).

[5] Yiu, P. *Introduction to the Geometry of the Triangle*. `http://math.fau.edu/Yiu/Geometry.html`, (2013).

[6] Coxeter, H. *Projective Geometry*. Springer, New York, (1987).

[7] Pamfilos, P. *A Characterization of Focals of Hyperbolas*. Elemente der Mathematik, N. 74 (2019): 131-134.

[8] Casey, J. *A treatise on the analytical geometry of the point, line, circle and conic sections*. Hodges Figgis and Co., Dublin, (1893).

[9] Kimberling, K. *Encyclopedia of Triangle Centers*. `https://faculty.evansville.edu/ck6/encyclopedia/ETC.htmll`.

[10] Patrascu, I. , Smarandache, F. *De Longchamps' point, line and circle*. `https://www.researchgate.net/publication/262036841`.

[11] Durell, C. *A course of plane Geometry II*. MacMillan and Co. London, (1910).

[12] Gheorghe, L. *Lemoine point and the inscribed conic*. arXiv:2201.05631v1 [Math.MG], N. 1 (2022): 1-8.

ESTIAS 4
IRAKLEIO 71307
GREECE.
Email address: `pamfilos@uoc.gr`