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$N(k)$-CONTACT METRIC MANIFOLD AS $*$- GRADIENT ρ-EINSTEIN SOLITON

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ABSTRACT. The purpose of this present paper is to study $N(k)$ contact metric manifolds admits $*$-gradient ρ-Einstein soliton with the Einstein potential function f. Here, we proved that, (i) the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ and flat for $n = 1$. (ii) The manifold M is $*$-Ricci flat and (iii) the Einstein potential function f satisfies the Poisson's equation. Finally, the results are verified by an example.

1. INTRODUCTION

In 1959, $*$-Ricci tensor was introduced by S. Tachibana [17] on almost Hermitian manifolds and later T. Hamada [10] studied this on real hyper surfaces of non-flat complex space satisfied by

$$S^*(U_1, U_2) = g(Q^*U_1, U_2) = \frac{1}{2}(tr\{\phi \circ R(U_1, \phi U_2)\}), \tag{1.1}$$

for all vector fields $U_1, U_2 \in \chi(M)$, where Q^* denotes the (1,1) $*$-Ricci operator. r^* is the $*$-scalar curvature and defined as $r^* = tr(Q^*)$.

In 1982, Hamilton [11] introduced the concept of Ricci flow. This is an evolution equation for metrics on Riemannian manifold as:

$$\frac{\partial}{\partial t}g + 2S = 0,$$

g denotes the Riemannian metric, S is the Ricci tensor. A self similar solution of the Ricci flow is said to be Ricci soliton [12] (g, X, λ) and it is defined as

$$2S + \mathcal{L}_X g = 2\lambda g,$$

where, $\mathcal{L}_X g$ is the Lie derivative operator along the vector field X and r is a scalar curvature of the metric g, λ is a real constant. G. Kaimakamis and K. Panagiotidou [14] introduced the concept of $*$-Ricci soliton S^* instead of the Ricci tensor S. The Ricci soliton is said to be gradient Ricci soliton if X is a gradient of smooth function f on M which is defined as

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$$S + \nabla^2 f = \lambda g,$$

where $\nabla^2 f$ denotes Hessian of the function f.

In 2016, the concept of Einstein soliton was introduced by Catino and Mazzieri [5], which generates self similar solution of the Einstein flow

$$\frac{\partial}{\partial t} g + 2\Big(S - \frac{r}{2} g\Big) = 0,$$

and introduced the notation of gradient Einstein soliton on Riemannian manifold M as

$$S + \nabla^2 f = \lambda g + \frac{r}{2} g,$$

where r is scalar curvature and λ is a real constant.

The so called Ricci-Bourguignon flow was introduced by Bourguignon [4] in 1979 which is defined as

$$\frac{\partial}{\partial t} g + 2(S - \rho r g) = 0,$$

where $\rho(\neq 0)$ is a constant. Catino and Mazzieri [5] introduced and deeply studied of gradient ρ-Einstein soliton as

$$S + \nabla^2 f = \lambda g + \rho r g,$$

here the function f is said to be Einstein potential function. Later, Venkatesha et al. [18] studied gradient ρ-Einstein soliton on almost Kenmotsu manifolds.

Definition 1.1. [8] *An n-dimensional almost contact metric manifold is said to be* $*$*-gradient* ρ*-Einstein soliton if*

$$S^* + \nabla^2 f = \rho r^* g + \lambda g, \tag{1.2}$$

where f is the smooth function, ρ is non zero constant and λ is a real constant.

In 2021, the authors D. Dey and P. Majhi [8] have proved that if the metric of $(2n+1)$ dimensional $(k,\mu)'$-almost kenmotsu manifold admits $*$-gradient ρ-Einstein soliton, then the manifold is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. A. A. Shaikh and C. S. Bagewadi [16] have proved that $N(k)$-contact metric manifolds with certain conditions is either the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or the manifold is η-Einstein. In [9], the authors have proved that if $N(k)$- contact metric manifolds with $*$-critical point equation becomes locally isometric to $E^{n+1}(0) \times S^n(4)$ and flat for $n = 1$. The authors A. Bhattacharyya and S. Pahan [1] have studied on a class of $N(k)$-mixed generalized quasi-Einstein manifolds. Hui and Lemence [13] have studied on some classes of $N(k)$-quasi Einstein manifolds.

From the above mentioned worked, a natural question arises

Question. Does the above results holds for $N(k)$ contact metric manifolds when the manifolds admits $*$-gradient ρ-Einstein soliton?

we have tried to find the answer which needs some preliminaries given in section-2 and is demonstrated In section-3 by studying $*$-gradient ρ-Einstein soliton in the framework of $N(k)$ contact metric manifolds. Finally, we set up an example to verify our main result

2. Preliminaries

A smooth manifold M^{2n+1} has an almost contact structure (ϕ, ξ, η, g), if it admits a Reeb vector field ξ, (1,1) tensor field ϕ and a 1-form η satisfying

$$\phi^2 U_1 = -U_1 + \eta(U_1)\xi, \qquad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi U_1, \phi U_2) = g(U_1, U_2) - \eta(U_1)\eta(U_2), \tag{2.2}$$

$$g(\phi U_1, U_2) = -g(U_1, \phi U_2), \tag{2.3}$$

$$g(U_1, \xi) = \eta(U_1), \tag{2.4}$$

for all vector fields U_1, U_2 on M. For details we refer to Blair [2, 3].

If $g(\phi U_1, U_2) = d\eta(U_1, U_2)$ for all U_1, U_2 on M, then the almost contact metric structure becomes contact metric structure. h is a (1,1) tensor field and it's defined as $h = \frac{1}{2}\mathcal{L}_\xi\phi$. h is the symmetric operator and satisfies

$$h\phi + \phi h = 0, \; tr(h) = tr(\phi h) = 0, \; h\xi = 0 \tag{2.5}$$

$$\nabla_{U_1}\xi = -\phi U_1 - \phi h U_1. \tag{2.6}$$

The k-nullity distribution of a Riemannian manifold was introduced by Tanno [17], defined as

$$N(k) = \{U_3 \in T(M) : R(U_1, U_2)U_3 = k[g(U_2, U_3)U_1 - g(U_1, U_3)U_2],$$

where k is a real number and $T(M)$ is the Lie algebra for all vector fields on M. If the reeb vector field $\xi \in N(k)$, then the contact metric manifold is called $N(k)$- contact metric manifold.

An $(2n+1)$- dimensional $N(k)$- contact metric manifolds satisfies the following relations

$$h^2 = (k-1)\phi^2, \tag{2.7}$$

$$R(U_1, U_2)\xi = k[\eta(U_2)U_1 - \eta(U_1)U_2], \tag{2.8}$$

$$R(\xi, U_1)U_2 = k[g(U_1, U_2)\xi - \eta(U_2)U_1], \tag{2.9}$$

$$(\nabla_{U_1}\eta)U_2 = g(U_1 + hU_1, \phi U_2), \tag{2.10}$$

$$(\nabla_{U_1}\phi)U_2 = g(U_1 + hU_1, U_2)\xi - \eta(U_2)(U_1 + hU_1), \tag{2.11}$$

$$\begin{aligned}(\nabla_{U_1}\phi h)U_2 = \quad & [g(U_1 + hU_2) + (k-1)g(U_1, -U_2 + \eta(U_2)\xi)]\xi \\ & +\eta(U_2)[hU_1 + (k-1)(-U_1 + \eta(U_1)\xi)],\end{aligned} \tag{2.12}$$

for any vector fields U_1, U_2 on M, R is Riemannian curvature.

For further details of $N(k)$- contact manifolds, we refer the reader to see the references [6, 15].

3. MAIN RESULTS

In this portion, we consider the notation of $*$- Gradient ρ-Einstein Soliton in the framework of $N(k)$-contact metric manifolds. Before going to our main results, we now state the following lemmas which are needed for proving our results.

Lemma 3.1. *An odd dimensional contact metric manifolds satisfying the condition $R(U_1, U_2)\xi = 0$ for all U_1, U_2 is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4, i.e., $E^{n+1} \times S^n(4)$ for $n > 0$ and flat for $n = 1$.*

Lemma 3.2. *A $(2n+1)$-dimensional $N(k)$- contact metric manifold is $*$-η-Einstein and the $*$-Ricci tensor is given by*

$$S^*(U_1, U_2) = -k[g(U_1, U_2) - \eta(U_1)\eta(U_2)]. \tag{3.1}$$

Theorem 3.1. *If M be a $N(k)$-contact metric manifold with $(2n+1)$ dimensional admitting $*$-Gradient ρ-Einstein Soliton with Einstein potential f. Then,*

1. *The manifold M is locally isometric to $E^{n+1}(0) \times S^n(4)$ and flat for $n = 1$.*
2. *The manifold M is $*$-Ricci flat.*
3. *The Einstein potential f satisfies the Poisson's equation $\triangle f = (2n+1)\lambda$*

Proof. Tracing (3.1) yields

$$r^* = -2nk. \tag{3.2}$$

Substituting the values of (3.1) and (3.2) in (1.2), we get

$$\nabla^2 f(U_1, U_2) = [\lambda - 2nk\rho + k]g(U_1, U_2) - k\eta(U_1)\eta(U_2),$$

which implies that

$$\nabla_{U_1} Df = [\lambda - 2nk\rho + k]U_1 - k\eta(U_1)\xi. \tag{3.3}$$

Differentiating (3.3) covariantly along the vector field U_2 and using (2.6), we obtain

$$\begin{aligned}\nabla_{U_2}\nabla_{U_1} Df = \quad & (\lambda - 2nk\rho + k)\nabla_{U_2} U_1 - k(\nabla_{U_2}\eta(U_1))\xi \\ & -k\eta(U_1)(-\phi U_2 - \phi h U_2).\end{aligned} \tag{3.4}$$

Interchanging U_1 and U_2 in the previous equation, we have

$$\begin{aligned}\nabla_{U_1}\nabla_{U_2} Df = \quad & (\lambda - 2nk\rho + k)\nabla_{U_1} U_2 - k(\nabla_{U_1}\eta(U_2))\xi \\ & -k\eta(U_2)(-\phi U_1 - \phi h U_1).\end{aligned} \tag{3.5}$$

Also,

$$\begin{aligned}\nabla_{[U_1,U_2]} Df = \quad & (\lambda - 2nk\rho + k)(\nabla_{U_1} U_2 - \nabla_{U_2} U_1) - k\eta(\nabla_{U_1} U_2)\xi \\ & +k\eta(\nabla_{U_2} U_1)\xi\end{aligned} \tag{3.6}$$

Now, it is well known that

$$R(U_1, U_2)Df = \nabla_{U_1}\nabla_{U_2} Df - \nabla_{U_2}\nabla_{U_1} Df - \nabla_{[U_1,U_2]} Df. \tag{3.7}$$

Substituting (3.4)-(3.6) in the equation (3.7) and using (2.10), we get

$$\begin{aligned}R(U_1, U_2)Df &= k[(\nabla_{U_2}\eta)U_1 - (\nabla_{U_1}\eta)U_2]\xi \\ &\quad +k[\eta(U_2)(\phi U_1 - \phi h U_1) - \eta(U_1)(\phi U_2 + \phi h U_2)] \\ &= k[g(U_2 + hU_2, \phi U_1) - g(U_1 + hU_1, \phi U_2)]\xi \\ &\quad +k[\eta(U_2)(\phi U_1 - \phi h U_1) - \eta(U_1)(\phi U_2 + \phi h U_2)]\end{aligned} \tag{3.8}$$

Taking $U_1 = \xi$ in (3.8) we get

$$R(\xi, U_2)Df = -k[(\phi U_2 + \phi h U_2)]. \tag{3.9}$$

Taking inner product in the previous equation with U_1, we obtain

$$g(R(\xi, U_2)Df, U_1) = -k[g(\phi U_2, \phi U_1) - g(\phi h U_2, U_1)]. \tag{3.10}$$

Again since

$$\begin{aligned} g(R(\xi, U_2)Df, U_1) &= -g(R[\xi, U_2]U_1, Df) \\ &= -kg(U_1, U_2)(\xi f) + k\eta(U_1)(U_2 f) \end{aligned} \tag{3.11}$$

from (3.10) and (3.11), we have

$$-k[g(\phi U_2, \phi U_1) - g(\phi h U_2, U_1)] = -kg(U_1, U_2)(\xi f) + k\eta(U_1)(U_2 f) \tag{3.12}$$

Anti-symmetrizing the previous equation, we get

$$2kg(\phi U_1, U_2) = k\eta(U_1)(U_2 f) + k\eta(U_1)(U_2 f) \tag{3.13}$$

Replacing U_1 by ϕU_1 and U_2 by ϕU_2, we obtain

$$2kg(\phi U_1, U_2) = 0, \tag{3.14}$$

which implies $k = 0$, so, from (3.1), we get $S^*(U_1, U_2) = 0$, therefore the manifold M is $*$-Ricci flat and from (3.2) $r^* = 0$. Again, from (2.7), we get $R^*(U_1, U_2) = 0$ and from Lemma-3.1, it follows that the manifold M is locally isometric to $E^{(n+1)}(0) \times S^n(4)$ and flat for $n = 1$.

From (1.2) $\nabla^2 f = \lambda g$, which is implies $\Delta f = (2n+1)\lambda$. Hence, the Einstein potential f satisfies the poisson' s equation.
This complete the proof. □

4. Example

In [7], the authors gave an example of three dimensional $N(1-\beta^2)$ contact metric manifolds, where β is a real number. Using this example, we can calculate the followings

$$S^*(e_1, e_1) = 0, S^*(e_2, e_2) = -(1-\beta^2), S^*(e_3, e_3) = -(1-\beta^2).$$

The $*$-scalar curvature $r^* = S^*(e_1, e_1) + S^*(e_2, e_2) + S^*(e_3, e_3) = -2(1-\beta^2)$

Tracing (1.2) we have $r^* + \nabla f = 3\lambda g + 3\rho r^*$

If $\beta = 1$, then the manifold reduces to flat manifold and $\Delta f = 3\lambda$
So, our main results are verified.

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