Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.12, (2023), Issue 2, pp.226-228

DARBOUX'S THEOREM OF LOCALLY CONFORMALLY SYMPLECTIC MANIFOLDS

SERVAIS CYR GATSE

ABSTRACT. Our purpose is to show simply the Darboux's theorem of locally conformally symplectic manifolds.

1. INTRODUCTION

A locally conformally symplectic manifold, l.c.s. for short, is a triple (M, α, Ω), where M is a finite dimensional smooth manifold, Ω an almost symplectic form and α is a Lee form, such that $d_\alpha \Omega = d\Omega + \alpha \wedge \Omega = 0$ (see [3], [4] and the references given there). We will always assume that M is connected and $\partial M \neq \emptyset$. It is easily seen that, $d_\alpha \circ d_\alpha = 0$. Thus one obtains a twisted de Rham cohomology of the locally constant sheave of d_α-constant functions. Note that since, $d_\alpha \Omega = 0$, Ω, defines a d_α-cohomology class $[\Omega] \in \mathcal{H}^2_{d_\alpha}(M)$. For some examples of l.c.s. manifolds see [1]. The standard model for locally conformally symplectic manifolds is construct in [2]. The theorem asserts that

Theorem 1.1. *Let $(e_1, ..., e_{2n})$ be the canonical basis of $\mathbb{R}^{2n}$ and $(e_1^*, ..., e_{2n}^*)$ the dual basis. For $i = 1, ..., 2n$, e_i^* is the canonical projection. Let $\alpha_0 = de_{2n}^*$ and $\Omega_0 = \sum_{i=1}^{n} d_{\alpha_0} e_i^* \wedge de_{n+i}^*$. The triple $(\mathbb{R}^{2n}, \alpha_0, \Omega_0)$ is a locally conformally symplectic manifold.*

Our main result reads:

Theorem 1.2. *When (M, α, Ω) is a locally conformally symplectic manifold with dimension $2n\,(n \geq 1)$, then for any point $p \in M$, there exist an open neighborhood U of p in M and coordinate functions $(x_1, ..., x_{2n})$ such that*

$$\begin{aligned} \alpha_{|U} &= dx_{2n}, \\ \Omega_{|U} &= \sum_{i=1}^{n} d_{\alpha_{|U}} x_i \wedge dx_{n+i}. \end{aligned}$$

2020 *Mathematics Subject Classification.* Primary 53C18, 53D05; Secondary 53D22, 53D35.

Key words and phrases. Locally conformally symplectic manifold, Darboux's theorem, symplectomorphism.

2. Proof of the main theorem

The following assumption will be needed thoughout the paper.

Lemma 2.1. *Let $(\Omega_t)_{0\leq t\leq 1}$ be a smooth family of l.c.s. forms on a compact manifold M, satisfying $\frac{d}{dt}\Omega_t = d\beta_t$. Suppose that for all $t \in [0,1]$, the Lee form of Ω_t is the same 1-form α_t and $\Omega_t - \Omega_0$ is d_α-exact. Then there is an isotopy ψ_t such that $\psi_t^*\Omega_t = \Omega_0$ with $\psi_0 = id$.*

Proof. of Theorem 1.2. By the local nature of geometric structures, for any point $p \in M$, there exists a local chart (U, φ), where U is a neighborhood of p such that

$$\varphi : U \longrightarrow \varphi(U) \subset \mathbb{R}^{2n}$$

is a symplectomorphism of $\left(U, \alpha_{|U}, \Omega_{|U}\right)$ on $\left(\varphi(U), \alpha_{0|\varphi(U)}, \Omega_{0|\varphi(U)}\right)$. Thus

$$\begin{aligned} \varphi^*(\alpha_{0|\varphi(U)}) &= \alpha_{|U}, \\ \varphi^*(\Omega_{0|\varphi(U)}) &= \Omega_{|U}. \end{aligned}$$

When $(x_1, ..., x_{2n})$ is the system of local coordinates of the chart (U, φ) with $x_i = e_i^* \circ \varphi$, we have

$$\begin{aligned} \alpha_{|U} &= \varphi^*\left[\alpha_{0|\varphi(U)}\right] \\ &= dx_{2n} \end{aligned}$$

and

$$\begin{aligned} \Omega_{|U} &= \varphi^*(\Omega_{0|\varphi(U)}) \\ &= \sum_{i=1}^{n} d_{\alpha_{|U}} x_i \wedge dx_{n+i}. \end{aligned}$$

We shall solve there equations by the lemma 2.1, as desired. □

Since

$$\begin{aligned} \Omega_{|U} &= \sum_{i=1}^{n} d_{\alpha_{|U}} x_i \wedge dx_{n+i} \\ &= \omega_0 + \sum_{i=1}^{n} x_i \cdot \alpha_{|U} \wedge dx_{n+i} \end{aligned}$$

then if $\alpha_{|U} = 0$, we have $\Omega_{|U} = \omega_0$, where ω_0 is a symplectic form on M. In this case M is a symplectic manifold.
We see that

$$\begin{aligned} d\Omega_{|U} &= -\alpha_{|U} \wedge \Omega_{|U} \\ &= -\alpha_{|U} \wedge \omega_0. \end{aligned}$$

3. Graphs of d_α-closed 1-forms in cotangent bundles

Let N be any manifold. Consider the cotangent bundle $M = T^*N$ of N with the natural projection

$$\pi : T^*N \longrightarrow N, (x, y) \longmapsto y.$$

Define the following 1-form λ on M. For $y \in N, (x, y) \in T^*N$ and $\gamma \in T_{(x,y)}(T^*N)$ we set $\lambda(\gamma) = \langle x, \pi_*\gamma\rangle$ where $\langle , \rangle$ is the natural pairing between T^*N and TN. We claim that $\Omega = d_\alpha \lambda$ is a locally conformally symplectic form on M. We use local coordinates $(x_1, ..., x_{2n})$ on T^*N and write $\gamma = \left(\dot{x}_1, ..., \dot{x}_{2n}\right)$, so

$$\pi_* \gamma = \left(\dot{x}_{n+1}, ..., \dot{x}_{2n}\right).$$

With this notation the pairing reads

$$\langle x, \pi_* \gamma \rangle = \sum_{i=1}^{n} x_i \cdot \dot{x}_{n+i}$$

which implies that

$$\lambda(\gamma) = \sum_{i=1}^{n} x_i \cdot dx_{n+i}.$$

Therefore

$$\begin{aligned} \Omega &= d_\alpha \lambda \\ &= \sum_{i=1}^{n} d_\alpha x_i \wedge dx_{n+i}. \end{aligned}$$

The 1-form λ is intrinsically defined and hence the 2-form Ω is also intrinsically defined.

References

[1] Dragomir, S. and Ornea, L. *Locally conformal Kaehler geometry*. Progress in Math., **55** (1988), Birkhaüser.

[2] Gatsé, S. C. *An example of locally conformally symplectic manifolds*. Advances in Mathematics: Scientific Journal, **12**(1) (2023): 187-192.

[3] Lee, H. C. *A kind of even-dimensional differential geometry and its application to exterior calculus*. Amer. J. Math., **65**(19) (1943): 433-438.

[4] Vaisman, I. *Locally conformal symplectic manifolds*. Internat. J. Math. & Math. Sci., **8**(3) (1985): 521-536.

Université Marien Ngouabi, Faculté des Sciences et Techniques, BP.69, Brazzaville, Congo.

Email address: `servais.gatse@umng.cg`