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ON GENERALIZED DOUGLAS CURVATURE OF FINSLER METRICS

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ABSTRACT. One of the important quantities in Finsler geometry is Douglas' curvature which is an invariant tensor by a projective change $\phi : F \to \bar{F}$. The aim of this paper is to study a class of Finsler metrics that includes the classes of Douglas metrics. Finsler metrics in this class are known as generalized Douglas metrics. We prove that every generalized Douglas metric with vanishing $\widetilde{\mathbf{B}}$-stretch tensor is a Douglas metric under the condition that the mean Berwald curvature is horizontally constant along geodesics of F. Then, we show that if (M, F) is a Douglas Finsler manifold then the Finsler metric F is an $\mathbf{H}$-stretch metric if and only if it is a $\widetilde{\mathbf{B}}$-metric.

1. INTRODUCTION

Let $\gamma = \gamma(t)$ be a geodesic curve of a Finsler metric F on a C^∞-manifold M which is determined by the system of second order ordinary differential equations $\ddot{\gamma}^i(t) + 2G^i(\gamma(t), \dot{\gamma}(t)) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients of F. A Finsler metric F is said to be projective to another Finsler metric $\bar{F}$ if any geodesic of F is a geodesic of $\bar{F}$ and vice versa. The condition for it is written as

$$\bar{G}^i = G^i + Py^i,$$

where $P = P(x, y)$ is a positive homogeneous scalar function of degree one. The change $\phi : F \to \bar{F}$ is called projective. In Finsler geometry projective invariants play an important role. Namely, if a Finsler metric F has a curvature, then any Finsler metric projectively equivalent to F must have the same curvature. Douglas curvature is one among them, which proposed by Bácsó-Matsumoto as a Berwald curvature extension [7].

L. Berwald was the first to introduce the Berwald connection, which effectively expanded the concept of Riemann curvature of Finsler metrics. Through his connection, he also introduced some non-Riemannian quantities in Finsler geometry. Berwald metrics are the most Riemannian-like of the Finsler metrics and are characterized by quadratic $G^i(x, y)$ in directions [5].

A Finsler metric is a stretch-type metric if the length of a vector does not change when traveling parallel along an infinitesimal parallelogram. Berwald coined the term called stretch curvature in 1924 as an extension of Landsberg curvature which is denoted by $\mathbf{\Sigma}_y$. A Finsler metric is called *stretch metric* if $\mathbf{\Sigma} = 0$ [4].

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Let $y \in T_pM$ be an arbitrary vector and $\gamma = \gamma(t)$ be the geodesic with $\gamma(0) = p, \dot{\gamma}(0) = y$. The $\widetilde{\mathbf{B}}$-curvature of F is given by

$$\widetilde{\mathbf{B}}_y(u,v,w) := \frac{d}{dt}\Big[\mathbf{B}_{\dot{\gamma}(t)}(U(t),V(t),W(t))\Big]_{t=0},$$

where $U = U(t), V = V(t)$ and $W = W(t)$ are linearly parallel vector fields along $\gamma(t)$ with $U(0) = u, V(0) = v$ and $W(0) = w$. The Finsler metric F is called $\widetilde{\boldsymbol{B}}$*-metric* if and only if $\widetilde{\mathbf{B}} = 0$ (see [9, Proposition 9.3.1]).
Very recently, S. A. Abbas and L. Kozma in [1] introduced the *stretch* $\widetilde{\boldsymbol{B}}$*-curvature* which is defined

$$\mathcal{K}_y : T_pM \times T_pM \times T_pM \times T_pM \to T_pM,$$

by $\mathcal{K}_y(u,v,w,z) = \mathcal{K}^i_{jklm}(y)u^jv^kw^lz^m\frac{\partial}{\partial x^i}\mid_x, u,v,w,z \in T_pM$, with

$$\mathcal{K}^i_{jklm} := \widetilde{B}^i_{jkl,m} - \widetilde{B}^i_{jkm,l}, \tag{1.1}$$

where $\widetilde{B}^i_{jkl,m}$ is the horizontal derivatives of $\widetilde{B}^i_{jkl}$ with respect to the Berwald connection of Finsler metric F. A Finsler metric F is said to be $\widetilde{\boldsymbol{B}}$*-stretch metric* if and only if $\mathcal{K} = 0$. Every Berwald metric is a $\widetilde{\mathbf{B}}$-stretch metric.
Let us consider a non-trivial example.

Example 1.1. *The interesting family of protectively flat Finsler metrics called quadratic metrics introduced by Berwald in* [6] *which is defined on the unit ball* $\mathbb{B}^n$ *where* $|.|$ *and* $\langle .,. \rangle$ *are the Euclidean norm and the inner product in* $\mathbb{R}^n$ *respectively.*

$$F(x,y) = \frac{\left(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \varepsilon\langle x,y\rangle\right)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)\,|y|^2 - \langle x,y\rangle^2}}.$$

The Finsler metric F is a $\widetilde{\mathbf{B}}$-stretch metric, not $\widetilde{\mathbf{B}}$-metric, not Berwaldian, this can be shown using the Finsler package and Maple program [14].
Numerous metrics such as Riemannian metrics, Berwald metrics, and locally projectively flat Finsler metrics are special Douglas metrics. Thus, Douglas metrics form a rich class of metrics in Finsler geometry (see [10, 3, 11, 13])
In this paper, we study a new class of Finsler metric which contains the class of Douglas metric as special case called *generalized Douglas metric*.
The $\widetilde{\mathbf{B}}$-curvature of this class is given by

$$\widetilde{B}^i_{jkl} := B^i_{jkl} + \omega_{jk}\delta^i_l + \omega_{jl}\delta^i_k + \omega_{kl}\delta^i_j + E_{jk;l}y^i, \tag{1.2}$$

where ω is a smooth map $M \to \wedge^2 T^*_pM$ given by $\omega(p) := \omega_{ij}(p)dx^i \wedge dx^j$ at any point $p \in M$.
We show that a generalized Douglas Finsler manifold reduces to a Douglas manifold if its stretch $\widetilde{\mathbf{B}}$-curvature vanishes as follows

Theorem 1.1. *Let* (M,F) *be a generalized Douglas Finsler manifold. Suppose that* F *is a* $\widetilde{\boldsymbol{B}}$-*stretch metric. Then* F *is a Douglas metric.*

The converse of Theorem 1.1 is not true. Let us introduce a famous example of Finsler metrics introduced by a physicist G. Randers in 1949 [8].

Example 1.2. *Let $|.|$ and $\langle .,. \rangle$ be the Euclidean norm and the inner product in $\mathbb{R}^n$ respectively. Let $F = \alpha + \beta$ be a Finsler metric with $\|\beta_x\|_\alpha < 1$ where*

$$\alpha = \frac{\sqrt{|y|^2 + \varepsilon\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)}}{1 + \varepsilon |x|^2}, \quad \beta = \frac{\sqrt{-\varepsilon}\langle x, y\rangle}{1 + \varepsilon |x|^2}, \qquad \varepsilon < 0,$$

and $x \in \mathbb{B}^n(\delta) \subset \mathbb{R}^n$, $\delta = \frac{1}{\sqrt{-\varepsilon}}$, $y \in T_x\mathbb{B}^n(\delta)$. Putting $\varepsilon = -1$, we get a metric called Funk metric which satisfies $F_{x^s} - FF_{y^s} = 0$. A Funk metric F is a Douglas metric but it is not $\widetilde{\mathbf{B}}$-stretch.

H. Akbar-Zadeh in [2] defined the important non-Riemannian quantity called $\mathbf{H}$-curvature arisen from the mean Berwald curvature $\mathbf{E}$ by the covariant horizontal differentiation along geodesics.

For a non-zero vector $y \in T_pM$ defined

$$\mathbf{H}_y : T_pM \times T_pM \to \mathbb{R},$$

by $\mathbf{H}_y(u, v) = H_{jk}(y)u^j v^k, u, v \in T_pM$, with

$$\mathbf{H}_y(u, v) := \frac{d}{dt}\left[\mathbf{E}_{\dot{\gamma}(t)}(U(t), V(t))\right]_{t=0},$$

where $U = U(t)$ and $V = V(t)$ are linearly parallel vector fields along $\gamma(t)$ with $U(0) = u, V(0) = v$. The Finsler metric F is called ***H**-metric* if and only if $\mathbf{H} = 0$.

By taking a trace of (1.1) we get the quantity called *stretch **H**-curvature* which is $\boldsymbol{\kappa} = \{\boldsymbol{\kappa}_y : y \in T_pM\}$, where

$$\boldsymbol{\kappa}_y : T_pM \times T_pM \times T_pM \to \mathbb{R}$$

by $\boldsymbol{\kappa}_y(u, v, w) = \kappa_{jkl}(y)u^j v^k w^l, u, v, w \in T_pM$, with

$$\kappa_{jkl} := H_{jk,l} - H_{jl,k}.$$

If $\boldsymbol{\kappa} = 0$, then a Finsler metric F is called ***H**-stretch metric*.

Every Finsler metric with vanishing $\widetilde{\mathbf{B}}$-curvature has vanishing $\mathbf{H}$-curvature. Thus, every $\widetilde{\mathbf{B}}$-metric is an $\mathbf{H}$-metric then it is $\mathbf{H}$-stretch metric. But the converse might not hold. A. Tayebi et al. in [12] showed that it is true on Finsler surfaces. We generalized this result as follows

Theorem 1.2. *Let (M, F) be a Douglas Finsler manifold with $n \geq 3$. Then every **H**-stretch metric is a $\widetilde{\mathbf{B}}$-metric.*

Exploiting the above statements, the next result gives a formula involving the relation between the quantities stretch $\widetilde{\mathbf{B}}$-curvature and stretch $\mathbf{H}$-curvature. Namely, we have the following

Theorem 1.3. *Let (M, F) be a Douglas Finsler manifold. Then, the stretch $\widetilde{\mathbf{B}}$-curvature of F is given by*

$$\mathcal{K}^i_{jklm} := \frac{2}{n+1}\left\{\kappa_{jlm}h^i_k + \kappa_{klm}h^i_j\right\}. \tag{1.3}$$

2. Definitions and Preliminaries

This section reviews some important definitions and concepts of Finsler manifolds that are extensively used in this work. For more detail see [9, 7, 12].
Let M be a C^∞-manifold of dimension n, and let $TM := \cup_{p\in M} T_pM$ be the tangent bundle of M, where T_pM is the tangent space at $p \in M$. We set $TM_\star = TM \setminus \{0\}$ where $\{0\}$ stands for $\{(p, 0) \mid p \in M, 0 \in T_pM\}$. A Finsler metric on a manifold M is a function $F : TM \to [0, \infty)$ with the following properties:

(1) F is C^∞ mapping over $TM_\star$;
(2) at each point $p \in M$, the restriction $F_p := F\mid_{T_pM}$ is a Minkowski norm on T_pM.

A C^∞-manifold M with Finsler metric F are called Finsler manifold denoted by the pair (M, F).
Let (M, F) be a Finsler manifold. A global vector field $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$ is induced by Finsler metric F on $TM_\star$, where $G^i(x, y)$ are local functions on $TM_\star$ satisfying $G^i(x, ky) = k^2 G^i(x, y)$, for all $k > 0$. The functions G^i are given by

$$G^i := \frac{1}{4} g^{il}\Big\{2\frac{\partial}{\partial x^k}(g_{jl}) - \frac{\partial}{\partial x^l}(g_{jk})\Big\} y^j y^k. \tag{2.1}$$

$\mathbf{G}$ is called the associated spray to (M, F).
For $y \in T_pM_\star$, define the Cartan torsion of F as follows:

$$\mathbf{C}_y : T_pM \times T_pM \times T_pM \to \mathbb{R},$$

by

$$\mathbf{C}_y(v_1, v_2, v_3) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tv_3}(v_1, v_2)\Big]_{t=0}, \quad v_1, v_2, v_3 \in T_pM.$$

For a vector $y \in T_pM_\star$, the mean Cartan torsion of a Finsler metric F is defined by $\mathbf{I}_y : T_pM \to \mathbb{R}$ where:

$$\mathbf{I}_y(v) := \sum_{i=1}^{n} \mathbf{C}_y\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, v\right) g^{jk}(y), \quad v \in T_pM,$$

where $\{\frac{\partial}{\partial x^j}\}$ is a basis for T_pM at $p \in M$.
For $y \in T_pM_\star$ the Berwald curvature of F is defined by $\mathbf{B}_y(u, v, w) = B^i_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}\mid_x$, where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

The Finsler metric F is called a Berwald metric if and only if $\mathbf{B} = 0$.
The mean Berwald curvature is defined by $\mathbf{E}_y(u, v) = E_{jk}(y)u^j v^k$, where

$$E_{jk} := \frac{1}{2} B_m{}^m{}_{jk} = \frac{1}{2}\frac{\partial^2}{\partial y^j \partial y^k}\left(\frac{\partial G^m}{\partial y^m}\right).$$

For $y \in T_pM_\star$, and $\gamma = \gamma(t)$ be the geodesic with $\gamma(0) = p, \dot{\gamma}(0) = y$.
Let $U = U(t), V = V(t)$ and $W = W(t)$ are linearly parallel vector fields along γ with $U(0) = u, V(0) = v$ and $W(0) = w$. The Landsberg curvature of Finsler metric F is given by

$$\mathbf{L}_y(u, v, w) := \frac{d}{dt}\Big[\mathbf{C}_{\dot{\gamma}(t)}\left(U(t), V(t), W(t)\right)\Big]_{t=0}.$$

A Finsler metric is said to be a Douglas metric if there exists positively 1-homogeneous function $P(x,y)$ on M where the spray coefficients $G^i = G^i(x,y)$ of a Finsler metric F can be expressed in the following form

$$G^i(y) = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x,y)y^i.$$

On an n-dimensional manifold M, for a non-zero vector $y \in T_pM$, the Douglas curvature $\mathbf{D}_y : T_pM \times T_pM \times T_pM \to T_pM$ is defined by $\mathbf{D}_y(u,v,w) := D^i_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$, where

$$D^i_{jkl} := B^i_{jkl} - \frac{2}{n+1}\left[\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(\frac{\partial G^s}{\partial y^s} y^i\right)\right]. \tag{2.2}$$

The Finsler metric F is called a Douglas metric if and only if $\mathbf{D} = 0$.
According to (2.2), the Douglas tensor can be written as follows

$$D^i_{jkl} = B^i_{jkl} - \frac{2}{n+1}\left\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk;l}y^i\right\},$$

where $E_{jk;l} = \frac{\partial E_{jk}}{\partial y^l}$.

3. Proof of Theorem 1.1

Now we can proceed to prove the main result of this paper, which was announced in the abstract.
Let F be a generalized Douglas metric,

$$\widetilde{B}^i_{jkl} = B^i_{jkl} + \omega_{jk}\delta^i_l + \omega_{jl}\delta^i_k + \omega_{kl}\delta^i_j + E_{jk;l}y^i. \tag{3.1}$$

Differentiating (3.1) along the direction $y^s\frac{\delta}{\delta x^s}$ yields,

$$\widetilde{B}^i_{jkl,s}y^s = \widetilde{B}^i_{jkl} + \omega'_{jk}\delta^i_l + \omega'_{jl}\delta^i_k + \omega'_{kl}\delta^i_j, \tag{3.2}$$

where $\omega'_{ij} := \frac{\delta \omega_{ij}}{\delta x^s}y^s$.
Plugging (3.1) into (3.2), we obtain

$$\widetilde{B}^i_{jkl,s}y^s = B^i_{jkl} + (\omega_{jk} + \omega'_{jk})\delta^i_l + (\omega_{jl} + \omega'_{jl})\delta^i_k + (\omega_{kl} + \omega'_{kl})\delta^i_j + E_{jk;l}y^i. \tag{3.3}$$

Since F is $\widetilde{\mathbf{B}}$-stretch, that is

$$\widetilde{B}^i_{jkl,s} = \widetilde{B}^i_{jks,l}, \tag{3.4}$$

contracting (3.4) with y^s yields

$$\widetilde{B}^i_{jkl,s}y^s = 0. \tag{3.5}$$

By (3.3) and (3.5), we have

$$B^i_{jkl} = -1\left[(\omega_{jk} + \omega'_{jk})\delta^i_l + (\omega_{jl} + \omega'_{jl})\delta^i_k + (\omega_{kl} + \omega'_{kl})\delta^i_j + E_{jk;l}y^i\right]. \tag{3.6}$$

Contracting (3.6) with h^m_i, one has

$$B^m_{jkl} = -1\left[(\omega_{jk} + \omega'_{jk})h^m_l + (\omega_{jl} + \omega'_{jl})h^m_k + (\omega_{kl} + \omega'_{kl})h^m_j\right]. \tag{3.7}$$

Multiplying (3.7) with $\frac{1}{2}g_{ms}g^{ls}$ and using the following relations

$$g^{ls}h_{ls} = n-1 \quad \textit{and} \quad g_{ms}g^{ls}(\omega_{jl}h^m_k) = g_{ms}g^{ls}(\omega_{kl}h^m_j) = \omega_{jk},$$

it implies that

$$E_{jk} = \frac{-(n+1)}{2}(\omega_{jk} + \omega'_{jk}). \tag{3.8}$$

Alternatively, we can write (3.8) as

$$\omega_{jk} + \omega'_{jk} = \frac{-2}{n+1} E_{jk}. \tag{3.9}$$

Plugging (3.9) in (3.6), we get

$$B^i_{jkl} = \frac{2}{n+1}\left\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk;l}y^i\right\}.$$

We clearly have that F is a Douglas metric, which concludes our proof.

4. Proof of Theorem 1.2

In this section, we are going to prove Theorem 1.2.
Let us begin with the auxiliary lemma which turns out to be the key tool in the proof of our theorem.

Lemma 4.1. *Let (M, F) be a Douglas manifold. Suppose that F is weakly $\mathbf{H}$-stretch metric. Then for any geodesic $\gamma(t)$ and any parallel vector field $V(t)$ along γ, the function $\boldsymbol{B} := \boldsymbol{B}_{\dot\gamma}(V(t))$ must be in the following form*

$$\boldsymbol{B}(s) = s\widetilde{\boldsymbol{B}}(0) + \boldsymbol{B}(0). \tag{4.1}$$

Proof. By definition, we have

$$\kappa_{jkl} := H_{jk,l} - H_{jl,k}.$$

By assumption F is $\mathbf{H}$-stretch metric, then

$$H_{jk,l} = H_{jl,k}. \tag{4.2}$$

Contracting (4.2) with y^l gives rise the constancy of the rate of change of the $\mathbf{H}$-curvature along geodesic of F, i.e.

$$H_{jk,l}y^l = 0. \tag{4.3}$$

Since F is Douglas metric, we have

$$B^i_{jkl} = \frac{2}{n+1}\left\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk;l}y^i\right\}. \tag{4.4}$$

Then

$$\widetilde{B}^i_{jkl} = \frac{2}{n+1}\left\{H_{jk}\delta^i_l + H_{jl}\delta^i_k + H_{kl}\delta^i_j + \mathcal{E}y^i\right\}, \tag{4.5}$$

where $\mathcal{E} = \frac{\delta E_{jk;l}}{\delta x^s}y^s$.
Taking a horizontal derivative of (4.5) along Finslerian geodesics implies that

$$\widetilde{B}^i_{jkl,n}y^n = \frac{2}{n+1}\left\{H_{jk,n}y^n\delta^i_l + H_{jl,n}y^n\delta^i_k + H_{kl,n}y^n\delta^i_j + \mathcal{E}'y^i\right\}, \tag{4.6}$$

where $\mathcal{E}' = \frac{\delta\mathcal{E}}{\delta x^n}y^n$.
Contracting (4.6) with h^m_i, one obtains

$$\widetilde{B}^m_{jkl,n}y^n = \frac{2}{n+1}\left\{H_{jk,n}y^n h^m_l + H_{jl,n}y^n h^m_k + H_{kl,n}y^n h^m_j\right\}, \tag{4.7}$$

By (4.3) and (4.7), we get

$$\widetilde{B}^{m}_{jkl,n}y^{n} = 0. \tag{4.8}$$

From our definition of $\widetilde{\mathbf{B}}_y$, we have $\widetilde{\mathbf{B}}(s) = \mathbf{B}'(s)$. We obtain

$$\mathbf{B}''(s) = \widetilde{\mathbf{B}}'(s) = 0.$$

Thus (4.1) follows. This completes the proof. □

In the following, we are going to prove Theorem 1.2. The proof involves applying Lemma 4.1 as follow

Proof of Theorem 1.2. Let (M, F) be a Finsler manifold. Take an arbitrary unit vector $y \in T_pM$ and an arbitrary vector $v \in T_pM$. Let $\gamma(t)$ be the geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = y$ and $V(t)$ be the parallel vector along γ with $V(0) = v$. Then by Lemma 4.1, we get

$$\mathbf{B}(s) = s\widetilde{\mathbf{B}}(0) + \mathbf{B}(0). \tag{4.9}$$

Suppose that $\mathbf{B}_y$ is bounded. i.e., there is a constant $D < 1$ such that

$$\|\mathbf{B}\|_p := \sup_{y\in T_pM_\circ}\sup_{v\in T_pM}\left[\frac{\mathbf{B}_y(v)}{\left[\mathbf{g}_y(v,v)\right]^{\frac{3}{2}}}\right] \leq D. \tag{4.10}$$

In view of Lemma 7.3.2 in [9], which yields that $|\mathbf{B}(s)| \leq DA^{\frac{3}{2}} < \infty$, for some constant A. Therefore, $\mathbf{B}(s)$ is a bounded function on $[0, \infty)$. (4.9) implies that $\widetilde{\mathbf{B}}_y(v) = \widetilde{\mathbf{B}}(0) = 0$. Hence, F is a $\widetilde{\mathbf{B}}$-metric. Thus, the proof is done. □

Bácsó and Matsumoto [7] showed that if the 1-form β is a closed, then any Randers metric $F = \alpha + \beta$ is a Douglas metric. Then by Theorem 1.2, we get the following.

Corollary 4.1. *Let $F = \alpha + \beta$ is a Randers metric on a manifold M with closed 1-form β. Then F is a $\widetilde{\mathbf{B}}$-metric if and only if it is a $\mathbf{H}$-stretch metric.*

5. Proof of Theorem 1.3

Proof. Since F is a Douglas metric, we have

$$B^{i}_{jkl} = \frac{2}{n+1}\left\{E_{jk}\delta^{i}_{l} + E_{jl}\delta^{i}_{k} + E_{kl}\delta^{i}_{j} + E_{jk;l}y^{i}\right\}. \tag{5.1}$$

By taking horizontal derivatives of (5.1) and contracting with h^{m}_{i} then replacing indices, one has

$$\widetilde{B}^{m}_{jkl,s} = \frac{2}{n+1}\{H_{jk,s}h^{m}_{l} + H_{jl,s}h^{m}_{k} + H_{kl,s}h^{m}_{j}\}, \tag{5.2}$$

and

$$\widetilde{B}^{m}_{jks,l} = \frac{2}{n+1}\{H_{jk,l}h^{m}_{s} + H_{js,l}h^{m}_{k} + H_{ks,l}h^{m}_{j}\}. \tag{5.3}$$

Subtracting (5.2) from (5.3), we get

$$\left(\widetilde{B}^{m}_{jkl,s} - \widetilde{B}^{m}_{jks,l}\right) = \frac{2}{n+1}\left\{(H_{jl,s} - H_{js,l})h^{m}_{k} + (H_{kl,s} - H_{ks,l})h^{m}_{j}\right\}.$$

By definition of $\widetilde{\mathbf{B}}$-stretch and $\mathbf{H}$-stretch metrics, we get (1.3). □

References

[1] S. A. Abbas and L. Kozma, On new classes of stretch Finsler metrics, *Journal of Finsler Geometry and its Applications*, **3** (2022), no. 1, 86–99.
[2] H. Akbar-Zadeh, On Finsler spaces with constant sectional curvature, *Acad. Roy. Belg. Bull. Cl. Sci. (5)*, **74** (1988), no. 10, 281–322.
[3] M. Atashafrouz, B. Najafi and A. Tayebi, Weakly Douglas Finsler metrics, *Period. Math. Hungar.*, **81** (2020), no. 2, 194–200.
[4] L. Berwald, Über Parallelübertragung in Räumen mit allgemeiner Maßbestimmung, *Jahresber. Dtsch. Math.-Ver.*, **34**(1926), 213–220.
[5] L. Berwald, Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus, *Math. Z.*, **26** (1927), no. 1, 176-176.
[6] L. Berwald, Über die n-dimensionalen Geometrien konstanter Krümmung, in denen die Geraden die kürzesten sind, *Math. Z.*, **30** (1929), no. 1, 449–469.
[7] S. Bácsó and M. Matsumoto, On Finsler spaces of Douglas type, A generalization of notion of Berwald space, *Publ. Math. Debrecen*, **51** (1997), no. 3-4, 385–406.
[8] G. Randers, On an asymmetrical metric in the fourspace of general relativity, *Phys. Rev. (2)*, **59** (1941), 195–199.
[9] Z. Shen, Differential geometry of spray and Finsler spaces (Kluwer Academic Publishers, Dordrecht, 2001).
[10] T. Tabatabaeifar, N. Behzad and M. Rafie-Rad, On generalized Douglas-Weyl Randers metrics, *Czechoslovak Math. J.*, **71** (2021), no. 1, 155–172.
[11] A. Tayebi and N. Izadian, Douglas-square metrics with vanishing mean stretch curvature, *Int. Electron. J. Geom.*, **12** (2019), no. 2, 188–201.
[12] A. Tayebi, M. Faghfouri and N. Jazer, On mean stretch curvatures of Finsler metrics, *Acta Math. Univ. Comenian. (N.S.)*, **89** (2020), no. 2, 335–342.
[13] V. S. Matveev and S. Saberali, Conformally related Douglas metrics in dimension two are Randers, *Arch. Math. (Basel)*, **116** (2021), no. 2, 221–231.
[14] L. N. Youssef and S. G. Elgendi, New Finsler package, *Comput. Phys. Commun.*, **185** (2014), no. 3, 986-997.

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