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QUADRILATERALS THAT ALLOW CLOSED LIGHT-RAY PATHS

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ABSTRACT. We describe first the quadrilaterals for which there are light rays that hit each of their sides (at a certain angle) and reflect (at an equal angle) to form closed quadrilaterals. The amazing fact is that not all quadrilaterals possess such closed light ray paths, but only the acute cyclic quadrilaterals. These closed quadrilateral light-ray paths, when they exist, are the inscribed quadrilaterals with the minimum perimeter. Moreover, unlike the case of acute triangles for which there is only one inscribed triangle of minimal perimeter, the acute cyclic quadrilaterals possess infinitely many inscribed quadrilaterals that have a minimal perimeter. All of these minimal perimeter inscribed quadrilaterals have corresponding parallel sides. The opposite sides of a closed light-ray path are symmetric with respect to the opposite diagonal of the acute cyclic quadrilateral.

1. INTRODUCTION AND MOTIVATIONS

In this section we present the motivation and a well known result related to the paper. Fermat principle of minimality of time for the path traveled by a light ray, which for a medium with a constant index of refraction becomes the principle of minimality of distance, is responsible for the Law of Reflection, which states that the angle of incidence is equal to the angle of reflection. This fact leads to the result known as Fagnano theorem, which states that among all triangles inscribed in a given acute triangle, the one that has the smallest perimeter is the orthic triangle, that means, the triangle made by the feet of the altitudes of the given triangle. The orthic triangle is the only inscribed triangle that has the minimum perimeter. We can imagine the orthic triangle as the triangular path created by a light ray that hits each side of an acute triangle at a certain angle and reflects from that side at an equal angle to form a closed triangular circuit. The orthic triangle is the only closed triangular light-ray path inscribed in an acute triangle.

Theorem 1.1. *(Fagnano 1775) In an acute triangle, there is only one closed light ray path, namely the one formed by the feet of the heights. If ABC is an acute triangle, then the inscribed triangle MNP with the minimum perimeter is the orthic triangle.*

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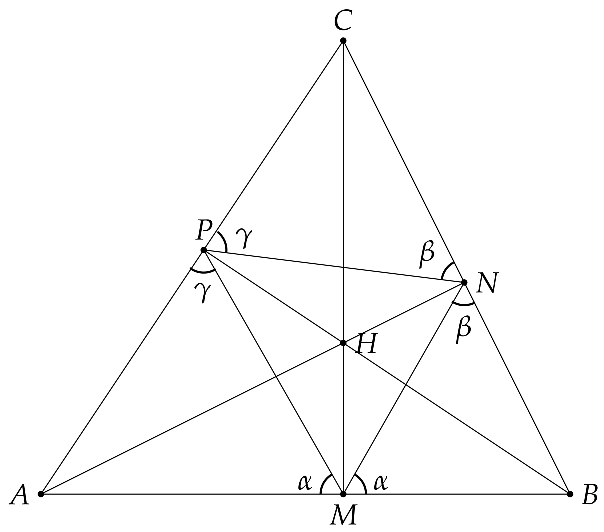

Figure 1. M1

In Fig. 1 above, we can view the orthic triangle, MNP, as the path of a light-ray starting at the point P on the side CA of the triangle ABC, hitting the side AB at an angle α and reflecting from this side at an equal angle α. Then the light-ray hits the side BC at an angle β and reflects from this side at an equal angle β. Finally the light-ray returns exactly to the point P from where it started, and repeats this path forever. This is what we mean by a triangular closed light-ray path inscribed in a triangle.

In this paper, we investigate the following problem. Given a quadrilateral $ABCD$, does it exist a quadrilateral closed light-ray path $MNPQ$ inscribed in it? If so, does this quadrilateral have the minimum perimeter among all inscribed quadrilaterals?
Let us explain first what we mean by an inscribed quadrilateral closed light-ray path $MNPQ$ in a quadrilateral $ABCD$. It means the path described by a light-ray starting at a point Q on the side DA, hitting the side AB, at a point M, at an angle α and reflecting at an equal angle α, then hitting the side BC, at a point N, at an angle β and reflecting at an equal angle β, then hitting the side CD, at a point P, at an angle γ and reflecting at an equal angle γ, and returning exactly at the point Q from where it started, and continuing this path forever. See Fig. 2.

2. MAIN RESULTS

In this section we present the main results of the paper.
Let us suppose that the quadrilateral $ABCD$ has a closed light ray path $MNPQ$, with $M \in (AB)$, $N \in (BC)$, $P \in (CD)$, and $Q \in (DA)$, where for any two distinct points X and Y in the plane, we denote by (XY) the open segment with the margins X and Y, that means, the set of all points Z in the plane collinear with X and Y, such that Z is strictly in between X and Y. Let $m(\sphericalangle AMQ) = m(\sphericalangle BMN) = \alpha$, $m(\sphericalangle BNM) = m(\sphericalangle CNP) = \beta$, $m(\sphericalangle CPN) = m(\sphericalangle DPQ) = \gamma$, and $m(\sphericalangle DQP) = m(\sphericalangle AQM) = \delta$.

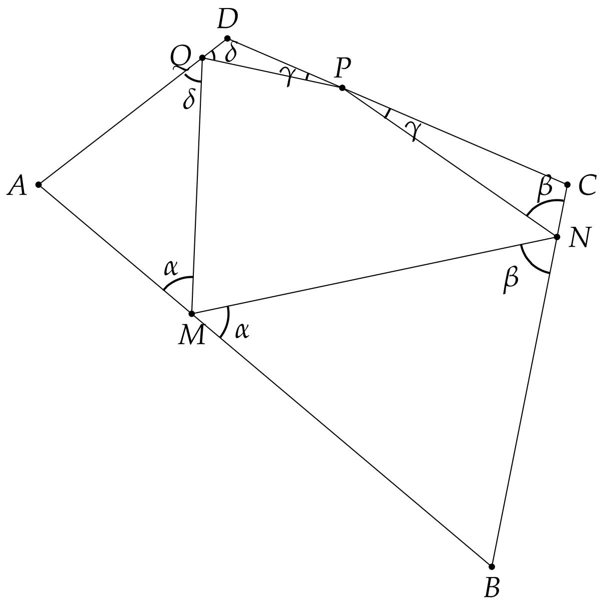

Figure 2. M2

The sum of the angles in triangle AMQ is 180°. Thus:

$$m(\sphericalangle A) \quad = \quad 180^\circ - \delta - \alpha. \tag{2.1}$$

The sum of the angles in triangle BNM is 180°. Thus:

$$m(\sphericalangle B) \quad = \quad 180^\circ - \alpha - \beta. \tag{2.2}$$

The sum of the angles in triangle CPN is 180°. Thus:

$$m(\sphericalangle C) \quad = \quad 180^\circ - \beta - \gamma. \tag{2.3}$$

The sum of the angles in triangle DQP is 180°. Thus:

$$m(\sphericalangle D) \quad = \quad 180^\circ - \gamma - \delta. \tag{2.4}$$

Adding first (2.1) and (2.3) together, and then (2.2) and (2.4), we obtain:

$$\begin{aligned} m(\sphericalangle A) + m(\sphericalangle C) &= m(\sphericalangle B) + m(\sphericalangle D) && (2.5) \\ &= 360^\circ - \alpha - \beta - \gamma - \delta. && (2.6) \end{aligned}$$

Since the sum of the measures of the angles of quadrilateral $ABCD$ is:

$$m(\sphericalangle A) + m(\sphericalangle B) + m(\sphericalangle C) + m(\sphericalangle D) \quad = \quad 360^0, \tag{2.7}$$

we conclude that:

$$\begin{aligned} m(\sphericalangle A) + m(\sphericalangle C) &= \frac{1}{2} \cdot 360^\circ \\ &= 180^\circ. && (2.8) \end{aligned}$$

Thus a necessary condition for the existence of the light-ray path $M - N - P - Q - M$ is that the quadrilateral $ABCD$ must be cyclic, which means there exists a circle passing through all the four vertices A, B, C, and D of $ABCD$. See Fig. 3.

We denote the lengths of the sides of $ABCD$ as follows:

$$a := AB, \quad b := BC, \quad c := CD, \quad \text{and} \quad d := DA.$$

Let $AM = x$, where by AM we denote the length of the segment (AM).
In the triangle QAM, we apply the Law of Sines:

$$\frac{AQ}{\sin(\alpha)} = \frac{AM}{\sin(\delta)}. \tag{2.9}$$

Since $AM = x$, solving for AQ, we obtain:

$$AQ = \frac{x\sin(\alpha)}{\sin(\delta)}. \tag{2.10}$$

Since $AM + MB = AB = a$, we have:

$$MB = a - x. \tag{2.11}$$

In the triangle MBN, we apply the Law of Sines:

$$\frac{BN}{\sin(\alpha)} = \frac{MB}{\sin(\beta)}. \tag{2.12}$$

Since $MB = a - x$, solving for BN, we obtain:

$$BN = \frac{(a-x)\sin(\alpha)}{\sin(\beta)}. \tag{2.13}$$

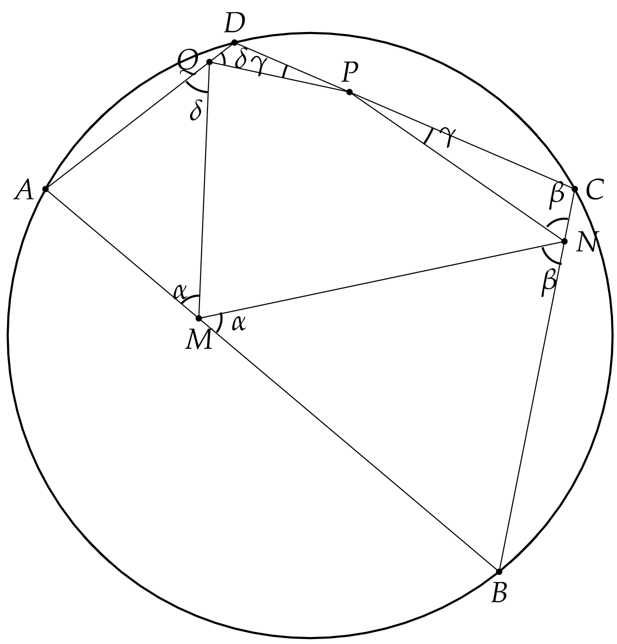

Figure 3. M3

Since $BN + NC = BC = b$, we obtain:

$$\begin{aligned} NC &= b - BN \\ &= b - \frac{(a-x)\sin(\alpha)}{\sin(\beta)} \\ &= \frac{b\sin(\beta) - a\sin(\alpha) + x\sin(\alpha)}{\sin(\beta)}. \end{aligned} \tag{2.14}$$

In the triangle NCP, we apply the Law of Sines:

$$\frac{CP}{\sin(\beta)} = \frac{NC}{\sin(\gamma)}. \tag{2.15}$$

Since $NC = [b\sin(\beta) - a\sin(\alpha) + x\sin(\alpha)]/\sin(\beta)$, solving for CP, we obtain:

$$CP = \frac{b\sin(\beta) - a\sin(\alpha) + x\sin(\alpha)}{\sin(\gamma)}. \tag{2.16}$$

Since $CP + PD = CD = c$, we obtain:

$$\begin{aligned} PD &= c - CP \\ &= c - \frac{b\sin(\beta) - a\sin(\alpha) + x\sin(\alpha)}{\sin(\gamma)} \\ &= \frac{c\sin(\gamma) - b\sin(\beta) + a\sin(\alpha) - x\sin(\alpha)}{\sin(\gamma)}. \end{aligned} \tag{2.17}$$

In the triangle PDQ, we apply the Law of Sines:

$$\frac{DQ}{\sin(\gamma)} = \frac{DP}{\sin(\delta)}. \tag{2.18}$$

Since $DP = [c\sin(\gamma) - b\sin(\beta) + a\sin(\alpha) - x\sin(\alpha)]/\sin(\gamma)$, solving for DQ, we obtain:

$$DQ = \frac{c\sin(\gamma) - b\sin(\beta) + a\sin(\alpha) - x\sin(\alpha)}{\sin(\delta)}. \tag{2.19}$$

Since $DQ + AQ = DA = d$, using formulas (2.10) and (2.19), we obtain:

$$\frac{x\sin(\alpha)}{\sin(\delta)} + \frac{c\sin(\gamma) - b\sin(\beta) + a\sin(\alpha) - x\sin(\alpha)}{\sin(\delta)} = d, \tag{2.20}$$

which is equivalent to:

$$a\sin(\alpha) + c\sin(\gamma) = b\sin(\beta) + d\sin(\delta). \tag{2.21}$$

Let us denote the measures of the following angles made by the diagonals and sides of the cyclic quadrilateral $ABCD$ (see Fig. 4) as follows:

$$m(\sphericalangle CAD) = m(\sphericalangle CBD) = m, \tag{2.22}$$

$$m(\sphericalangle ABD) = m(\sphericalangle ACD) = n, \tag{2.23}$$

$$m(\sphericalangle ACB) = m(\sphericalangle ADB) = p, \tag{2.24}$$

$$m(\sphericalangle BDC) = m(\sphericalangle BAC) = q. \tag{2.25}$$

Formulas (2.1), (2.2), (2.3), and (2.4) become now:

$$q + m = 180^\circ - \delta - \alpha, \tag{2.26}$$

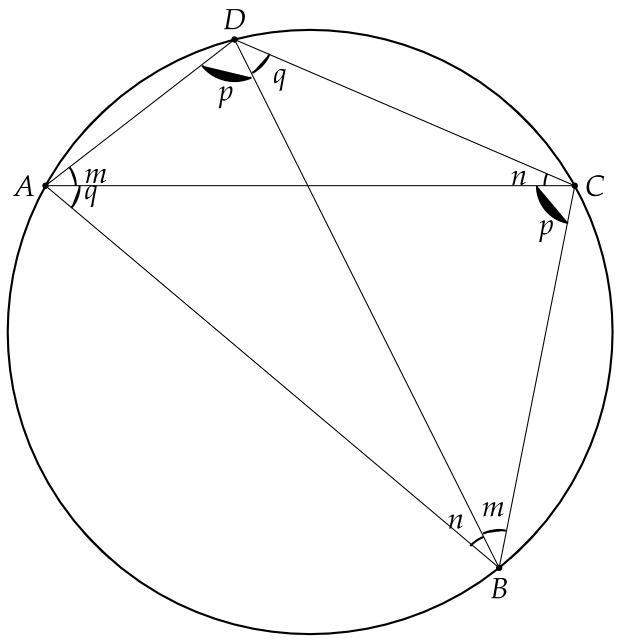

Figure 4. M4

$$m + n = 180^\circ - \alpha - \beta, \tag{2.27}$$

$$n + p = 180^\circ - \beta - \gamma, \tag{2.28}$$

$$p + q = 180^\circ - \gamma - \delta. \tag{2.29}$$

Let R be the radius of the circle that is circumscribed to the cyclic quadrilateral $ABCD$. Using the Law of Sines, we have:

$$a = 2R\sin(p), \tag{2.30}$$

$$b = 2R\sin(q), \tag{2.31}$$

$$c = 2R\sin(m), \tag{2.32}$$

$$d = 2R\sin(n). \tag{2.33}$$

Substituting the last four formulas into the equation (2.21), we obtain:

$$2R\sin(p)\sin(\alpha) + 2R\sin(m)\sin(\gamma) = 2R\sin(q)\sin(\beta) + 2R\sin(n)\sin(\delta). \tag{2.34}$$

Dividing both sides of this equation by R, we get:

$$2\sin(p)\sin(\alpha) + 2\sin(m)\sin(\gamma) = 2\sin(q)\sin(\beta) + 2\sin(n)\sin(\delta). \tag{2.35}$$

Using the formula of changing the product into a sum:

$$2\sin(u)\sin(v) = \cos(u - v) - \cos(u + v), \tag{2.36}$$

the necessary condition (2.35) becomes:

$$\begin{aligned} & \cos(\alpha - p) - \cos(\alpha + p) + \cos(\gamma - m) - \cos(\gamma + m) \\ = \; & \cos(\beta - q) - \cos(\beta + q) + \cos(\delta - n) - \cos(\delta + n). \end{aligned} \tag{2.37}$$

Moving the terms around in the last equation, we obtain:

$$\begin{aligned} & \cos(\alpha - p) - \cos(\beta - q) + \cos(\gamma - m) - \cos(\delta - n) \\ = \; & \cos(\alpha + p) - \cos(\beta + q) + \cos(\gamma + m) - \cos(\delta + n). \end{aligned} \tag{2.38}$$

Using the formula of changing the product into a sum:

$$\cos(u) - \cos(v) \;=\; 2\sin\left(\frac{v-u}{2}\right)\sin\left(\frac{u+v}{2}\right), \tag{2.39}$$

formula (2.38) becomes:

$$\begin{aligned} & 2\sin\left(\frac{\beta-\alpha+p-q}{2}\right)\sin\left(\frac{\alpha+\beta-p-q}{2}\right) \\ & +2\sin\left(\frac{\delta-\gamma+m-n}{2}\right)\sin\left(\frac{\gamma+\delta-m-n}{2}\right) \\ = \; & 2\sin\left(\frac{\beta-\alpha+q-p}{2}\right)\sin\left(\frac{\alpha+\beta+p+q}{2}\right) \\ & +2\sin\left(\frac{\delta-\gamma+n-m}{2}\right)\sin\left(\frac{\gamma+\delta+m+n}{2}\right). \end{aligned}$$

Let us observe that the left-hand side of the last equation is 0, since:

$$\begin{aligned} \alpha + \beta &= 180^\circ - m(\sphericalangle B) \\ &= m(\sphericalangle D) \\ &= p + q \end{aligned}$$

and

$$\begin{aligned} \gamma + \delta &= 180^\circ - m(\sphericalangle D) \\ &= m(\sphericalangle B) \\ &= m + n. \end{aligned}$$

Thus, we have:

$$\sin\left(\frac{\alpha+\beta-p-q}{2}\right) = \sin\left(\frac{\gamma+\delta-m-n}{2}\right) = \sin\left(0^\circ\right) = 0.$$

Therefore, we conclude that:

$$\begin{aligned} & \sin\left(\frac{\beta-\alpha+q-p}{2}\right)\sin\left(\frac{\alpha+\beta+p+q}{2}\right) \\ = \; & -\sin\left(\frac{\delta-\gamma+n-m}{2}\right)\sin\left(\frac{\gamma+\delta+m+n}{2}\right). \end{aligned} \tag{2.40}$$

We have:

$$\begin{aligned} \frac{(\alpha+\beta)+(p+q)}{2} &= \frac{m(\sphericalangle D)+m(\sphericalangle D)}{2} \\ &= m(\sphericalangle D), \\ \frac{(\gamma+\delta)+(m+n)}{2} &= \frac{m(\sphericalangle B)+m(\sphericalangle B)}{2} \\ &= m(\sphericalangle B), \end{aligned}$$

and

$$\begin{aligned} \sin(\sphericalangle D) &= \sin(\sphericalangle B) \\ &\neq 0. \end{aligned}$$

Thus, dividing both sides of (2.40) by $\sin((\alpha+\beta+p+q)/2) = \sin((\gamma+\delta+m+n)/2)$, we obtain:

$$\sin\left(\frac{\beta-\alpha+q-p}{2}\right) = \sin\left(\frac{-\delta+\gamma-n+m}{2}\right).$$

Moving all terms to the left, we obtain:

$$\sin\left(\frac{\beta-\alpha+q-p}{2}\right) - \sin\left(\frac{-\delta+\gamma-n+m}{2}\right) = 0.$$

This is equivalent to:

$$2\sin\left(\frac{\beta+\delta-\alpha-\gamma+q+n-p-m}{4}\right)\cos\left(\frac{\beta+\gamma-\delta-\alpha+q+m-p-n}{4}\right) = 0.$$

Since $\beta+\gamma = q+m = m(\sphericalangle A)$ and $\delta+\alpha = p+n = m(\sphericalangle C)$, we obtain:

$$2\sin\left(\frac{\beta+\delta-\alpha-\gamma+q+n-p-m}{4}\right)\cos\left(\frac{2m(\sphericalangle A)-2m(\sphericalangle C)}{4}\right) = 0.$$

That means, we have:

$$\sin\left(\frac{\beta+\delta-\alpha-\gamma+q+n-p-m}{4}\right)\cos\left(\frac{m(\sphericalangle A)-m(\sphericalangle C)}{2}\right) = 0. \quad (2.41)$$

Due to the fact that:

$$\begin{aligned} \left|\frac{m(\sphericalangle A)-m(\sphericalangle C)}{2}\right| &< \frac{m(\sphericalangle A)+m(\sphericalangle C)}{2} \\ &= 90^\circ, \end{aligned} \quad (2.42)$$

we conclude that:

$$\cos\left(\frac{m(\sphericalangle A)-m(\sphericalangle C)}{2}\right) \neq 0.$$

Thus, equation (2.41) implies:

$$\sin\left(\frac{\beta+\delta-\alpha-\gamma+q+n-p-m}{4}\right) = 0. \quad (2.43)$$

Since we obviously have:

$$\begin{aligned} \frac{\beta+\delta-\alpha-\gamma+q+n-p-m}{4} &> \frac{-\alpha-\gamma-p-m}{4} \\ &> \frac{-180^\circ-180^\circ-180^\circ-180^\circ}{4} \\ &= -180^\circ \end{aligned}$$

and

$$\begin{aligned}\frac{\beta+\delta-\alpha-\gamma+q+n-p-m}{4} &< \frac{\beta+\delta+q+n}{4}\\ &< \frac{180^\circ+180^\circ+180^\circ+180^\circ}{4}\\ &= 180^\circ,\end{aligned}$$

we conclude from equation (2.43) that:

$$\frac{\beta+\delta-\alpha-\gamma+q+n-p-m}{4} = 0^\circ. \tag{2.44}$$

Thus, we have:

$$\beta-\alpha+q-p = \gamma-\delta+m-n. \tag{2.45}$$

We solve first for δ and β, in terms of α, from equations (2.26) and (2.27), and obtain:

$$\delta = 180^\circ-\alpha-q-m \tag{2.46}$$

$$\beta = 180^\circ-\alpha-m-n. \tag{2.47}$$

We solve now for γ, first in terms of β, from equation (2.28), and then substitute β in terms of α, from formula (2.47), obtaining:

$$\begin{aligned}\gamma &= 180^\circ-\beta-n-p\\ &= 180^\circ-(180^\circ-\alpha-m-n)-n-p\\ &= \alpha+m-p.\end{aligned} \tag{2.48}$$

Substituting now β, γ, and δ, from formulas (2.47), (2.48), and (2.46), into formula (2.45), we obtain:

$$(180^\circ-\alpha-m-n)-\alpha+q-p = (\alpha+m-p)-(180^\circ-\alpha-q-m)+m-n.$$

This equation is equivalent to:

$$360^\circ = 4\alpha+4m, \tag{2.49}$$

from which it follows that:

$$\alpha = 90^\circ-m. \tag{2.50}$$

Similarly, we obtain:

$$\beta = 90^\circ-n, \tag{2.51}$$

$$\gamma = 90^\circ-p, \tag{2.52}$$

and

$$\delta = 90^\circ-q. \tag{2.53}$$

Of course, formulas (2.50), (2.51), (2.52), and (2.53) make sense if and only if $m < 90^\circ$, $n < 90^\circ$, $p < 90^\circ$, and $q < 90^\circ$, that means if and only if the quadrilateral $ABCD$ is acute, where we introduce the following definition:

Definition 2.1. *A quadrilateral is called* acute *if each interior angle made by a diagonal with a side of that quadrilateral is acute.*

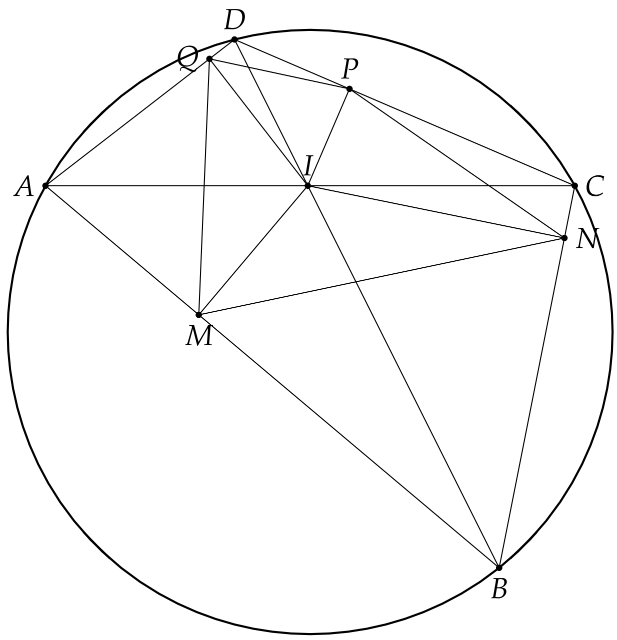

Figure 5. M5

Let $AC \cap BD = \{I\}$. We draw the perpendiculars from I to AB, BC, CA, and AB, and denote the feet of these perpendiculars by M, N, P, and Q, respectively. We will show that $MNPQM$ is a closed light-ray path, and we will call this particular closed light-ray path, for reasons that will become obvious later, *the circum-light-ray path* of the cyclic acute quadrilateral $ABCD$. See Fig. 5 above.

Indeed, because the angles $\sphericalangle IAB$ and $\sphericalangle IBA$ are both acute, the foot, M, of the altitude IM of the triangle IAB, belongs to the interior (AB) of the side $[AB]$ of the quadrilateral $ABCD$. Similarly, $N \in (BC)$, $P \in (CD)$, and $Q \in (DA)$. Since $m(\sphericalangle IMA) + m(\sphericalangle IQA) = 90^\circ + 90^\circ = 180^\circ$, the quadrilateral $IQAM$ is cyclic. Thus we have:

$$\begin{aligned} m(\sphericalangle AMQ) &= m(\sphericalangle AIQ) \\ &= 90^\circ - m. \end{aligned}$$

Similarly, we have $m(\sphericalangle BMN) = 90^\circ - m$, $m(\sphericalangle BNM) = m(\sphericalangle CNP) = 90^\circ - n$, $m(\sphericalangle CPN) = m(\sphericalangle DPQ) = 90^\circ - p$, and $m(\sphericalangle DQP) = m(\sphericalangle AQM) = 90^\circ - q$.
Hence, $MNPQ$ is a quadrilateral closed light-ray path.
Therefore, we have proven the following theorem:

Theorem 2.1. *A quadrilateral admits a quadrilateral closed light-ray path if and only if it is acute and cyclic.*

Proposition 2.1. *Let $ABCD$ be an acute cyclic quadrilateral with sides $AB = a$, $BC = b$, $CD = c$ and $DA = d$, and let R be the radius of its circumscribed circle. Let us denote the angles made by the diagonals with the sides in the following way: $m(\sphericalangle ACB) = m(\sphericalangle ADB) = u_{AB}$, $m(\sphericalangle BDC) = m(\sphericalangle BAC) = u_{BC}$, $m(\sphericalangle CAD) = m(\sphericalangle CBD) = u_{CD}$, and $m(\sphericalangle DBA) = m(\sphericalangle DCA) = u_{DA}$. Then, we can choose a point M on (AB), such that $AM = x$, and construct a closed ray-path starting at M, $MNPQM$, with $N \in (BC)$, $P \in (CD)$, and $Q \in (DA)$, if and* *only if:*

$$\begin{aligned} & 2R \cdot \max\left\{0, \frac{\sin(u_{AB})\cos(u_{CD}) - \sin(u_{BC})\cos(u_{DA})}{\cos(u_{CD})}\right\} \\ < & \ x \\ < & \ 2R \cdot \min\left\{\sin(u_{AB}), \frac{\cos(u_{BC})\sin(u_{DA})}{\cos(u_{CD})}\right\}. \end{aligned}$$

Proof. Of course, we have:

$$\begin{aligned} x &= AM \\ &< AB \\ &= 2R\sin(u_{AB}). \end{aligned} \tag{2.54}$$

We know that $\alpha = m(\sphericalangle BMN) = 90^\circ - u_{AB}$. For N to belong to the interior (BC) of the side $[BC]$, we must have $BN < BC$. Thus, according to equation (2.13), we have:

$$\begin{aligned} BN &= \frac{(a-x)\sin(\alpha)}{\sin(\beta)} \\ &< BC \\ &= b. \end{aligned} \tag{2.55}$$

Solving this inequality for x, we obtain:

$$\begin{aligned} x &> \frac{a\sin(\alpha) - b\sin(\beta)}{\sin(\alpha)} \\ &= \frac{2R\sin(u_{AB})\sin(90^\circ - u_{CD}) - 2R\sin(u_{BC})\sin(90^\circ - u_{DA})}{\sin(90^\circ - u_{CD})} \\ &= 2R \cdot \frac{\sin(u_{AB})\cos(u_{CD}) - \sin(u_{BC})\cos(u_{DA})}{\cos(u_{CD})}. \end{aligned} \tag{2.56}$$

For P to belong to (CD), we must have $CP < CD$. Thus, using (2.16), we have:

$$\begin{aligned} CP &= \frac{b\sin(\beta) - a\sin(\alpha) + x\sin(\alpha)}{\sin(\gamma)} \\ &< CD \\ &= c. \end{aligned} \tag{2.57}$$

Solving this inequality for x, we obtain:

$$\begin{aligned} x &< \frac{a\sin(\alpha) + c\sin(\gamma) - b\sin(\beta)}{\sin(\alpha)} \\ &= 2R \cdot \frac{\sin(u_{AB})\sin(90^\circ - u_{CD}) + \sin(u_{CD})\sin(90^\circ - u_{AB})}{\sin(90^\circ - u_{CD})} \\ &\quad - \frac{\sin(u_{BC})\sin(90^\circ - u_{DA})}{\sin(90^\circ - u_{CD})} \\ &= 2R \cdot \frac{\sin(u_{AB})\cos(u_{CD}) + \sin(u_{CD})\cos(u_{AB}) - \sin(u_{BC})\cos(u_{DA})}{\cos(u_{CD})} \\ &= 2R \cdot \frac{\sin(u_{AB} + u_{CD}) - \sin(u_{BC})\cos(u_{DA})}{\cos(u_{CD})}. \end{aligned}$$

Since $u_{AB} + u_{BC} + u_{CD} + u_{DA} = 180°$ (due to the fact that they are measures of angles with the vertices on the circumscribed circle of the cyclic quadrilateral $ABCD$, and together they subtend the entire circumscribed circle of this quadrilateral), we have $\sin(u_{AB} + u_{CD}) = \sin(u_{BC} + u_{DA})$. Thus, the last inequality becomes:

$$\begin{aligned} x &< 2R \cdot \frac{\sin(u_{AB} + u_{CD}) - \sin(u_{BC})\cos(u_{DA})}{\cos(u_{CD})} \\ &= 2R \cdot \frac{\sin(u_{BC} + u_{DA}) - \sin(u_{BC})\cos(u_{DA})}{\cos(u_{CD})} \\ &= 2R\frac{\cos(u_{BC})\sin(u_{DA})}{\cos(u_{CD})}. \end{aligned} \tag{2.58}$$

Finally, for Q to belong to the interior (DA) of the side $[DA]$, we must have $DQ < DA$. Thus, according to equation (2.19), we have:

$$\begin{aligned} DQ &= \frac{c\sin(\gamma) - b\sin(\beta) + a\sin(\alpha) - x\sin(\alpha)}{\sin(\delta)} \\ &< DA \\ &= d. \end{aligned} \tag{2.59}$$

Solving this inequality for x, we obtain:

$$\begin{aligned} x &> \frac{a\sin(\alpha) + c\sin(\gamma) - b\sin(\beta) - d\sin(\delta)}{\sin(\alpha)} \\ &= 2R \cdot \frac{\sin(u_{AB})\cos(u_{CD}) + \sin(u_{CD})\cos(u_{AB})}{\cos(u_{CD})} \\ &\quad - \frac{\sin(u_{BC})\cos(u_{DA}) + \sin(u_{DA})\cos(u_{BC})}{\cos(u_{CD})} \\ &= 2R \cdot \frac{\sin(u_{AB} + u_{CD}) - \sin(u_{BC} + u_{DA})}{\cos(u_{CD})} \\ &= 2R \cdot \frac{0}{\cos(u_{CD})} \\ &= 2R \cdot 0 \\ &= 0. \end{aligned}$$

So, this inequality is automatically satisfied if $x > 0$.
Therefore, x must satisfy conditions (2.54), (2.56), and (2.58), which means:

$$\begin{aligned} & 2R \cdot \max\left\{0, \frac{\sin(u_{AB})\cos(u_{CD}) - \sin(u_{BC})\cos(u_{DA})}{\cos(u_{CD})}\right\} \\ < \; & x \\ < \; & 2R \cdot \min\left\{\sin(u_{AB}), \frac{\cos(u_{BC})\sin(u_{DA})}{\cos(u_{CD})}\right\}. \end{aligned}$$

□

The proposition that we have just proven shows that there are infinitely many closed light ray-path in an acute cyclic quadrilateral, each of them being determined uniquely

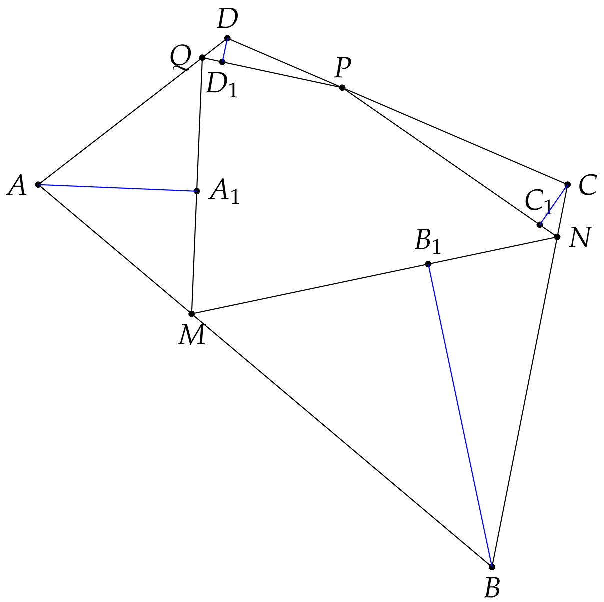

Figure 6. M6

by the length $x = AM$.
We prove now that all of these closed light-ray paths have the same total length.

Proposition 2.2. *Given an acute cyclic quadrilateral $ABCD$, then for every number $x = AM$ greater than $2R \cdot \max\{0, [\sin(u_{AB})\cos(u_{CD}) - \sin(u_{BC})\cos(u_{DA})] / \cos(u_{CD})\}$ and less than $2R \cdot \min\{\sin(u_{AB}), [\cos(u_{BC})\sin(u_{DA})] / \cos(u_{CD})\}$, the total length of the closed light-ray path $MNPQM$ (that means, the perimeter of the quadrilateral $MNPQ$) is:*

$$MN + NP + PQ + QM = \frac{AB \cdot CD + BC \cdot DA}{R} \tag{2.60}$$

$$= \frac{AC \cdot BD}{R}. \tag{2.61}$$

This perimeter is independent of the value of x.

Proof. Let us draw the perpendiculars AA_1, BB_1, CC_1, and DD_1 to the sides QM, MN, NP, and PQ, respectively, of the closed light-ray path $MNPQM$, where $A_1 \in QM$, $B_1 \in MN$, $C_1 \in NP$, and $D_1 \in PQ$. See Fig. 6. Since both angles $\sphericalangle AMQ$ and $\sphericalangle AQM$ of the triangle AMQ are acute, the foot, A_1, of the altitude AA_1, belongs for sure to the interior (QM) of the side $[QM]$. Similarly, $B_1 \in (MN)$, $C_1 \in (NP)$, and $D_1 \in (PQ)$. Due to this fact, the perimeter of the quadrilateral $MNPQ$ can be broken up as:

$$MN + NP + PQ + QM = (A_1M + MB_1) + (B_1N + NC_1) + (C_1P + PD_1) + (D_1Q + QA_1). \tag{2.62}$$

Using the definition of cosine in the right traingles AA_1M and BB_1M, and the Law of Sines in the triangle ACD, we obtain:

$$\begin{aligned} A_1M + MB_1 &= AM \cdot \cos(\sphericalangle AMA_1) + MB \cdot \cos(\sphericalangle BMB_1) \\ &= AM\cos(\alpha) + MB\cos(\alpha) \\ &= (AM + MB)\cos(\alpha) \\ &= AB\cos(\alpha) \\ &= AB\cos\left(90^\circ - u_{CD}\right) \\ &= AB\sin\left(u_{CD}\right) \\ &= AB \cdot \frac{CD}{2R} \\ &= \frac{AB \cdot CD}{2R}. \end{aligned} \tag{2.63}$$

Similarly, we have:

$$B_1N + NC_1 = \frac{BC \cdot DA}{2R}, \tag{2.64}$$

$$C_1P + PD_1 = \frac{CD \cdot AB}{2R}, \tag{2.65}$$

and

$$D_1Q + QA_1 = \frac{DA \cdot BC}{2R}. \tag{2.66}$$

Summing up the equations (2.63), (2.64), (2.65), and (2.66), and using the decomposition equation (2.62), we obtain:

$$\begin{aligned} MN + NP + PQ + QM &= 2 \cdot \frac{AB \cdot CD}{2R} + 2 \cdot \frac{BC \cdot DA}{2R} \\ &= \frac{AB \cdot CD + BC \cdot DA}{R}. \end{aligned}$$

Since the quadrilateral $ABCD$ is cyclic, by Ptolemy theorem, we have:

$$AB \cdot CD + BC \cdot DA = AC \cdot BD. \tag{2.67}$$

Thus, the perimeter of the quadrilateral $MNPQ$ can also be written as:

$$MN + NP + PQ + QM = \frac{AC \cdot BD}{R}.$$

□

We show now that if a quadrilateral $ABCD$ admits a closed light-ray path, then among all quadrilaterals $MNPQ$ inscribed in $ABCD$, the cloosed light-ray path quadrialterals have the smallest perimeter. Our proof is inspired by the proof of Fagnano Theorem from [1].

Theorem 2.2. *Let $ABCD$ be an acute cyclic quadrilateral. Then for any point $M \in (AB)$, $N \in (BC)$, $P \in (CD)$, and $Q \in (DA)$, we have:*

$$MN + NP + PQ + QM \geq \frac{AC \cdot BD}{R}. \tag{2.68}$$

The equality holds in inequality (2.68) if and only if $MNPQM$ is a closed light-ray path.

Proof. Applying the Law of Cosines in triangle AQM, we have:

$$\begin{aligned} QM^2 &= AQ^2 + AM^2 - 2AQ \cdot AM \cdot \cos(\sphericalangle QAM) \\ &= AQ^2 + AM^2 - 2AQ \cdot AM \cdot \cos(u_{BC} + u_{CD}) \\ &= AQ^2 \sin^2(u_{BC}) + AM^2 \sin^2(u_{CD}) + 2AQ \cdot AM \cdot \sin(u_{BC}) \sin(u_{CD}) \\ &\quad + AQ^2 \cos^2(u_{BC}) + AM^2 \cos^2(u_{CD}) - 2AQ \cdot AM \cdot \cos(u_{BC}) \cos(u_{CD}) \\ &= [AQ \sin(u_{BC}) + AM \sin(u_{CD})]^2 + [AQ \cos(u_{BC}) - AM \cos(u_{CD})]^2 \\ &\geq [AQ \sin(u_{BC}) + AM \sin(u_{CD})]^2 . \end{aligned} \tag{2.69}$$

Taking the square root from both sides of inequality (2.69), we obtain:

$$QM \geq AQ \sin(u_{BC}) + AM \sin(u_{CD}) . \tag{2.70}$$

The equality in inequality (2.70) (or equivalently in (2.69)) holds if and only if:

$$AQ \cos(u_{BC}) - AM \cos(u_{CD}) = 0$$

which is equivalent to:

$$\begin{aligned} \frac{AM}{AQ} &= \frac{\cos(u_{BC})}{\cos(u_{CD})} \\ &= \frac{\sin(90^\circ - u_{BC})}{\sin(90^\circ - u_{CD})}. \end{aligned} \tag{2.71}$$

That means, if $M'N'P'Q'M'$ is a closed light-ray path, with $M' \in (AB)$, $N' \in (BC)$, $P' \in (CD)$, and $Q' \in (DA)$, we have:

$$\begin{aligned} \frac{AM}{AQ} &= \frac{AM'}{AQ'} \qquad (2.72) \\ &= \frac{\sin(90^\circ - u_{BC})}{\sin(90^\circ - u_{CD})}. \end{aligned}$$

Therefore, by the reciprocal of Thales Theorem (or by the similarity of the triangles AMQ and $AM'Q'$), we conclude that the lines QM and $Q'M'$ are parallel. Thus, for the equality case, we must have:

$$\begin{aligned} m(\sphericalangle AMQ) &= m\left(\sphericalangle AM'Q'\right) \\ &= 90^\circ - u_{CD} \end{aligned} \tag{2.73}$$

and

$$\begin{aligned} m(\sphericalangle AQM) &= m\left(\sphericalangle AQ'M'\right) \\ &= 90^\circ - u_{BC}. \end{aligned} \tag{2.74}$$

Similarly, we have:

$$MN \geq BM \sin(u_{CD}) + BN \sin(u_{DA}) , \tag{2.75}$$

with equality if and only if

$$m(\sphericalangle BMN) = 90^\circ - u_{CD} \tag{2.76}$$

and

$$m(\sphericalangle BNM) = 90^\circ - u_{DA}. \quad (2.77)$$

We also have:

$$NP \geq CN\sin(u_{DA}) + CP\sin(u_{AB}), \quad (2.78)$$

with equality if and only if

$$m(\sphericalangle CNP) = 90^\circ - u_{DA} \quad (2.79)$$

and

$$m(\sphericalangle CPN) = 90^\circ - u_{AB}. \quad (2.80)$$

Finally, we have:

$$PQ \geq DP\sin(u_{AB}) + DQ\sin(u_{BC}), \quad (2.81)$$

with equality if and only if

$$m(\sphericalangle DPQ) = 90^\circ - u_{AB} \quad (2.82)$$

and

$$m(\sphericalangle DQP) = 90^\circ - u_{BC}. \quad (2.83)$$

Summing up inequalities (2.70), (2.75), (2.78), and (2.81), we obtain:

$$\begin{aligned} QM + MN + NP + PQ \geq\ & AQ\sin(u_{BC}) + AM\sin(u_{CD}) \\ & +BM\sin(u_{CD}) + BN\sin(u_{DA}) \\ & +CN\sin(u_{DA}) + CP\sin(u_{AB}) \\ & +DP\sin(u_{AB}) + DQ\sin(u_{BC}). \end{aligned} \quad (2.84)$$

We can rearrange the terms in the right side of this inequality as:

$$\begin{aligned} QM + MN + NP + PQ &\geq (AM + BM)\sin(u_{CD}) + (BN + CN)\sin(u_{DA}) \\ & \quad (CP + DP)\sin(u_{AB}) + (DQ + AQ)\sin(u_{BC}) \\ &= AB\sin(u_{CD}) + BC\sin(u_{DA}) \\ & \quad +CD\sin(u_{AB}) + DA\sin(u_{BC}). \end{aligned} \quad (2.85)$$

Using the Law of Sines in the cyclic quadrilateral $ABCD$, the last inequality can be rewritten as:

$$\begin{aligned} QM + MN + NP + PQ &\geq AB \cdot \frac{CD}{2R} + BC \cdot \frac{DA}{2R} \\ &= CD \cdot \frac{AB}{2R} + DA \cdot \frac{BC}{2R} \\ &= \frac{AB \cdot CD + BC \cdot DA}{R}. \end{aligned} \quad (2.86)$$

From the previous proposition, we can see that the right side of inequality (2.86) is equal to the perimeter of any closed light-ray path of the quadrilateral $ABCD$.
To have equality in (2.86), we must have equality in all inequalities that we used. That means:

$$m(\sphericalangle AQM) = 90^\circ - u_{BC},$$

$$m(\sphericalangle AMQ) = 90^\circ - u_{CD},$$

$$m(\sphericalangle BMN) = 90^\circ - u_{CD},$$

$$m(\sphericalangle BNM) = 90^\circ - u_{DA},$$

$$m(\sphericalangle CNP) = 90^\circ - u_{DA},$$

$$m(\sphericalangle CPN) = 90^\circ - u_{AB},$$

$$m(\sphericalangle DPQ) = 90^\circ - u_{AB},$$

and

$$m(\sphericalangle DQP) = 90^\circ - u_{BC}.$$

That means, the equality holds if and only if $MNPQM$ is a closed light-ray path. □

2.1. **Some geometric properties.** In this last subsection of the paper, we present some geometric properties involving the quadrilateral closed light-ray paths. We introduce the following terminology:

Definition 2.2. *If $ABCD$ is an acute cyclic quadrilateral and $MNPQM$ is a closed light-ray path, with $M \in (AB)$, $N \in (BC)$, $P \in (CD)$, and $Q \in (DA)$, then we say that:*

- *the diagonal AC is* opposite *to and the diagonal BD is* transverse *to the opposite sides MN and PQ of the closed light-ray path $MNPQM$.*
- *the diagonal BD is* opposite *to and the diagonal AC is* transverse *to the opposite sides NP and QM of the closed light-ray path $MNPQM$.*

We introduce also the following convention:

Definition 2.3. *Given three lines d, g, and h in the plane, we say that d, g, and h are* concurent *if one of the following two situtions occurs:*

- *$d \cap g \cap h \neq \emptyset$, that means the three lines share a common point in the regular plane.*
- *$d \parallel g \parallel h$, that means the three lines are parallel, and so they intersect at a point situated on the line at infinity of the projective plane.*

With this terminology, we have the following proposition:

Proposition 2.3. *Every two opposite sides of any closed light-ray path in an acute cyclic quadrilateral and their opposite diagonal are concurrent.*
More precisely, we have two possibilities:

- *If an angle of the acute cyclic quadrilateral is not right (and consequently its opposite angle is also not right), then the diagonal joining the vertices of these two opposite angles and the two sides of any closed light-ray path, that are opposite to it, intersect at a point, in the regular plane.*
- *If the acute cyclic quadrilateral has a right angle (and consequently its opposite angle is also right), then the diagonal joining the vertices of these two opposite right angles is parallel to the two sides of any closed light-ray path that are opposite to it.*

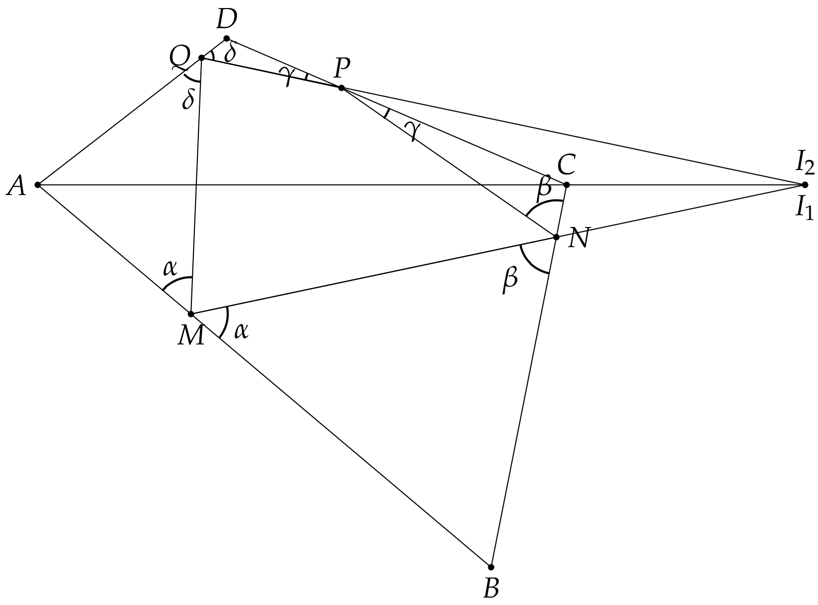

Figure 7. M7

Proof. Let $ABCD$ be an acute cyclic quadrilateral and $MNPQM$ a closed light-ray path, with $M \in (AB)$, $N \in (BC)$, $P \in (CD)$, and $Q \in (DA)$. Let us define the points:

$$\{I_1\} := MN \cap AC \quad \text{and} \quad \{I_2\} := QP \cap AC, \tag{2.87}$$

with the convention that if $MN \parallel AC$, then $I_1 := \infty_{AC}$, and if $QP \parallel AC$, then $I_2 := \infty_{AC}$, where ∞_{AC} denotes the infinity point of the line AC in the projective plane.
We want to show that $I_1 = I_2$.
Applying Menelaus Theorem in the triangle ABC cut by the transversal I_1NM, we have:

$$\frac{I_1C}{I_1A} \cdot \frac{MA}{MB} \cdot \frac{NB}{NC} = +1, \tag{2.88}$$

where we are working with oriented segments, and the + sign in front of the number 1 was put to remind us about this agreement. In (2.88), if $I_1 = \infty_{AC}$, then we make the convention that $I_1C/I_1A = \infty_{AC}C/\infty_{AC}A := 1$, in which case Menelaus Theorem becomes Thales Theorem.
Applying Menelaus Theorem in the triangle ADC cut by the transversal I_2PQ, we have:

$$\frac{I_2C}{I_2A} \cdot \frac{QA}{QD} \cdot \frac{PD}{PC} = +1, \tag{2.89}$$

with similar conventions as before.
It follows from (2.88) and (2.89) that:

$$\frac{I_1C}{I_1A} = \frac{MB}{MA} \cdot \frac{NC}{NB} \tag{2.90}$$

and

$$\frac{I_2C}{I_2A} = \frac{QD}{QA} \cdot \frac{PC}{PD}. \tag{2.91}$$

To show that $I_1 = I_2$, since we are working with oriented segments, we must prove that:

$$\frac{I_1C}{I_1A} = \frac{I_2C}{I_2A}. \tag{2.92}$$

This means, via equations (2.90) and (2.91), that me must prove that:

$$\frac{MB}{MA} \cdot \frac{NC}{NB} = \frac{QD}{QA} \cdot \frac{PC}{PD}, \tag{2.93}$$

which, after dividing both sides by the right side, is equivalent to:

$$\frac{MB}{MA} \cdot \frac{NC}{NB} \cdot \frac{QA}{QD} \cdot \frac{PD}{PC} = 1. \tag{2.94}$$

The last equation, that we must prove, can be rewritten as:

$$\frac{MB}{NB} \cdot \frac{NC}{PC} \cdot \frac{PD}{QD} \cdot \frac{QA}{MA} = 1, \tag{2.95}$$

and in this moment, we are no longer working with oriented segments.
Indeed, using the Law of Sines in the triangles BMN, CNP, DPQ, and AQM, equation (2.95) is equivalent to:

$$\frac{\sin(\beta)}{\sin(\alpha)} \cdot \frac{\sin(\gamma)}{\sin(\beta)} \cdot \frac{\sin(\delta)}{\sin(\gamma)} \cdot \frac{\sin(\alpha)}{\sin(\delta)} = 1, \tag{2.96}$$

which is obviously true.
Similarly, we can prove that the opposite sides NP and QM, of the closed light-ray path $MNPQM$, and their opposite diagonal BD are concurrent.
It remains to discuss the case when MN and PQ are parallel.
We have $MN \parallel PQ$ if and only if

$$m(\sphericalangle MNP) + m(\sphericalangle NPQ) = 180^\circ, \tag{2.97}$$

as interior supplementary angles of the same side of the secant.
Equation (2.97) is equivalent to:

$$[180^\circ - m(\sphericalangle BNM) - m(\sphericalangle CNP)] + [180^\circ - m(\sphericalangle CPN) - m(\sphericalangle DPQ)] = 180^\circ,$$

which reduces further to the equation:

$$2\beta + 2\gamma = 180^\circ. \tag{2.98}$$

Since $\beta = 90^\circ - u_{DA}$ and $\gamma = 90^\circ - u_{AB}$, equation (2.98) is equivalent to:

$$u_{DA} + u_{AB} = 90^\circ, \tag{2.99}$$

which means that:

$$m(\sphericalangle BCD) = 90^\circ, \tag{2.100}$$

and since $ABCD$ is a cyclic quadrilateral, we also have:

$$\begin{aligned} m(\sphericalangle DAB) &= 180^\circ - m(\sphericalangle BCD) \\ &= 90^\circ. \end{aligned} \tag{2.101}$$

That means, the diagonal BD is a diameter of the circumscribed circle of the cyclic quadrilateral $ABCD$.
Similarly, the case $NP \parallel QM$ is equivalent to $m(\sphericalangle ABC) = m(\sphericalangle CDA) = 90^\circ$. □

We also have the following lemma:

Lemma 2.1. *In an acute cyclic quadrilateral, the bisectors of any two consecutive interior angles, of a closed light-ray path, intersect at a point located on the diagonal, of the quadrilateral, that is transverse to their common side. That means, if $ABCD$ is an acute cyclic quadrilateral and $MNPQM$ is a closed light-ray path, with $M \in (AB)$, $N \in (BC)$, $P \in (CD)$, and $Q \in (DA)$, and if b_M and b_Q denote the bisectors of the angles $\sphericalangle QMN$ and $\sphericalangle MQP$, respectively, then we have:*

$$b_M \cap b_Q \cap AC \quad \neq \quad \emptyset. \tag{2.102}$$

Proof. Let us denote:

$$b_Q \cap b_M \quad = \quad \{J_{QM}\}. \tag{2.103}$$

Since b_M is the interior bisector of $\sphericalangle QMN$, and AB is the exterior bisector of $\sphericalangle QMN$ (because $m(\sphericalangle AMQ) = m(\sphericalangle BMN) = \alpha$), we have $b_M \perp AB$.
Similarly, we have $b_Q \perp DA$.
Thus, we obtain:

$$\begin{aligned} m(\sphericalangle AMJ_{QM}) + m(\sphericalangle AQJ_{QM}) &= 90^\circ + 90^\circ \\ &= 180^\circ. \end{aligned} \tag{2.104}$$

Therefore, the quadrilateral $AMJ_{QM}Q$ is cyclic. This implies:

$$\begin{aligned} m\left(\sphericalangle MAJ_{QM}\right) &= m\left(\sphericalangle MQJ_{QM}\right) \\ &= m\left(\sphericalangle AQJ_{QM}\right) - m\left(\sphericalangle AQM\right) \\ &= 90^\circ - \delta \\ &= 90^\circ - (90^\circ - u_{BC}) \\ &= u_{BC} \\ &= m(\sphericalangle BAC). \end{aligned} \tag{2.105}$$

Since $m(\sphericalangle MAJ_{QM}) = m(\sphericalangle BAC)$, we conclude that the points A, J_{QM}, and C are collinear. □

Corollary 2.1. *In an acute cyclic quadrilateral, every two opposite sides of a closed light-ray path are symmetric about their opposite diagonal.*
Moreover, the circum-light-ray path, formed by the feet of the perpendiculars dropped from the point of intersection of the diagonals to the sides of the acute cyclic quadrileteral is the only closed light-ray path whose sides form a tangential (circumscriptible) quadrilateral, that means a quadrilateral for which there exists a circle that is tangent to all of its four sides.

Proof. Indeed, with the notations from before, we know from the previous lemma that the point J_{QM} belongs to the diagonal AC of the acute cyclic quadrilateral $ABCD$.
We distinguish between two cases:
Case 1. If MN and PQ are not parallel, then let $MN \cap PQ := \{I_1\}$.
Since b_M and b_Q are the bisectors of the angles of the quadrilateral $MNPQ$, we have two possibilities:

- If N is located in between M and I_1, and P is located in between Q and I_1, then J_{QM} is the center of the incircle of the triangle I_1QM. Thus the point J_{QM} is located on the bisector of the angle $\sphericalangle MI_1Q$. Therefore, the line I_1J_{QM} is the bisector of the angle $\sphericalangle MI_1Q$. Since the line I_1J_{QM} coincides with the line AC, we conclude

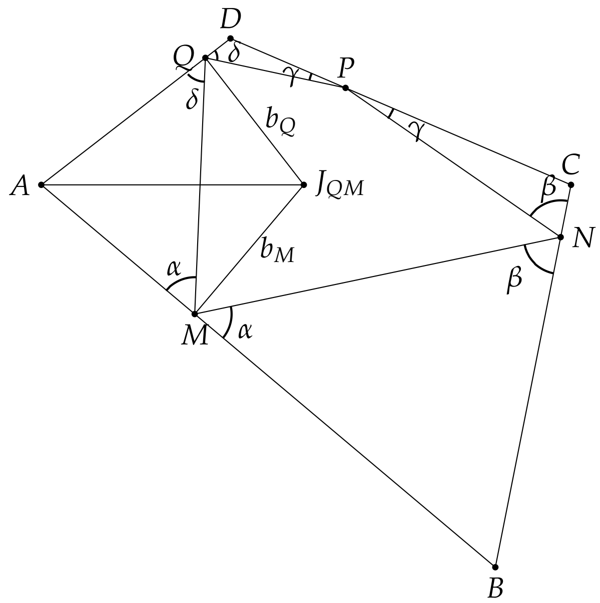

Figure 8. M8

that the line AC is the bisector of the angle $\sphericalangle MI_1Q$. Therefore, the lines MN and QP are symmetric about the line AC.

- If M is located in between N and I_1, and Q is located in between P and I_1, then J_{QM} is the center of the excircle of the triangle I_1QM that touches the side QM in its interior. Thus again the point J_{QM} is located on the bisector of the angle $\sphericalangle MI_1Q$, and as before we conclude that the lines MN and QP are symmetric about the line AC.

Case 2. If $MN \parallel QP$, then we know that AC is also parallel to MN and QP, and the point of intersection of the bisectors b_M and b_Q, J_{QM}, is located on the diagonal AC. Therefore, the line AC is the parallel drawn from the point J_{QM} to the lines MN and QP.
Let K, L, and R be the feet of the perpendiculars dropped from J_{QM} to the lines MN, QP, and MQ. See Fig. 10. Since $MN \parallel QP$, the points K, J_{QM}, and L are collinear.
Since J_{QM} is on the bisector b_M of the angle $\sphericalangle QMN$, J_{QM} is equally far away from the sides MN and MQ of this angle. Thus, we have:

$$J_{QM}K \;=\; J_{QM}R. \tag{2.106}$$

Since J_{QM} is on the bisector b_Q of the angle $\sphericalangle PQM$, J_{QM} is equally far away from the sides QP and QM of this angle. Thus, we have:

$$J_{QM}L \;=\; J_{QM}R. \tag{2.107}$$

It follows from equations (2.106) and (2.107), by trasitivity, that:

$$J_{QM}K \;=\; J_{QM}L. \tag{2.108}$$

Since KL is perpendicular to both MN and QP, it follows from equation (2.108) that J_{QM} is equally far away from the lines MN and QP, and thus since AC is the parallel drawn from the point J_{QM} to the lines MN and QP, the line QP is the symmetric of the line MN with respect to the line (mirror) AC. □

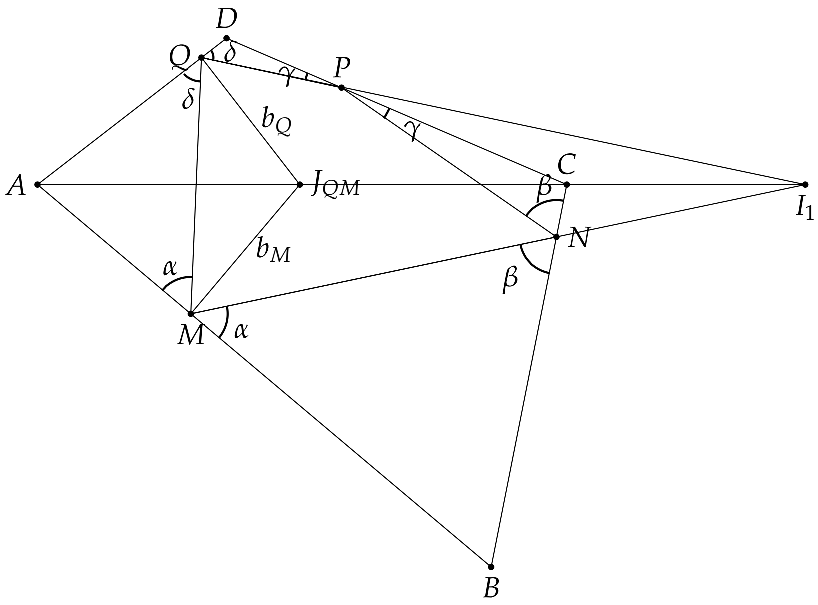

Figure 9. M9

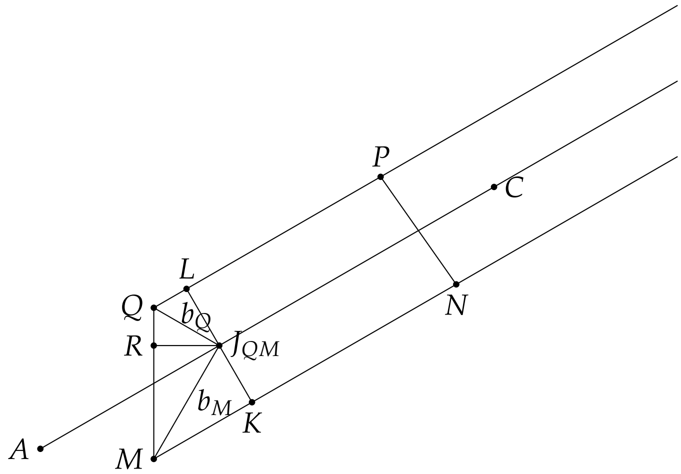

Figure 10. M10

Finally, we can see that the quadrilateral $MNPQ$ is tangential if and only if the bisectors b_M, b_N, b_P, and b_Q of its angles are all concurrent at a point I, that is the center of the inscribed circle in the quadrilateral $MNPQ$. Since b_M and b_N intersect at a point J_{MN} that belongs to the diagonal BD of $ABCD$, while b_N and b_P intersect at a point J_{NP} that belongs to the diagonal AC of $ABCD$, in order for the points J_{MN} and J_{NP} to coincide, we must have that $I := J_{MN} = J_{NP}$ is the point of the intersection of the diagonals BD and AC of the acute cyclic quadrilateral $ABCD$. Thus, M, N, P, and Q must be the feet of the perpendiculars dropped from the point I of intersection of the diagonals AC and BD to the sides AB, BC, CD, and DA of $ABCD$. Therefore, $MNPQM$ must be the circum-light-ray path.

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