Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.12, (2023), Issue 2, pp.237-254

HARMONICITY OF VECTOR FIELDS ON 1-DIMENSIONAL EXTENSIONS OF LORENTZIAN SOLVABLE LIE GROUPS

FERDINAND H. KOUDJO

ABSTRACT. Our paper studied the harmonicity properties of vector fields on 1-dimensional extensions of Lorentzian solvables Lie groups. From the harmonicity condition on warped products, we characterize harmonic vector fields on such extensions and we construct some examples. Finally we determine a vector fields that are harmonic as maps from an 1-dimensional extension of Lorentzian solvable group onto its tangent bundle equipped with the Sasaki metric.

1. INTRODUCTION

Let (M, g) be a smooth pseudo-Riemannian manifold. A smooth vector field X on M determines a section of the tangent bundle TM and therefore it is natural to investigate vector fields from the points of view of the corresponding maps $X : (M, g) \longrightarrow (TM, g^s)$, where (TM, g^s) is equipped with the Sasaki metric.
The energy of X is, by definition, the energy of the corresponding map. A natural and general problem is to decide which vector fields X on (M, g) qualify as "better than the rest" with regard to harmonicity properties. The most general among these criteria are the following.

(i) X is harmonic map if it is a critical point for the energy functional $E : C^\infty(M, TM) \longrightarrow \mathbb{R}$
(ii) X is harmonic section if it is a critical point for the energy functional $E \mid_{\mathfrak{X}(M)}$ restricted to maps defined by vector fields

Generally, these two predefined notions are not equivalent as shown in the case of the remark 6.5 of our work, where a harmonic vector field is not a harmonic map.
In [11], Milnor considered a special class of solvable Lie groups such that for any x, y in their Lie algebras, $[x, y]$ is a linear combination of x and y. For convenience, we call such a Lie group, a LCS Lie group. It was proved that every left-invariant Riemannian metric on such a Lie group is of constant negative sectional curvature. Then in [12] Nomizu proved that every left-invariant Lorentzian metric on a LCS Lie group is also of constant sectional curvature. However, depending on the choice of left invariant Lorentz metrics, the sign of the constant sectional curvature may be positive or negative. Later, Albuquerque generalized this result to the semi-Riemannian case in [1].

2010 *Mathematics Subject Classification.* Primary 53C15. Secondary: 58E20;53C25.
Key words and phrases. Harmonic maps, Lorentzian Lie groups, Warped product.

Harmonicity properties of vector fields on this Lorentzian solvable Lie groups have been studied in [10]. In this paper, first, we determine harmonicity conditions for vector fields on the warped product $B \times_f F$(Theorem 3.1) by using a characteristic variational condition relatively to Sasaki metric, where (B, g_B) and (F, g_F) are two pseudo-Riemannian manifolds with $f : B \longrightarrow]0, +\infty[$. We apply this to the case $B = I \subset \mathbb{R}$ and F a Lorentzian solvable Lie groups G to investigate the harmonicity properties of vector fields on $I \times_f G$. Finally we construct some examples.

2. PRELIMINARIES

The tangent bundle TM of a pseudo-Riemannian manifold (M, g) can be endowed in a natural way with a pseudo-Riemannian metric g^s, the *Sasaki metric*, depending only on the pseudo-Riemannian structure g of the base manifold M. It is uniquely determined by

$$g^s(X^h, Y^h) = g^s(X^v, Y^v) = g(X, Y) \circ \pi, \quad g^s(X^h, Y^v) = 0 \tag{2.1}$$

Let V denote a smooth vector field on M and $\{e_i\}_{i=1,\dots,n}$ a pseudo-orthonormal basis of (M, g). The energy of V is, the energy of the corresponding smooth map $V : (M, g) \longrightarrow (TM, g^s)$, that is

$$E(V) = \frac{p}{2} Vol(D) + \frac{1}{2} \int_D \|\nabla V\|^2 v_g \quad p = \sum_{i=1}^{m} g(e_i, e_i)$$

(assuming M compact; in the non-compact case, one works over relatively compact domains). The Euler–Lagrange equations characterize vector fields V defining harmonic maps as the ones whose tension field $\tau(V) = \mathrm{tr}(\nabla^2 V)$ vanishes, where

$$\tau(V) = \{\mathrm{tr}[R(\nabla_. V, V).]\}^h + \{\nabla^* \nabla V\}^v$$

and with respect to a pseudo-orthonormal local frame $\{e_1, \dots, e_n\}$ on (M, g), with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices i, we have $\nabla^* \nabla V = \sum_{i=1}^{m} \varepsilon_i \{\nabla_{\nabla_{e_i} e_i} V - \nabla_{e_i} \nabla_{e_i} V\}$. Thus, V defines a harmonic map from (M, g) to (TM, g^s) if and only if

$$S(V) = \mathrm{tr}[R(\nabla_. V, V).] = 0 \quad \text{and} \quad \nabla^* \nabla V = 0. \tag{2.2}$$

A smooth vector field V is said to be a harmonic section if and only if it is a critical point of E^v where E^v is the vertical energy. The corresponding Euler-Lagrange equations are given by $\nabla^* \nabla V = 0$.

3. HARMONIC VECTOR FIELDS ON WARPED PRODUCTS

In this section, we consider $K = M \times_f N$ where $f : M \longrightarrow]0, +\infty[$, where M is m-dimensional pseudo-Riemannian manifold and N n- dimensional pseudo-Riemannian manifolds and we determine the harmonicity conditions for vector fields on K.

Let $\{e'_i\}_{i=1,\dots,m}$ be an pseudo-orthonormal basis of (M, g_M) and $\{e''_i\}_{i=1,\dots,n}$ an pseudo-orthonormal basis of (N, g_N). Then $\{e_i\}_{i=1,\dots,m+n}$ is an pseudo-orthonormal basis of (K, g) with $e_i = (e'_i, 0)$ for $i = 1, \dots, m$ and $e_{i+m} = \frac{1}{f}(0, e''_i)$ for $i = 1, \dots, n$. Hence, we obtain

Theorem 3.1. *Let (M, g_M) and (N, g_N) be a pseudo-Riemannian manifolds and $f : M \longrightarrow \mathbb{R}_+^*$ a smooth function on B. Let $\{e_i'\}_{i=1,\cdots,m}$ be an pseudo-orthonormal basis of (M, g_M) and $\{e_i''\}_{i=1,\cdots,n}$ an pseudo-orthonormal basis of (N, g_N). Then a vector field $V = V_1 + V_2$ on $K = M \times_f N$ is a harmonic vector field if and only if*

$$\begin{cases} \nabla^*\nabla V_1 - \frac{p}{f}\nabla^M_{\operatorname{grad}^M f} V_1 + \frac{2}{f}(div^N V_2)\operatorname{grad}^M f + \\ p\frac{V_1(f)}{f^2}\operatorname{grad}^M f = 0 \\ \frac{1}{f^2}\nabla^*\nabla V_2 + \frac{\Delta^M(f)}{f}V_2 + (1-p)\frac{\operatorname{grad}^M f(f)}{f^2}V_2 = 0 \end{cases} . \tag{3.1}$$

where $\Delta^M(f) = -\operatorname{tr} H^f$, $p = \sum_{i=1}^{n} \varepsilon_i$, $\varepsilon_i = g^N(e_i'', e_i'')$,

The proof of this theorem use the theorem 3.1 of [8], by considering here two pseudo Riemannian manifolds.

Assume that $M = I \subset \mathbb{R}$, $V_1 = \phi(t)\partial_t$ on I and V_2 a vector field on (N, g), then a vector field $V = V_1 + V_2$ on $K = I \times_f N$ is a harmonic vector field if and only if

$$\begin{cases} \phi''(t) + p\frac{f'(t)}{f(t)}\phi'(t) - 2\frac{f'(t)}{f(t)}\operatorname{div}(V_2) - p\frac{f'(t)^2}{f(t)^2}\phi(t) = 0 \\ \nabla^*\nabla V_2 - f(t)f''(t)V_2 + (1-p)f'(t)^2 V_2 = 0; t \in I \end{cases} \tag{3.2}$$

Next, we recall some results on the structure of LCS Lie groups and its Lie algebras.

Definition 3.1. *A n-dimensional Lorentzian LCS Lie group G $(n > 1)$ is said to be of type A if the induced metric is non degenerate when it is restricted to the commutative ideal $\mathfrak{u}$ of co-dimension 1 in Lie algebra $\mathfrak{g}$ of G. A Lorentzian LCS Lie group G is said to be of type B if the induced metric is degenerate when it is restricted to the commutative ideal $\mathfrak{u}$ of co-dimension 1 in Lie algebra $\mathfrak{g}$ of G.*

Type A: If $\mathfrak{u}$ is non degenerate, then by [12] we have the following equations:

$$\mathfrak{g} = \mathbb{R}b \oplus \mathfrak{u}, < b, \mathfrak{u} >= 0, [b, x] = x, \forall x \in \mathfrak{u}.$$

In this case, a Lie group G is said to be of type A_1 if b is timelike, and G is said to be of type A_2 if b is spacelike.
Type A_1 : b is timelike, i.e., $< b, b >= -\lambda^2$ with $\lambda > 0$. Fix an orthonormal basis $\{e_1, e_2, \cdots, e_{n-1}\}$ of $\mathfrak{u}$ and set $b = \lambda e_n$. Then we have the following equations:

$$[e_n, e_i] = \frac{1}{\lambda}e_i, \quad < e_i, e_i >= 1, \quad i = 1, \cdots, n-1, \quad < e_n, e_n >= -1 \tag{3.3}$$

Type A_2 : b is spacelike, i.e., $< b, b >= \lambda^2$ with $\lambda > 0$.Similarly as above, fix an orthonormal basis $\{e_1, e_2, \cdots, e_{n-1}$ of u and set $b = \lambda e_n$. Then we have the following equations:

$$[e_n, e_i] = \frac{1}{\lambda}e_i, \quad < e_1, e_1 >= -1, \quad < e_j, e_j >= 1 \quad i = 2, \cdots, n, \tag{3.4}$$

Type B: If $\mathfrak{u}$ is degenerate, then by [12] $\mathfrak{u}$ contains a lightlike vector c and an $(n-2)$-dimensional subspace $\mathfrak{u}_1$ on which the metric is positive definite such that $\mathfrak{u} = \mathbb{R}c \oplus \mathfrak{u}_1$ and $< c, \mathfrak{u}_1 >= 0$. Moreover, by [12]we have the following equations:

$$\mathfrak{g} = \mathbb{R}b \oplus \mathbb{R}c \oplus \mathfrak{u}_1, \quad \mathfrak{u} = \mathbb{R}c \oplus \mathfrak{u}_1, \quad < b, b >=< c, c >= 0,$$

$$< b, c >= -1, \quad < b, \mathfrak{u}_1 >=< c, \mathfrak{u}_1 >= 0, \quad [b, c] = c, \quad [b, y] = y, \quad \forall y \in \mathfrak{u}_1.$$

fix an orthonormal basis $\{e_1, e_2, \cdots, e_{n-2}\}$ of u_1 and set

$$e_{n-1} = \frac{\sqrt{2}}{2}(b - c), \quad e_n = \frac{\sqrt{2}}{2}(b + c).$$

Then we have the following equations:

$$[e_{n-1}, e_i] = [e_n, e_i] = \frac{\sqrt{2}}{2} e_i \quad \text{for} \quad i = 1, 2, \cdots, n-2; \quad [e_{n-1}, e_n] = \frac{\sqrt{2}}{2}(e_n - e_{n-1})$$

$$< e_i, e_i >= 1, \quad \text{for} \quad i = 1, \cdots, n-1, \quad < e_n, e_n >= -1 \tag{3.5}$$

4. Harmonic vector fields on $I \times G$: G is type A

Consider an n-dimensional connected simply connected Lorentzian Lie group G of type A_1. Using 3.3 and the well-known Koszul formula one can determine the Levi–Civita connection as follows:

$$\nabla_{e_i} e_i = -\frac{1}{\lambda} e_n, \nabla_{e_i} e_n = -\frac{1}{\lambda} e_i, \quad i = 1, \cdots, n-1$$

$$\nabla_{e_i} e_i = 0, \quad \text{in other case}$$

The Riemann curvature tensor is given by

$$R(e_n, e_i)e_i = -R(e_i, e_n)e_i = \frac{1}{\lambda^2} e_n, \quad R(e_n, e_i)e_n = -R(e_i, e_n)e_i = \frac{1}{\lambda^2} e_i, \quad i = 1, \cdots, n-1$$

$$R(e_i, e_j)e_k = 0 \quad \text{in other cases}$$

Given a left invariant vector fields $V = \sum_{i=1}^{n} K_i e_i$ with $\left(\sum_{i=1}^{n} K_i^2 - K_n^2 = 1 \right)$, we have

$$\nabla_{e_i} V = -\frac{1}{\lambda}(K_i e_n + K_n e_i), i = 1, \cdots, n-1 \quad \text{and} \quad \nabla_{e_n} V = 0$$

$$\nabla^* \nabla V = -\frac{1}{\lambda^2} \left((n-1) K_n e_n + \sum_{i=1}^{n-1} K_i e_i \right)$$

Also $\operatorname{div}(V_2) = -\frac{n-1}{\lambda} K_n$, Hence ,a vector field $V = V_1 + V_2$ on $I \times_f G$ is a harmonic vector field if and only if

$$\begin{cases} \phi''(t) + (n-2)\frac{f'(t)}{f(t)}\phi'(t) - (n-2)\frac{f'(t)^2}{f(t)^2}\phi(t) = -2\frac{(n-1)}{\lambda} K_n \frac{f'(t)}{f(t)} \\ -\frac{1}{\lambda^2}\left((n-1) K_n e_n + \sum_{i=1}^{n-1} K_i e_i \right) - \left(f(t) f''(t) - (3-n) f'(t)^2 \right) \left(K_n e_n + \sum_{i=1}^{n-1} K_i e_i \right) = 0; t \in I \end{cases}$$

$$\begin{cases} \phi''(t) + (n-2)\dfrac{f'(t)}{f(t)}\phi'(t) - (n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t) = -2\dfrac{(n-1)}{\lambda}K_n\dfrac{f'(t)}{f(t)} \\ K_n\left[f(t)f''(t) - (3-n)f'(t)^2 + \dfrac{n-1}{\lambda^2}\right] = 0 \\ K_i\left[f(t)f''(t) - (3-n)f'(t)^2 + \dfrac{n-1}{\lambda^2}\right] = 0, i = 1, \cdots, n-1; t \in I \end{cases}$$

Because of $\sum_{i=1}^{n-1} K_i^2 - K_n^2 = 1$, hence V is harmonic if and only if

$$\begin{cases} \phi''(t) + (n-2)\dfrac{f'(t)}{f(t)}\phi'(t) - (n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t) = -2\dfrac{(n-1)}{\lambda}K_n\dfrac{f'(t)}{f(t)} \\ f(t)f''(t) - (3-n)f'(t)^2 = -\dfrac{n-1}{\lambda^2}, t \in I \end{cases}$$

Theorem 4.1. *Let $V = \sum_{i=1}^{n} K_i e_i$ denote a left-invariant unit vector field on a group G of type A_1 where $\{e_1, \cdots, e_n\}$ be a pseudo orthonormal basis of its Lie algebra. A vector field $V = V_1 + V_2$ on the warped product $I \times_f G$, $I \subset \mathbb{R}$ $f : I \longrightarrow]0, +\infty[$ with $V_1 = \phi(t)\partial_t$ is a harmonic vector field if and only if $V = \sum_{i=1}^{n} K_i e_i$ and*

$$\begin{cases} \phi''(t) + (n-2)\dfrac{f'(t)}{f(t)}\phi'(t) - (n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t) = -2\dfrac{(n-1)}{\lambda}K_n\dfrac{f'(t)}{f(t)} \\ f(t)f''(t) - (3-n)f'(t)^2 = -\dfrac{n-1}{\lambda^2}; t \in I \end{cases}$$

Remark 4.2. *A vector field $V = V_1 + V_2$ on $I \times_f G$ with G is of type A_1 cannot be a harmonic vector field if f is a constant on $I = \mathbb{R}$*

Next we consider an n-dimensional connected simply connected Lorentzian Lie group G of type A_2. We obtain

$$\nabla_{e_1}e_1 = -\frac{1}{\lambda}e_n, \nabla_{e_i}e_n = -\frac{1}{\lambda}e_i, \quad i = 1, \cdots, n-1$$

$$\nabla_{e_i}e_i = \frac{1}{\lambda}e_n, \quad i = 2, \cdots, n-1$$

$$\nabla_{e_i}e_j = 0, \quad \text{in other case}$$

The Riemann curvature tensor is given by

$$R(e_i, e_1)e_1 = -R(e_i, e_n)e_n = \frac{1}{\lambda^2}e_i, \quad R(e_1, e_n)e_1 = -R(e_i, e_n)e_i = -\frac{1}{\lambda^2}e_n,$$

$$R(e_1, e_n)e_n = -R(e_i, e_1)e_i = -\frac{1}{\lambda^2}e_1 \quad i = 2, 3, \cdots, n-1$$

$$R(e_i, e_j)e_k = 0 \quad \text{in other cases}$$

$$\nabla_{e_1}V = -\frac{1}{\lambda}(K_1e_n + K_ne_1) \quad \nabla_{e_n}V = 0, \quad \nabla_{e_i}V = -\frac{1}{\lambda}(K_ne_i - K_ie_n) \quad \forall i = 2, \cdots, n-1$$

$$\nabla^*\nabla V = \frac{1}{\lambda^2}\left((n-1)K_n e_n + \sum_{i=1}^{n-1} K_i e_i\right) \text{ and } \operatorname{div}(V) = -\frac{n-1}{\lambda}K_n$$

A vector field V on $M \times_f N$ is a harmonic vector field if and only if

$$\begin{cases} \phi''(t) + (n-2)\frac{f'(t)}{f(t)}\phi'(t) - (n-2)\frac{f'(t)^2}{f(t)^2}\phi(t) = -2\frac{n-1}{\lambda}K_n\frac{f'(t)}{f(t)} \\ \frac{1}{\lambda^2}\left((n-1)K_n e_n + \sum_{i=1}^{n-1} K_i e_i\right) - \left(f(t)f''(t) - (3-n)f'(t)^2\right)\left(K_n e_n + \sum_{i=1}^{n-1} K_i e_i\right) = 0; t \in I \end{cases}$$

$$\begin{cases} \phi''(t) + (n-2)\frac{f'(t)}{f(t)}\phi'(t) - (n-2)\frac{f'(t)^2}{f(t)^2}\phi(t) = -2\frac{(n-1)}{\lambda}K_n\frac{f'(t)}{f(t)} \\ K_n\left[f(t)f''(t) - (3-n)f'(t)^2 - \frac{n-1}{\lambda^2}\right] = 0 \\ K_i\left[f(t)f''(t) - (3-n)f'(t)^2 - \frac{1}{\lambda^2}\right] = 0, i = 1, \cdots, n-1; t \in I \end{cases}$$

If $K_n = 0$, Hence $\sum_{i=1}^{n-1} K_i^2 = 1$ and V is harmonic if and only if

$$\begin{cases} \phi''(t) + (n-2)\frac{f'(t)}{f(t)}\phi'(t) - (n-2)\frac{f'(t)^2}{f(t)^2}\phi(t) = 0 \\ f(t)f''(t) - (3-n)f'(t)^2 = \frac{1}{\lambda^2}; t \in I \end{cases}$$

If $K_n \neq 0$, V is harmonic if and only if

$$\begin{cases} \phi''(t) + (n-2)\frac{f'(t)}{f(t)}\phi'(t) - (n-2)\frac{f'(t)^2}{f(t)^2}\phi(t) = -2\frac{(n-1)}{\lambda}K_n\frac{f'(t)}{f(t)} \\ f(t)f''(t) - (3-n)f'(t)^2 = \frac{n-1}{\lambda^2} \\ K_i\frac{2-n}{\lambda^2} = 0, \quad i = 2, \cdots, n-1; t \in I \end{cases}$$

$$\begin{cases} \phi''(t) + (n-2)\frac{f'(t)}{f(t)}\phi'(t) - (n-2)\frac{f'(t)^2}{f(t)^2}\phi(t) = -2\frac{(n-1)}{\lambda}K_n\frac{f'(t)}{f(t)} \\ f(t)f''(t) - (3-n)f'(t)^2 = \frac{n-1}{\lambda^2} \\ K_i(2-n) = 0, \quad i = 2, \cdots, n-1; t \in I \end{cases}$$

hence

$$\begin{cases} V_2 = K_1 e_1 + K_2 e_2, K_2 \neq 0 \\ f(t)f''(t) + f'(t)^2 = \frac{1}{\lambda^2} \\ f(t)\lambda\phi''(t) = -2K_2 f'(t); t \in I \end{cases}$$

or

$$\begin{cases} V_2 = K_n e_n, K_n \neq 0 \\ \phi''(t) + (n-2)\dfrac{f'(t)}{f(t)}\phi'(t) - (n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t) = -2\dfrac{(n-1)}{\lambda}K_n\dfrac{f'(t)}{f(t)} \\ f(t)f''(t) - (3-n)f'(t)^2 = \dfrac{n-1}{\lambda^2}; t \in I \end{cases}$$

Theorem 4.3. *Let $V = \sum_{i=1}^{n} K_i e_i$ denote a left-invariant unit vector field on a group G of type A_2 where $\{e_1, \cdots, e_n\}$ be an pseudo orthonormal basis of its Lie algebra. A vector field $V = V_1 + V_2$ on the warped product $I \times_f G$, $I \subset \mathbb{R}$ $f : I \longrightarrow]0, +\infty[$ with $V_1 = \phi(t)\partial_t$ is a harmonic vector field if and only if*

$$\begin{cases} V_2 = K_1 e_1 + K_2 e_2, K_2 \neq 0 \\ f(t)f''(t) + f'(t)^2 = \dfrac{1}{\lambda^2} \\ f(t)\lambda\phi''(t) = -2K_2 f'(t), t \in I \end{cases} \quad or \quad \begin{cases} V_2 = K_n e_n, K_n \neq 0 \\ \phi''(t) + (n-2)\dfrac{f'(t)}{f(t)}\phi'(t) - \\ (n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t) = -2\dfrac{(n-1)}{\lambda}K_n\dfrac{f'(t)}{f(t)} \\ f(t)f''(t) - (3-n)f'(t)^2 = \dfrac{n-1}{\lambda^2}, t \in I \end{cases}$$

Remark 4.4. *A vector field $V = V_1 + V_2$ on $I \times_f G$ with G is of type A_2 cannot be a harmonic vector field if f is a constant on $I = \mathbb{R}$*

5. Harmonic vector fields on $I \times_f G$: G is type B

Consider an n-dimensional simply connected Lorentzian Lie group G of the type B. We obtain

$$\nabla_{e_i} e_i = \frac{\sqrt{2}}{2}(e_{n-1} - e_n), \nabla_{e_i} e_{n-1} = \nabla_{e_i} e_n = -\frac{\sqrt{2}}{2} e_i \quad i = 1, 2 \cdots, n-2$$

$$\nabla_{e_{n-1}} e_{n-1} = \nabla_{e_n} e_{n-1} = -\frac{\sqrt{2}}{2} e_n, \nabla_{e_n} e_{n-1} = \nabla_{e_n} e_n = -\frac{\sqrt{2}}{2} e_{n-1},$$

$$\nabla_{e_i} e_j = 0 \quad \text{in all other cases}$$

$$\forall i, j, k \in \{1, 2, 3, \cdots, n\}, \quad R(e_i, e_j)e_k = 0$$

For an arbitrary left invariant vector field $V = \sum_{i=1}^{n} K_i e_i$, we have

$$\nabla_{e_i} V = \frac{\sqrt{2}}{2} K_i (e_{n-1} - e_n) - \frac{\sqrt{2}}{2}(K_{n-1} + K_n) e_i, \quad i = 1, 2 \cdots, n-2$$

$$\nabla_{e_{n-1}} V = \nabla_{e_n} V = \frac{\sqrt{2}}{2}(K_{n-1} e_n - K_n e_{n-1})$$

We now determine $\nabla_{e_i} \nabla_{e_i} V$ and $\nabla_{\nabla_{e_i} e_i} V$ for all indices i. We obtain

$$\nabla_{e_i} \nabla_{e_i} V = \frac{1}{2}(K_{n-1} + K_n)(e_{n-1} - e_n), \quad i = 1, \cdots, n-2$$

$$\nabla_{e_{n-1}}\nabla_{e_{n-1}}V = \nabla_{e_n}\nabla_{e_n}V = \frac{1}{2}(K_{n-1}e_{n-1} + K_n e_n)$$
$$\nabla_{\nabla_{e_i}e_i}V = 0, \quad i = 1, 2, \cdots, n-2$$
$$\nabla_{\nabla_{e_{n-1}}e_{n-1}}V = \nabla_{\nabla_{e_n}e_n}V = \frac{1}{2}(K_{n-1}e_n + K_n e_{n-1})$$

Thus we get $\nabla^*\nabla V = \frac{n-2}{2}(K_{n-1} + K_n)(e_n + e_{n-1})$.
We have $\operatorname{div}(V_2) = -\frac{\sqrt{2}}{2}(n-1)(K_n + K_{n-1})$
Then a vector field V on $I \times_f G$ is a harmonic vector field if and only if

$$\begin{cases} \phi''(t) + (n-2)\frac{f'(t)}{f(t)}\phi'(t) - (n-2)\frac{f'(t)^2}{f(t)^2}\phi(t) = -\sqrt{2}(n-1)(K_n + K_{n-1})\frac{f'(t)}{f(t)} \\ \frac{n-2}{2}(K_n + K_{n-1})(e_{n-1} + e_n) = \Big(f(t)f''(t) + (n-3)f'(t)^2\Big)\Big(K_{n-1}e_{n-1} + K_n e_n + \sum_{i=1}^{n-2} K_i e_i\Big) \\ t \in I \end{cases}$$

$$\begin{cases} \phi''(t) + (n-2)\frac{f'(t)}{f(t)}\phi'(t) - (n-2)\frac{f'(t)^2}{f(t)^2}\phi(t) = -\sqrt{2}(n-1)(K_n + K_{n-1})\frac{f'(t)}{f(t)} \\ \frac{n-2}{2}(K_n + K_{n-1}) - \Big(f(t)f''(t) + (n-3)f'(t)^2\Big)K_{n-1} = 0 \\ \frac{n-2}{2}(K_n + K_{n-1}) - \Big(f(t)f''(t) + (n-3)f'(t)^2\Big)K_n = 0 \\ K_i\Big(f(t)f''(t) + (n-3)f'(t)^2\Big) = 0, i = 1, \cdots, n-2, t \in I \end{cases}$$

Suppose that $f(t)f''(t) + (n-3)f'(t)^2 = 0$, then $(n-2)(K_n + K_{n-1}) = 0$

- If $n = 2$, we have $\begin{cases} f(t)f''(t) = f'(t)^2 \\ \phi''(t) = -\sqrt{2}(K_1 + K_2)\frac{f'(t)}{f(t)} \\ V = K_1 e_1 + K_2 e_2, t \in I \end{cases}$
- If $K_n = -K_{n-1}, n \geq 3$, we have $\begin{cases} \phi''(t) + (n-2)\frac{f'(t)}{f(t)}\phi'(t) - (n-2)\frac{f'(t)^2}{f(t)^2}\phi(t) = 0 \\ f(t)f''(t) + (n-3)f'(t)^2 = 0 \\ V_2 = \sum_{i=1}^{n-2} K_i e_i + K_{n-1}\Big(e_{n-1} - e_n\Big), t \in I \end{cases}$

Suppose that $\forall\, i = 1, 2, \cdots, n-2$, $K_i = 0$, we have necessary $K_n \neq 0$, $K_{n-1} \neq 0$ and

$$\begin{cases} \phi''(t) + (n-2)\frac{f'(t)}{f(t)}\phi'(t) - (n-2)\frac{f'(t)^2}{f(t)^2}\phi(t) = -2\sqrt{2}(n-1)K_n\frac{f'(t)}{f(t)} \\ f(t)f''(t) + (n-3)f'(t)^2 = n-2 \\ V_2 = K_n(e_{n-1} + e_n), t \in I \end{cases}$$

Theorem 5.1. *Let $V = \sum_{i=1}^{n} K_i e_i$ denote a left-invariant unit vector field on a group G of type B where $\{e_1, \cdots, e_n\}$ be an pseudo orthonormal basis of its Lie algebra. A vector field $V = V_1 + V_2$ on the warped product $I \times_f G$, $I \subset \mathbb{R}$ $f : I \longrightarrow]0, +\infty[$ with $V_1 = \phi(t)\partial_t$ is a harmonic vector field if and only if*

(1)

$$\begin{cases} n = 2 \\ f(t)f''(t) = f'(t)^2 \\ \phi''(t) = -\sqrt{2}(K_1 + K_2)\dfrac{f'(t)}{f(t)} \\ V_2 = K_1 e_1 + K_2 e_2, t \in I \end{cases}$$

(2)

$$\begin{cases} n \geq 3, K_n = -K_{n-1} \\ \phi''(t) + (n-2)\dfrac{f'(t)}{f(t)}\phi'(t) - (n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t) = 0 \\ f(t)f''(t) + (n-3)f'(t)^2 = 0 \\ V_2 = \sum_{i=1}^{n-2} K_i e_i + K_{n-1}\Big(e_{n-1} - e_n\Big), t \in I \end{cases}$$

(3)

$$\begin{cases} \phi''(t) + (n-2)\dfrac{f'(t)}{f(t)}\phi'(t) - (n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t) = -2\sqrt{2}(n-1)K_n\dfrac{f'(t)}{f(t)} \\ f(t)f''(t) + (n-3)f'(t)^2 = n-2 \\ V_2 = K_n(e_{n-1} + e_n), t \in I \end{cases}$$

Remark 5.2. *If f is constant function on $\mathbb{R}$, then a vector field V is harmonic vector field is and only if ϕ if affine function and $(n-2)(K_n + K_{n-1}) = 0$*

6. Harmonic vectors fields which are harmonic maps on warped product $I \times_f G$

We now determine the horizontal part $S(V)$ of the tension fields on the warped product $M \times_f N$. We write $S(V) = S_1(V) + S_2(V)$ where

$$S_1(V) = \sum_{i=1}^{m} R(\nabla_{e_i} V, V)e_i \quad \text{and} \quad S_2(V) = \sum_{i=m+1}^{m+n} \epsilon_i R(\nabla_{e_i} V, V)e_i.$$

Theorem 6.1. *Let (M, g_M) and (N, g_N) be a pseudo-Riemannian manifolds and $f : M \longrightarrow \mathbb{R}_+^*$ a smooth function on M. Let $\{e'_i\}_{i=1,\cdots,m}$ be an pseudo-orthonormal basis of (M, g_M) and $\{e''_i\}_{i=1,\cdots,n}$ an pseudo-orthonormal basis of (N, g_N). Then a harmonic vector field $V = V_1 + V_2$ on $K = M \times_f N$ is a harmonic map field if and only if*

$$
\begin{cases}
S(V_1) + p\dfrac{V_1(f)}{f^2}\nabla^M_{V_1}\operatorname{grad}^M f + \|V_2\|^2\nabla^M_{\operatorname{grad}^M f}\operatorname{grad}^M f + \dfrac{1}{f}\operatorname{div}(V_2)\nabla^M_{V_1}\operatorname{grad} f = 0 \\
\dfrac{1}{f^2}S(V_2) - \sum\limits_{i=1}^{m}\left(\dfrac{e_i'(f)}{f^2}H^f(V_1, e_i') - \dfrac{1}{f}H^f(\nabla^M_{e_i'}V_1, e_i')\right)V_2 \\
+\dfrac{\operatorname{grad} f(f)}{f^2}\left(\operatorname{div}(V_2)V_2 - \nabla^N_{V_2}V_2\right) \\
+\dfrac{V_1(f)}{f^3}\sum\limits_{i=1}^{n}\varepsilon_i R^N(e_i'', V_2)e_i'' + \dfrac{V_1(f)}{f^3}\operatorname{grad}^M f(f)(p-1)V_2 = 0
\end{cases}
\tag{6.1}
$$

By Considering $M = I \subset \mathbb{R}$, $V_1 = \phi(t)\partial_t$ on I and V_2 a vector field on (N, g),

$$
\begin{aligned}
S(V) = \Bigg(& p\frac{\phi(t)^2 f'(t)f''(t)}{f(t)2}\partial_t + \|V_2\|^2 f'(t)f''(t)\partial_t + \frac{1}{f(t)}\operatorname{div}(V_2)\phi(t)f''(t)\partial_t; \\
& \frac{1}{f(t)^2}S(V_2) - \frac{f'(t)\phi(t)f''(t)}{f(t)^2}V_2 + \frac{\phi'(t)f''(t)}{f(t)}V_2 + \frac{f'(t)^2}{f(t)^2}\operatorname{div}(V_2)V_2 - \\
& \frac{f'(t)^2}{f(t)^2}\nabla_{V_2}V_2 + \frac{\phi(t)f'(t)^3}{f(t)^3}(p-1)V_2 + \frac{\phi(t)f'(t)}{f(t)^3}\sum_{i=1}^{n}R^N(e_i'', V_2)e_i''\Bigg), t \in I
\end{aligned}
$$

Then (6.1) is equivalently to,

$$
\begin{cases}
p\dfrac{\phi(t)^2 f'(t)f''(t)}{f(t)^2} + \|V_2\|^2 f'(t)f''(t) + \dfrac{1}{f(t)}\operatorname{div}(V_2)\phi(t)f''(t) = 0, t \in I \\
\dfrac{1}{f(t)^2}S(V_2) - \dfrac{f'(t)\phi(t)f''(t)}{f(t)^2}V_2 + \dfrac{\phi'(t)f''(t)}{f(t)}V_2 + \dfrac{f'(t)^2}{f(t)^2}\operatorname{div}(V_2)V_2 - \dfrac{f'(t)^2}{f(t)^2}\nabla_{V_2}V_2 + \\
\dfrac{\phi(t)f'(t)}{f(t)^3}\sum\limits_{i=1}^{n}\varepsilon_i R^N(e_i'', V_2)e_i'' + \dfrac{\phi(t)f'(t)^3}{f(t)^3}(p-1)V_2 = 0;
\end{cases}
$$

Consider $V_2 = \sum\limits_{i=1}^{n} K_i V_i$ a unit vector field on G of type A_1 and we can determine on this type of lie group the harmonic vector field which are harmonic map. We have

$$
\begin{aligned}
S(V) &= \sum_{i=1}^{n-1} R(\nabla_{e_i}V, V)e_i - R(\nabla_{e_n}V, V)e_n \\
&= -\frac{1}{\lambda^3}\left(\sum_{i=1}^{n-1} K_i^2 e_n - K_n^2 e_n\right) \\
S(V) &= -\frac{1}{\lambda^3}e_n
\end{aligned}
$$

$$\begin{aligned}
\nabla_{V_2}V_2 &= K_n\nabla_{e_n}V_2+\sum_{i=1}^{n-1}K_i\nabla_{e_i}V_2\\
&= -\frac{1}{\lambda}\sum_{i=1}^{n-1}K_i\Big(K_ie_n+K_ne_i\Big)\\
&= -\frac{1}{\lambda}\Big((1+K_n^2)e_n+K_n\sum_{i=1}^{n-1}K_ie_i\Big)\\
\sum_{i=1}^{n}\varepsilon_iR(e_i,V_2)e_i &= \sum_{i=1}^{n-1}R(e_i,V_2)e_i-R(e_n,V_2)e_n\\
&= -\frac{1}{\lambda^2}\Big[\sum_{i=1}^{n-1}K_ie_i+(n-1)K_ne_n\Big]
\end{aligned}$$

Then a vector field $V=V_1+V_2$ is harmonic map if and only if $V_2=\sum_{i=1}^{n}K_ie_i$, and

$$\begin{cases}
\lambda(n-2)f'(t)f''(t)\phi(t)^2+\lambda f(t)^2f'(t)f''(t)+(1-n)f(t)f''(t)\phi(t)K_n=0\\
\phi''(t)+(n-2)\dfrac{f'(t)}{f(t)}\phi'(t)-(n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t)=-2\dfrac{(n-1)}{\lambda}K_n\dfrac{f'(t)}{f(t)}\\
f(t)f''(t)-(3-n)f'(t)^2=-\dfrac{n-1}{\lambda^2}\\
-\dfrac{1}{\lambda^3}+\dfrac{1}{\lambda}f'(t)^2(1+K_n^2)-\dfrac{(n-1)f'(t)\phi(t)}{\lambda^2f(t)}K_n+f(t)f''(t)\phi'(t)K_n-f'(t)\phi(t)f''(t)K_n\\
-\dfrac{n-1}{\lambda}f'(t)^2K_n^2+\dfrac{\phi(t)f'(t)^3}{f(t)}(n-3)K_n=0\\
\Big(-f'(t)f''(t)\phi(t)+f(t)f''(t)\phi'(t)+\dfrac{1}{\lambda}(1-n)K_nf'(t)^2+\dfrac{1}{\lambda}K_nf'(t)^2-\\
\dfrac{\phi(t)f'(t)}{f(t)\lambda^2}+\dfrac{\phi(t)f'(t)^3}{f(t)}(n-3)\Big)K_i=0, i=1,\cdots,n-1\\
t\in I
\end{cases}$$

If $K_n=0$, we have $f'(t)f''(t)=0, \phi(t)f'(t)=0$ and $f'(t)^2=\frac{1}{\lambda^2}$(Not possible). Hence $V=V_1+V_2$ is harmonic map if and only if $V_2=\sum_{i=1}^{n}K_ie_i, K_n\neq 0$ and

$$\begin{cases} \lambda(n-2)f'(t)f''(t)\phi(t)^2 + \lambda f(t)^2 f'(t)f''(t) + (1-n)f(t)f''(t)\phi(t)K_n = 0 \\ \phi''(t) + (n-2)\dfrac{f'(t)}{f(t)}\phi'(t) - (n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t) = -2\dfrac{(n-1)}{\lambda}K_n\dfrac{f'(t)}{f(t)} \\ f(t)f''(t) - (3-n)f'(t)^2 = -\dfrac{n-1}{\lambda^2} \\ -\dfrac{1}{\lambda^3} + \dfrac{1}{\lambda}f'(t)^2(1+K_n^2) - \dfrac{(n-1)f'(t)\phi(t)}{\lambda^2 f(t)}K_n + f(t)f''(t)\phi'(t)K_n - f'(t)\phi(t)f''(t)K_n \\ -\dfrac{n-1}{\lambda}f'(t)^2K_n^2 + \dfrac{\phi(t)f'(t)^3}{f(t)}(n-3)K_n = 0 \\ \Big(-f'(t)f''(t)\phi(t) + f(t)f''(t)\phi'(t) + \dfrac{1}{\lambda}(1-n)K_nf'(t)^2 + \dfrac{1}{\lambda}K_nf'(t)^2 \\ -\dfrac{\phi(t)f'(t)}{f(t)\lambda^2} + \dfrac{\phi(t)f'(t)^3}{f(t)}(n-3)\Big)K_i = 0, i = 1, \cdots, n-1 \\ t \in I \end{cases}$$

Theorem 6.2. *Let* $V = \sum_{i=1}^{n} K_i e_i$ *denote a left-invariant unit vector field on a group* G *of type* A_1 *where* $\{e_1, \cdots, e_n\}$ *be a pseudo orthonormal basis of his Lie algebra. A vector field* $V_1 + V_2$ *on the warped product* $I \times_f G$, $I \subset \mathbb{R}$ $f : I \longrightarrow]0, +\infty[$ *with* $V_1 = \phi(t)\partial_t$ *is a harmonic map if and only if*
$V_2 = \sum_{i=1}^{n} K_i e_i, K_n \neq 0$ *and*

$$\begin{cases} \lambda(n-2)f'(t)f''(t)\phi(t)^2 + \lambda f(t)^2 f'(t)f''(t) + (1-n)f(t)f''(t)\phi(t)K_n = 0 \\ \phi''(t) + (n-2)\dfrac{f'(t)}{f(t)}\phi'(t) - (n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t) = -2\dfrac{(n-1)}{\lambda}K_n\dfrac{f'(t)}{f(t)} \\ f(t)f''(t) - (3-n)f'(t)^2 = -\dfrac{n-1}{\lambda^2} \\ -\dfrac{1}{\lambda^3} + \dfrac{1}{\lambda}f'(t)^2(1+K_n^2) - \dfrac{(n-1)f'(t)\phi(t)}{\lambda^2 f(t)}K_n + f(t)f''(t)\phi'(t)K_n - f'(t)\phi(t)f''(t)K_n \\ -\dfrac{n-1}{\lambda}f'(t)^2K_n^2 + \dfrac{\phi(t)f'(t)^3}{f(t)}(n-3)K_n = 0 \\ \Big(-f'(t)f''(t)\phi(t) + f(t)f''(t)\phi'(t) + \dfrac{1}{\lambda}(1-n)K_nf'(t)^2 + \\ \dfrac{1}{\lambda}K_nf'(t)^2 - \dfrac{\phi(t)f'(t)}{f(t)\lambda^2} + \dfrac{\phi(t)f'(t)^3}{f(t)}(n-3)\Big)K_i = 0, i = 1, \cdots, n-1 \\ t \in I \end{cases}$$

Consider $V_2 = \sum_{i=1}^{n} K_i V_i$ a unit vector field on G of type A_2 and we can determine on this type of lie group the harmonic vector field which are harmonic map. We have

$$S(V) \quad = \quad -R(\nabla_{e_1}V, V)e_1 + \sum_{i=2}^{n-1} R(\nabla_{e_i}V, V)e_i \quad \text{because} \quad \nabla_{e_n}V = 0$$

i) $R(e_n, V)e_1 = \frac{1}{\lambda^2} K_1 e_n$ and $R(e_1, V)e_1 = -\frac{1}{\lambda^2}\left(K_n e_n + \sum_{i=2}^{n-1} K_i e_i\right)$ Hence

$$R(\nabla_{e_1} V, V)e_1 = \frac{1}{\lambda^3}\left((K_n^2 - K_1^2)e_n + K_n \sum_{i=2}^{n-1} K_i e_i\right)$$

ii) $R(e_i, V)e_i = \frac{1}{\lambda^2}(K_1 e_1 + K_n e_n)$ and $R(e_n, V)e_i = 0$ Hence

$$R(\nabla_{e_i} V, V)e_i = \frac{1}{\lambda^3} K_n (K_1 e_1 + K_n e_n)$$

Then

$$S(V) = \frac{1}{\lambda^3}\left((n-2)K_n K_1 e_1 - K_n \sum_{i=2}^{n-1} K_i e_i\right)$$

$$\nabla_{V_2} V_2 = \frac{1}{\lambda}\left((1 + K_n^2)e_n - K_n \sum_{i=1}^{n-1} K_i e_i\right)$$

$$\begin{aligned}\sum_{i=1}^{n} \varepsilon_i R(e_i, V_2)e_i &= -R(e_1, V_2)e_1 + \sum_{i=2}^{n-1} R(e_i, V_2)e_i + R(e_n, V_2)e_n \\ &= \frac{1}{\lambda^2}\left[(n-1)K_1 e_1 - (n-3)K_n e_n + \sum_{i=2}^{n-1} K_i e_i\right]\end{aligned}$$

Using, $V = V_1 + V_2$ is harmonic map if and only if additional to harmonic vector condition , we have

$$\begin{cases} \lambda(n-2)f'(t)f''(t)\phi(t)^2 + \lambda f(t)^2 f'(t)f''(t) + (1-n)f(t)f''(t)\phi(t)K_n = 0 \\ K_1\Big(-\frac{n-2}{\lambda}K_n f'(t)^2 + \frac{\phi(t)f'(t)}{\lambda^2 f(t)}(n-1) + \\ f(t)f''(t)\phi'(t) - f'(t)\phi(t)f''(t) + \frac{1}{\lambda^3} + \frac{\phi(t)f'(t)^3}{f(t)}(n-3)\Big) = 0 \\ K_i\Big(\frac{2-n}{\lambda}K_n f'(t)^2 + \frac{\phi(t)f'(t)}{\lambda^2 f(t)} + f(t)f''(t)\phi'(t) - f'(t)\phi(t)f''(t) + \\ \frac{1}{\lambda^3} + \frac{\phi(t)f'(t)^3}{f(t)}(n-3)\Big) = 0, i = 2, \cdots, n-1 \\ \Big(-\frac{1+K_n^2}{\lambda}f'(t)^2 - \frac{\phi(t)f'(t)}{\lambda^2 f(t)}(n-3)K_n + f(t)f''(t)\phi'(t)K_n - \\ f'(t)\phi(t)f''(t)K_n + \frac{1}{\lambda^3}(n-1)K_n + \frac{\phi(t)f'(t)^3}{f(t)}(n-3)K_n\Big) = 0, t \in I \end{cases}$$

that is equivalently to,

$$\begin{cases} V_2 = K_2 e_2, K_2 \neq 0 \\ f(t)f''(t) + f'(t)^2 = \dfrac{1}{\lambda^2} \\ f(t)\lambda\phi''(t) = 2K_2 f'(t) \\ \lambda f(t)^2 f'(t) f''(t) - f(t)f''(t)\phi(t)K_2 = 0 \\ -\dfrac{2}{\lambda} f'(t)^2 + \dfrac{\phi f'(t)}{\lambda^2 f(t)} K_2 + f(t)f''(t)\phi'(t)K_2 - f'(t)\phi(t)f''(t)K_2 \\ +\dfrac{1}{\lambda^3} K_2 - \frac{\phi(t)f'(t)^3}{f(t)} K_2 = 0, t \in I \end{cases}$$

or

$$\begin{cases} V_2 = K_n e_n, K_n \neq 0 \\ \phi''(t) + (n-2)\dfrac{f'(t)}{f(t)}\phi'(t) - (n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t) = -2\dfrac{(n-1)}{\lambda} K_n \dfrac{f'(t)}{f(t)} \\ f(t)f''(t) - (3-n)f'(t)^2 = \dfrac{n-1}{\lambda^2} \\ \lambda(n-2)f'(t)f''(t)\phi^2(t) + \lambda f^2(t)f'(t)f''(t) + (1-n)f(t)f''(t)\phi(t)K_n = 0 \\ -\dfrac{1+K_n^2}{\lambda} f'(t)^2 - \dfrac{\phi(t)f'(t)}{\lambda^2 f(t)}(n-3)K_n + f(t)f''(t)\phi'(t)K_n - \\ f'(t)\phi(t)f''(t)K_n + \dfrac{1}{\lambda^3}(n-1)K_n + \dfrac{\phi(t)f'(t)^3}{f(t)}(n-3)K_n = 0, t \in I \end{cases}$$

Theorem 6.3. *Let $V = \sum_{i=1}^{n} K_i e_i$ denote a left-invariant unit vector field on a group G of type A_2 where $\{e_1, \cdots, e_n\}$ be an pseudo orthonormal basis of its Lie algebra. A vector field $V = V_1 + V_2$ on the warped product $I \times_f G$, $I \subset \mathbb{R}$ $f : I \longrightarrow]0, +\infty[$ with $V_1 = \phi(t)\partial_t$ is a harmonic vector field if and only if*

$$\begin{cases} V_2 = K_2 e_2, K_2 \neq 0 \\ f(t)f''(t) + f'(t)^2 = \dfrac{1}{\lambda^2} \\ f(t)\lambda\phi''(t) = -2K_2 f'(t) \\ \lambda f(t)^2 f'(t) f''(t) - f(t)f''(t)\phi(t)K_2 = 0 \\ -\dfrac{2}{\lambda} f'(t)^2 + \dfrac{\phi(t) f'(t)}{\lambda^2 f(t)} K_2 + f(t)f''(t)\phi'(t)K_2 - \\ f'(t)\phi(t)f''(t)K_2 + \dfrac{1}{\lambda^3} K_2 - \frac{\phi(t)f'(t)^3}{f(t)} K_2 = 0, t \in I \end{cases}$$

or

$$\begin{cases} V_2 = K_n e_n, K_n \neq 0 \\ \phi''(t) + (n-2)\dfrac{f'(t)}{f(t)}\phi'(t) - (n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t) = -2\dfrac{(n-1)}{\lambda}K_n\dfrac{f'(t)}{f(t)} \\ f(t)f''(t) - (3-n)f'(t)^2 = \dfrac{n-1}{\lambda^2} \\ \lambda(n-2)f'(t)f''(t)\phi(t)^2 + \lambda f(t)^2 f'(t)f''(t) + (1-n)f(t)f''(t)\phi(t)K_n = 0 \\ -\dfrac{1+K_n^2}{\lambda}f'(t)^2 - \dfrac{\phi(t)f'(t)}{\lambda^2 f(t)}(n-3)K_n + f(t)f''(t)\phi'(t)K_n - f'(t)\phi(t)f''(t)K_n + \\ \dfrac{1}{\lambda^3}(n-1)K_n + \dfrac{\phi(t)f'(t)^3}{f(t)}(n-3)K_n = 0, t \in I \end{cases}$$

Ended, we can determine on the type B of lie group G the harmonic vector field which are harmonic map. We have

$$S(V) = \sum_{i=1}^{n} \varepsilon_i R(\nabla_{e_i} V, V)e_i = 0.$$

$$\begin{aligned} \nabla_{V_2} V_2 &= -\frac{\sqrt{2}}{2}(K_{n-1} + K_n)\sum_{i=1}^{n-2} K_i e_i + \frac{\sqrt{2}}{2}\left(\sum_{i=1}^{n-2} K_i^2 - K_{n-1}K_n - K_n^2\right)e_{n-1} - \\ &\quad \frac{\sqrt{2}}{2}\left(\sum_{i=1}^{n-2} K_i^2 + K_{n-1}K_n + K_{n-1}^2\right)e_n \\ &= -\frac{\sqrt{2}}{2}(K_{n-1} + K_n)\sum_{i=1}^{n-2} K_i e_i + \frac{\sqrt{2}}{2}\left(1 - K_{n-1}K_n - K_{n-1}^2\right)e_{n-1} - \\ &\quad \frac{\sqrt{2}}{2}\left(1 + K_{n-1}K_n + K_n^2\right)e_n \end{aligned}$$

also

$$\sum_{i=1}^{n} \varepsilon_i R(e_i, V_2)e_i = S(V_2) = 0$$

Hence $V = V_1 + V_2$ is harmonic map if and only if additional to harmonic vector condition, we have

$$\begin{cases} (n-2)\phi(t)^2 f'(t)f''(t) + f(t)^2 f'(t)f''(t) - \dfrac{\sqrt{2}}{2}f(t)\phi(t)f''(t)(n-1)(K_n + K_{n-1}) = 0 \\ \left(-f'(t)\phi(t)f''(t) + f(t)\phi'(t)f''(t) - \dfrac{\sqrt{2}}{2}(n-1)(K_n + K_{n-1})f'(t)^2 + (n-3)\dfrac{\phi(t)f'(t)^3}{f(t)}\right)V_2 + \\ \dfrac{\sqrt{2}}{2}(K_{n-1} + K_n)\displaystyle\sum_{i=1}^{n-2} K_i e_i + \dfrac{\sqrt{2}}{2}\left(1 - K_{n-1}K_n - K_{n-1}^2\right)e_{n-1} - \dfrac{\sqrt{2}}{2}\left(1 + K_{n-1}K_n + K_n^2\right)e_n = 0 \\ t \in I \end{cases}$$

that is equivalently to
case 1:

$$\begin{cases} n=2, f(t)f''(t)=f'(t)^2 \\ \phi''(t)=-\sqrt{2}(K_1+K_2)\dfrac{f'(t)}{f(t)} \\ V_2=K_1e_1+K_2e_2 \\ \left[f(t)^2f'(t)-\dfrac{\sqrt{2}}{2}f(t)\phi(t)(K_1+K_2)\right]f''(t)=0 \\ \left[-2f'(t)\phi(t)+f(t)\phi'(t)-\dfrac{\sqrt{2}}{2}f(t)(K_1+K_2)\right]K_2f''(t)+\dfrac{\sqrt{2}}{2}f(t)f''(t)(1+K_2^2+K_1K_2)=0 \\ \left[-2f'(t)\phi(t)+f(t)\phi'(t)-\dfrac{\sqrt{2}}{2}f(t)(K_1+K_2)\right]K_1f''(t)-\dfrac{\sqrt{2}}{2}f(t)f''(t)(1-K_1^2-K_1K_2)=0 \\ t\in I \end{cases}$$

case 2:

$$\begin{cases} n\geq 3, K_n=-K_{n-1} \\ \phi''(t)+(n-2)\dfrac{f'(t)}{f(t)}\phi'(t)-(n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t)=0 \\ f(t)f''(t)(f(t)^2+(n-2)\phi(t)^2)=0 \\ \left(-f'(t)\phi(t)f''(t)+f(t)\phi'(t)f''(t)+(n-3)\phi(t)\dfrac{f'(t)^3}{f(t)}\right)K_i=0, \forall i=1,\cdots,n-2 \\ \left(-f'(t)\phi(t)f''(t)+f(t)\phi'(t)f''(t)+(n-3)\phi(t)\dfrac{f'(t)^3}{f(t)}\right)K_n+\dfrac{\sqrt{2}}{2}f'(t)^2=0 \\ -\left(-f'(t)\phi(t)f''(t)+f(t)\phi'(t)f''(t)+(n-3)\phi(t)\dfrac{f'(t)^3}{f(t)}\right)K_n-\dfrac{\sqrt{2}}{2}f'(t)^2=0 \\ V_2=\sum_{i=1}^{n-2}K_ie_i-K_n(e_{n-1}-e_n) \\ t\in I \end{cases}$$

that is equivalently to

$$\forall t\in I, f'(t)=0, \phi''(t)=0, n\geq 3, K_n=-K_{n-1}, V_2=\sum_{i=1}^{n-2}K_ie_i+K_{n-1}(e_{n-1}-e_n)$$

Case 3:

$$\begin{cases} \phi''(t) + (n-2)\dfrac{f'(t)}{f(t)}\phi'(t) - (n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t) = -2\sqrt{2}(n-1)K_n\dfrac{f'(t)}{f(t)} \\ f(t)f''(t) + (n-3)f'(t)^2 = n-2 \\ V_2 = K_n(e_{n-1}+e_n) \\ \Big(- f'(t)\phi(t)f''(t) + f(t)\phi'(t)f''(t) - \\ \sqrt{2}(n-1)K_n f'(t)^2 + (n-3)\phi(t)\dfrac{f'(t)^3}{f(t)}\Big) + \dfrac{\sqrt{2}}{2}f'(t)^2(1+K_n^2) = 0 \\ \Big(- f'(t)\phi(t)f''(t) + f(t)\phi'(t)f''(t) - \sqrt{2}(n-1)K_n f'(t)^2 + (n-3)\phi(t)\dfrac{f'(t)^3}{f(t)}\Big) - \\ \dfrac{\sqrt{2}}{2}f'(t)^2(1-K_n^2) = 0 \\ t \in I \end{cases}$$

$$\begin{cases} \phi''(t) + (n-2)\dfrac{f'(t)}{f(t)}\phi'(t) - (n-2)\dfrac{f'(t)^2}{f(t)^2}\phi(t) = -2\sqrt{2}(n-1)K_n\dfrac{f'(t)}{f(t)} \\ f(t)f''(t) + (n-3)f'(t)^2 = n-2 \\ V_2 = K_n(e_{n-1}+e_n) \\ f'(t)(1+K_n^2) = -f'(t)(1-K_n^2) \\ t \in I \end{cases}$$

So that we obtain $n = 2, I = \mathbb{R}, \forall t \in \mathbb{R}, f'(t) = 0, \phi''(t) = 0, V_2 = K_2(e_1+e_2)$

Theorem 6.4. *Let $V = \sum_{i=1}^{n} K_i e_i$ denote a left-invariant unit vector field on a group G of type B where $\{e_1, \cdots, e_n\}$ be a pseudo orthonormal basis of its Lie algebra. A vector field $V = V_1 + V_2$ on the warped product $I \times_f G$, $I \subset \mathbb{R}$, $f : I \longrightarrow]0, +\infty[$ with $V_1 = \phi(t)\partial_t$ is a harmonic vector field if and only if*

$$\begin{cases} n = 2, f(t)f''(t) = f'(t)^2 \\ \phi''(t) = -\sqrt{2}(K_1+K_2)\dfrac{f'(t)}{f(t)} \\ V_2 = K_1e_1 + K_2e_2 \\ \Big(f(t)^2f'(t) - \dfrac{\sqrt{2}}{2}f(t)\phi(t)(K_1+K_2)\Big)f''(t) = 0 \\ \Big(-2f'(t)\phi(t) + f(t)\phi'(t) - \dfrac{\sqrt{2}}{2}f(t)(K_1+K_2)\Big)K_2f''(t) + \dfrac{\sqrt{2}}{2}f(t)f''(t)(1+K_2^2+K_1K_2) = 0 \\ \Big(-2f'(t)\phi(t) + f(t)\phi'(t) - \dfrac{\sqrt{2}}{2}f(t)(K_1+K_2)\Big)K_1f''(t) - \dfrac{\sqrt{2}}{2}f(t)f''(t)(1-K_1^2-K_1K_2) = 0 \\ t \in I \end{cases}$$

or

$$\forall t \in I = \mathbb{R}, f'(t) = 0, \phi''(t) = 0, n \geq 3, K_n = -K_{n-1}, V_2 = \sum_{i=1}^{n-2} K_i e_i + K_{n-1}(e_{n-1} - e_n)$$

or

$$n = 2, \forall t \in I = \mathbb{R}, f'(t) = 0, \phi''(t) = 0, V_2 = K_2(e_1 + e_2)$$

Remark 6.5. *If f is constant on $I = \mathbb{R}$, then a harmonic vector field on $I \times_f G$ with G is type B, is harmonic map*

Proposition 6.1. *if $n = 2$, $f(t) = \frac{1}{\lambda}$, $\phi(t) = -\frac{2}{\lambda}K_2 t \ln|t|$, a vector fields $V = \phi(t)\partial_t + (K_1 e_1 + K_2 e_2)$ $(K_1, K_2 \in \mathbb{R}, K_1^2 + K_2^2 = 1)$ on the warped product $\mathbb{R}^* \times_f G$, with G is of type A is a harmonic vector field but cannot be define a harmonic map*

Proposition 6.2. *if $n = 2$, $f(t) = \exp(t)$, $\phi(t) = -\frac{\sqrt{2}}{2}(K_1 + K_2)t^2$, a vector field $V = \phi(t)\partial_t + (K_1 e_1 + K_2 e_2)$ $(K_1, K_2 \in \mathbb{R}, K_1^2 + K_2^2 = 1)$ on $\mathbb{R} \times_f G$, with G is of type B is a harmonic vector field but cannot be define a harmonic map*

7. Conclusion

The analysis performed in performed in this aims is the harmonicity of vector fields on extended Class of Lorentzian solvable Lie groups. After having determined the harmonicity condition on the warped product $M \times_f N$, by using a characteristic variational condition. We apply this to the case $I \times G$ where $I \subset \mathbb{R}$ and consider G a special class of solvable Lie groups, to investigate the harmonicity properties of vector fields on $I \times_f G$ called extended Class of a Lorentzian solvable Lie groups

References

[1] R. P. Albuquerque, *On Lie groups with left invariant semi-Riemannian metric*. In: Proceedings of the 1st International Meeting on Geometry and Topology (Braga, 1997), 1–13, Cent. Mat. Univ. Minho, Braga 1998. MR1694936 Zbl 0938.53022

[2] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Math. Soc. Monogr. No. **29**, Oxford University Press (2003).

[3] E. Boeckx and L. Vanhecke, *Characteristic reflections on unit tangent sphere bundles*, Houston J. Math. **23** (1997), 427–448.

[4] G. Calvaruso, *Harmonicity of vector fields on four-dimensional generalized symmetric spaces.* Cent. Eur. J. Math. 10 (2012), 411–425. MR2886549 Zbl 1246.53083

[5] G. Calvaruso, *Harmonicity properties of invariant vector fields on three-dimensional Lorentzian Lie groups*, Journal of Geometry and Physics **61** (2011) 498–515

[6] S. Dragomir and D. Perrone, *Harmonic Vector Fields: Variational Principles and Differential Geometry*, Elsevier (2011).

[7] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86**(1) (1964), 109–160.

[8] F. Koudjo E. Loubeau, L. Todjihoundé, *Harmonic vector fields on extended 3-dimensional Riemannian Lie groups*. Differential Geometry-Dynamical Systems.**24** (2022)92-118.

[9] T. Ishihara, *Harmonic sections of tangent bundles*, J. Math. Tokushima Univ. **13** (1979), 23–27.

[10] Ju Tan and Shaoqiang Deng *Harmonicity of vector fields on a class of Lorentzian solvable Lie groups* DOI 10.1515/advgeom-2017-0014. 1-9

[11] J. Milnor, *Curvature of left invariant metrics on Lie groups*, Adv. in Math. **21** (1976), 293–329.

[12] K. Nomizu, *Left-invariant Lorentz metrics on Lie groups*. Osaka J. Math. 16 (1979), 143–150. MR527022 Zbl 0397.53047

Institut de Mathematiques et de Sciences Physiques, Porto-Novo, Benin.
Email address: ferdinand.koudjo@imsp-uac.org