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DISTANCES INVOLVING NOTABLE POINTS F_+, F_-, J_+, J_-

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Abstract. In this paper, formulas are obtained for the distances of the points F_+, F_-, J_+, J_- at points O, G, H, N, K as well as between them. The formulae express these distances by Δ, l_+, l_- and f (see (1.1), (1.13)), and finally by a, b, c. As an application, one remark on Evans' conic and another on the triangle OKM are made.

Consider a reference triangle ABC and assume that $a > b > c$, without restricting the generality. Denote the Fermat (or isogonic) points of the triangle ABC by F_+ and F_-, and the isodynamic points by J_+ and J_-. We utilize the standard notations of triangle geometry. So, we consider known the meanings of the notations O, H, G, K or R and Δ. We also denote N the nine-point center and M the midpoint of HG. The purpose of this note is to find a lot of formulas for the distances of points F_+, F_-, J_+, J_- to points O, H, G, K and M, as well as between them, all these formulas expressed by a, b, c. Finally, we use the formulas found in two applications. *We do not use barycentric or trilinear coordinates; all problems are dealt with in an elementary way.*
The properties of the points used in this work are generally well known. There are many studies on these notable points. We quote a few: [11], [12], [1], [6], [10], [7], [8]. Recently, in this journal appeared the paper [9] which contains forty-five distances between various notable points of a triangle.

1. Preliminaries

Let A_+ and A_- be the vertices of the equilateral triangles built on the BC outside and inside the triangle ABC, respectively; similar for B_+, B_- and C_+, C_-(Fig. 1). It is known that $F_+ = AA_+ \cap BB_+ \cap CC_+$ and $F_- = AA_- \cap BB_- \cap CC_-$ and that $AA_+ = BB_+ = CC_+$ and $AA_- = BB_- = CC_-$ For the common lengths of these segments, denoted l_+ and l_-, we have [6, p. 220]:

$$l_+^2 = \frac{1}{2}\left(a^2 + b^2 + c^2 + 4\sqrt{3}\Delta\right), \quad l_-^2 = \frac{1}{2}\left(a^2 + b^2 + c^2 - 4\sqrt{3}\Delta\right). \tag{1.1}$$

Let $\varphi_A, \varphi_B, \varphi_C$ be the angles defined by

$$\varphi_A = \widehat{h_a, m_a}, \quad \varphi_B = \widehat{h_b, m_b}, \quad \varphi_C = \widehat{h_c, m_c} \tag{1.2}$$

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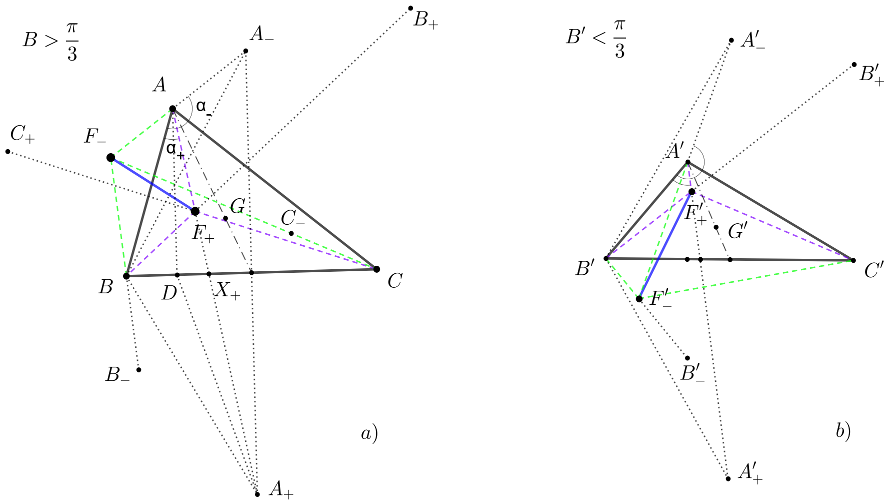

Figure 1

(h_a, m_a- lengths of the altitude and median corresponding to BC, etc.). Denote A', B', C' the midpoints of the sides BC, CA, AB and D, E, F the feet of the perpendicular from the vertices A, B, C on the oposite sides BC, CA, AB of the triangle ABC. We have:

$$DA' = \frac{|b^2 - c^2|}{2a}, \quad EB' = \frac{|c^2 - a^2|}{2b}, \quad FC' = \frac{|a^2 - b^2|}{2c}, \tag{1.3}$$

By applying the sine and cosine laws to triangle ADA' we deduce the formulas:

$$\sin\varphi_A = \frac{|b^2 - c^2|}{2am_a}, \qquad \cos\varphi_A = \frac{h_a}{m_a} = \frac{2\Delta}{am_a}, \tag{1.4}$$

and then their analogues for φ_B and φ_C.
In addition to the assumption $a > b > c$, we will consider two cases:

I. $A > B > \frac{\pi}{3} > C$,
II. $A > \frac{\pi}{3} > B > C$.

Then, it is easy to determine what is the position of the point F_- in the plane of the triangle in each of these cases (Fig. 1). Denote α_+ (resp. α_-) the measure of the counterclockwise oriented angle $\widehat{BAA_+}$ (resp. $\widehat{BAA_-}$); β_+, β_- and γ_+, γ_- are similarly defined. These angles, as well as the angles φ_A, φ_B and φ_C, were introduced in [4]. Their use allows for an elementary approach to the intended purpose. Due to the assumption $a > b > c$, we will only need the angles φ_A, α_+, α_-.The next two statements appear in the cited work; for the convenience of the rader, we again state and prove it.

Lemma 1.1. *We have:*

$$\sin\alpha_+ = \frac{4\Delta + \sqrt{3}\left(c^2 + a^2 - b^2\right)}{4cl_+}, \qquad \cos\alpha_+ = \frac{b^2 + 3c^2 - a^2 + 4\sqrt{3}\Delta}{4cl_+}; \tag{1.5}$$

$$\sin\alpha_- = \frac{4\Delta - \sqrt{3}\left(c^2 + a^2 - b^2\right)}{4cl_-}, \qquad \cos\alpha_- = \frac{b^2 + 3c^2 - a^2 - 4\sqrt{3}\Delta}{4cl_-}, \tag{1.6}$$

and formulas for β_+, γ_+ and β_-, γ_- cyclically obtained from them.

Proof. In both cases mentioned above, it is enough to apply the sine and cosine formulas to the triangles ABA_+ and ABA_-. For example,

$$\sin\alpha_+ = \frac{a\sin\left(B+\frac{\pi}{3}\right)}{l_+} = \frac{a\left(\sin B+\sqrt{3}\cos B\right)}{2l_+} = \frac{4\Delta+\sqrt{3}\left(c^2+a^2-b^2\right)}{4cl_+},$$

and

$$\cos\alpha_+ = \frac{l_+^2+c^2-a^2}{2cl_+} = \frac{b^2+3c^2-a^2+4\sqrt{3}\Delta}{4cl_+}$$

(I used the formulas $\sin B = \dfrac{2\Delta}{ca}$ and $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$). □

Lemma 1.2. *The distances of F_+ and F_- to the vertix A are given by the formulas*

$$F_+A = \frac{1}{2\sqrt{3}}\frac{4\Delta+\sqrt{3}\left(b^2+c^2-a^2\right)}{l_+},\ F_-A = \frac{1}{2\sqrt{3}}\frac{4\Delta-\sqrt{3}\left(b^2+c^2-a^2\right)}{l_-}. \tag{1.7}$$

Proof. Consider the triangle F_+AB.
Note that $\widehat{AF_+B} = \frac{2\pi}{3.}$ and $\widehat{ABF_+} = \pi-\left(\alpha_+ + \frac{2\pi}{3}\right)$. By the sine formula,

$$F_+A = \sin\left[\pi-\left(\alpha_+ + \frac{2\pi}{3}\right)\right]\frac{c}{\sin\frac{2\pi}{3}}.$$

Taking into account (1.5), we get the first formula. For the second we can do the same in the triangle F_-AB. □

We will routinely use the following identities:

$$16\Delta^2 = 2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4 \text{ (Heron)}, \tag{1.8}$$

$$l_+^2+l_-^2 = a^2+b^2+c^2, \qquad l_+^2-l_-^2 = 4\sqrt{3}\Delta, \tag{1.9}$$

$$\begin{aligned} 4l_+^2l_-^2 &= \left(a^2+b^2+c^2\right)^2-3\cdot 16\Delta^2 \\ &= 2\left[\left(a^2-b^2\right)^2+\left(a^2-c^2\right)^2+\left(b^2-c^2\right)^2\right], \end{aligned} \tag{1.10}$$

$$9a^2b^2c^2-16\Delta^2\left(a^2+b^2+c^2\right) = f(a,b,c), \tag{1.11}$$

$$\begin{aligned} a^8+b^8+c^8-a^6b^2-a^2b^6-a^6c^2-a^2c^6-b^6c^2-b^2c^6 \\ = 5a^2b^2c^2\sum a^2-8\Delta^2\sum\left(a^2+b^2\right)^2, \end{aligned} \tag{1.12}$$

where

$$f(a,b,c) = a^6+b^6+c^6+3a^2b^2c^2-a^4b^2-a^2b^4-a^4c^2-a^2c^4-b^4c^2-b^2c^4. \tag{1.13}$$

The points mentioned above are notable points of the triangle. They are located on three important axes of the triangle: the Euler line OH, the Brocard axis OK, and the Fermat axis F_+F_-. These axes determine the triangle OKM (Fig. 2).

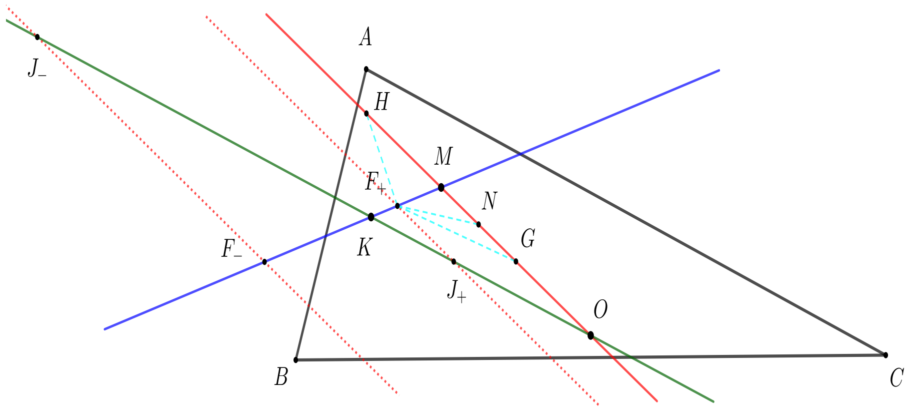

Figure 2

2. Distances between F_+, F_- and G, H, O, M, N, K

We will calculate these distances using the cosine law, median theorem or Stewart's theorem.
Denote X_+ the point at which AF_+ intersects the side BC. We need the following result:

Lemma 2.1. *The points A', X_+, D are placed on the side BC in the order $B - D - X_+ - A' - C$.*

Proof. Because $a > b > c$, we have $B - D - A' - C$. According to the same assumptions, it follows that the quadrilateral ADA_+A' is convex (Fig. 1). The point of intersection of its diagonals, X_+, is inside them, hence $D - X_+ - A'$. □

Proposition 2.1. *The distances between Fermat points F_+, F_- and the centroid G are given by*

$$F_+G = \frac{1}{3}l_-, \tag{2.1}$$

and

$$F_-G = \frac{1}{3}l_+. \tag{2.2}$$

Proof. By Lemma 2.1, and applying the cosine law to triangle AF_+G, we have:

$$F_+G^2 = F_+A^2 + AG^2 - 2F_+A \cdot AG \cdot \cos \widehat{F_+AG}.$$

In both of the cases mentioned above, I and II, we have (Fig. 1):

$$\widehat{F_+AG} = \widehat{BAG} - \widehat{BAA_+} = \left[\left(\frac{\pi}{2} - B\right) + \varphi_A\right] - \alpha_+ = \frac{\pi}{2} - (B + \alpha_+ - \varphi_A).$$

Hence

$$\cos \widehat{F_+AG} = \sin (B + \alpha_+ - \varphi_A).$$

Taking into account (1.4), (1.5), and the formula $4m_a^2 = 2b^2 + 2c^2 - a^2$, we have:

$$\begin{aligned} \cos \widehat{F_+AG} &= \sin (B + \alpha_+ - \varphi_A) = \sin (B + \alpha_+) \cos \varphi_A - \cos (B + \alpha_+) \sin \varphi_A \\ &= [\sin B \cos \alpha_+ + \cos B \sin \alpha_+] \frac{2\Delta}{am_a} \end{aligned}$$

$$\begin{aligned}
&-\left[\cos B\cos\alpha_+ - \sin B\sin\alpha_+\right]\frac{b^2-c^2}{2am_a}\\
=\ &\left[\frac{2\Delta}{ac}\frac{2\Delta}{am_a} - \frac{a^2+c^2-b^2}{2ac}\frac{b^2-c^2}{2am_a}\right]\cos\alpha_+\\
&+\left[\frac{a^2+c^2-b^2}{2ac}\frac{2\Delta}{am_a} + \frac{2\Delta}{ac}\frac{b^2-c^2}{2am_a}\right]\sin\alpha_+\\
=\ &\frac{1}{4cm_a}\left[\left(b^2+3c^2-a^2\right)\cos\alpha_+ + 4\Delta\sin\alpha_+\right]\\
=\ &\frac{1}{4m_al_+}\left(2b^2+2c^2-a^2+4\sqrt{3}\Delta\right)\\
=\ &\frac{1}{m_al_+}\left(\sqrt{3}\Delta+m_a^2\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
F_+G^2 =\ &\left(\frac{1}{2\sqrt{3}}\frac{4\Delta+\sqrt{3}\left(b^2+c^2-a^2\right)}{l_+}\right)^2 + \frac{4}{9}m_a^2 -\\
&-2\cdot\frac{1}{2\sqrt{3}}\frac{4\Delta+\sqrt{3}\left(b^2+c^2-a^2\right)}{l_+}\cdot\frac{2}{3}m_a\cdot\frac{1}{m_al_+}\left(\sqrt{3}\Delta+m_a^2\right),
\end{aligned}$$

and, after a routine calculation, we get:

$$F_+G^2 = \frac{1}{18l_+^2}\left[\left(a^2-b^2\right)^2+\left(a^2-c^2\right)^2+\left(b^2-c^2\right)^2\right] = \frac{1}{9}l_-^2;$$

hence, the formula (2.1) is proven.
Now, we calculate the distance F_-G in the same manner. By the cosine law applied to the triangle F_-AG, we have:

$$F_-G^2 = F_-A^2 + AG^2 - 2F_-A\cdot AG\cos\widehat{F_-AG}.$$

It is easy to see that in case I (Fig. 1a) $\widehat{F_-AG} = \widehat{F_-AB} + \widehat{BAG} = (\pi-\alpha_-) + \left[\left(\frac{\pi}{2}-B\right)+\varphi_A\right] = \frac{3\pi}{2} - (B+\alpha_- - \varphi_A)$, and in case II (Fig. 1b) $\widehat{F_-AG} = \widehat{BAG} - \widehat{BAF_-} = \left[\left(\frac{\pi}{2}-B\right)+\varphi_A\right] - (\alpha_- - \pi) = \frac{3\pi}{2} - (B+\alpha_- - \varphi_A)$ (Fig. 1). So, in both cases, we have $\cos\widehat{F_-AG} = -\sin(B+\alpha_- - \varphi_A)$, and as above, we get:

$$\cos\widehat{F_-AG} = \frac{1}{m_al_-}\left(\sqrt{3}\Delta - m_a^2\right).$$

It follows that

$$\begin{aligned}
F_-G^2 =\ &\left(\frac{1}{2\sqrt{3}}\frac{4\Delta-\sqrt{3}\left(b^2+c^2-a^2\right)}{l_-}\right)^2 + \frac{4}{9}m_a^2 -\\
&-2\cdot\frac{1}{2\sqrt{3}}\frac{4\Delta-\sqrt{3}\left(b^2+c^2-a^2\right)}{l_-}\cdot\frac{2}{3}m_a\cdot\frac{1}{m_al_-}\left(\sqrt{3}\Delta-m_a^2\right),
\end{aligned}$$

and finally

$$F_-G^2 = \frac{1}{18l_-^2}\left[\left(a^2-b^2\right)^2+\left(a^2-c^2\right)^2+\left(b^2-c^2\right)^2\right] = \frac{1}{9}l_+^2,$$

and the proof is complete. □

Remark 2.1. Formulas (2.1) appear in [13, p.110], written with other notations and demonstrated with Leibniz's identity.

Proposition 2.2. *The distances between Fermat points F_+, F_- and the orthocenter H are given by*

$$F_+H^2 = \frac{g(a,b,c)}{48\Delta^2 l_+^2} = \frac{1}{6l_+^2}\left[12R^2\left(a^2+b^2+c^2\right)-\sum\left(a^2+b^2\right)^2\right], \tag{2.3}$$

and

$$F_-H^2 = \frac{g(a,b,c)}{48\Delta^2 l_-^2} = \frac{1}{6l_-^2}\left[12R^2\left(a^2+b^2+c^2\right)-\sum\left(a^2+b^2\right)^2\right], \tag{2.4}$$

where

$$\begin{aligned} g(a,b,c) \;&=\; a^8+b^8+c^8+a^2b^2c^2\left(a^2+b^2+c^2\right) \\ &\quad -a^6b^2-a^2b^6-a^6c^2-a^2c^6-b^6c^2-b^2c^6. \end{aligned} \tag{2.5}$$

Proof. By the cosine law applied to triangle AF_+H (Fig. 1), we have:

$$F_+H^2 = F_+A^2 + AH^2 - 2F_+A\cdot AH\cdot\cos\widehat{F_+AH}.$$

But,

$$AH = 2R\cos A = R\frac{b^2+c^2-a^2}{bc} = \frac{a\left(b^2+c^2-a^2\right)}{4\Delta},$$

and

$$\begin{aligned}\cos\widehat{F_+AH} &= \cos\left(\widehat{BAF_+}-\widehat{BAD}\right) = \cos\left[\alpha_+-\left(\frac{\pi}{2}-B\right)\right] = \sin\left(B+\alpha_+\right) = ... \\ &= \frac{1}{2al_+}\left(4\Delta+\sqrt{3}a^2\right).\end{aligned}$$

Substituting the expression of F_+A given by (1.7), and the expressions found for AH and $\cos\widehat{F_+AH}$ in the previous equation and then making routine calculations, we will get:

$$F_+H^2 = \frac{1}{48\Delta^2 l_+^2}g(a,b,c).$$

Now, using the identity (1.12),

$$\begin{aligned} g(a,b,c) \;&=\; 6a^2b^2c^2\sum a^2 - 8\Delta^2\sum\left(a^2+b^2\right)^2 \\ &=\; 6\cdot 16\Delta^2R^2\cdot\sum a^2 - 8\Delta^2\sum\left(a^2+b^2\right)^2 \\ &=\; 8\Delta^2\left[12R^2\left(a^2+b^2+c^2\right)-\sum\left(a^2+b^2\right)^2\right]. \end{aligned}$$

Hence,

$$F_+H^2 = \frac{1}{6l_+^2}\left[12R^2\left(a^2+b^2+c^2\right)-\sum\left(a^2+b^2\right)^2\right],$$

and the proof of (2.3) is finished. The formula (2.4) is established in the same way. The proof is complete. □

Proposition 2.3. *The distances between* F_+, F_- *and* O *are given by the formulas*

$$F_+O^2 = \frac{1}{144\Delta^2 l_+^2}\left[32\Delta^2 l_+^2 l_-^2 + \left(2l_+^2 - l_-^2\right) f\right], \tag{2.6}$$

and

$$F_-O^2 = \frac{1}{144\Delta^2 l_-^2}\left[32\Delta^2 l_+^2 l_-^2 + \left(2l_-^2 - l_+^2\right) f\right]. \tag{2.7}$$

Proof. Applying the cosine law to the triangle AF_+O,

$$F_+O^2 = F_+A^2 + AO^2 - 2F_+A \cdot AO \cos \widehat{F_+AO}.$$

But $\cos \widehat{F_+AO} = \cos\left(A - \widehat{BAF_+} - \widehat{OAC}\right) = \cos\left[A - \alpha_+ - \left(\frac{\pi}{2} - B\right)\right]$
$= \sin(C + \alpha_+) = \sin C \cos \alpha_+ + \cos C \sin \alpha_+$. By (1.5), we get in the end

$$\cos \widehat{F_+AO} = \frac{1}{4abcl_+}\left\{\sqrt{3}\left[a^2\left(b^2+c^2\right) - \left(b^2-c^2\right)^2\right] + 4\Delta\left(b^2+c^2\right)\right\}.$$

Then, to find F_+O, it remains to substitute in the previous equation F_+A, AO and $\cos \widehat{F_+AO}$ by their expressions. To obtain F_+O in the form (2.6), during the calculation we must always take care to enter the lengths l_+ and l_-.
We do the same to establish formula (2.7). The proof is complete. □

Remarks 2.2. Since we know the distances between Fermat points F_+, F_- and the points G and H, the formulas (2.6) and (2.7) can also be obtained by applying Stewart's theorem to the triangles GF_+H and GF_-H. More, if the distances between F_+, F_- and the points G and H are known, we can calculate the distances between F_+, F_- at other points on the Euler line using the median theorem or Stewart's theorem. Such is the case with O, M, N, DeLongchamps point, and many other points making a constant distance-ratios on the Euler line [7, p. 140]. When possible, it is preferable to use these theorems instead of the cosine law.

Proposition 2.4. *The distances between* F_+, F_- *and* M *are given by*

$$F_+M = \frac{\sqrt{f}}{12\Delta} \cdot \frac{l_-}{l_+}, \tag{2.8}$$

and

$$F_-M = \frac{\sqrt{f}}{12\Delta} \cdot \frac{l_+}{l_-}. \tag{2.9}$$

Proof. It is known that $HG = \frac{2}{3}OH$ and that $OH^2 = 9R^2 - \left(a^2+b^2+c^2\right)$. But, we have:

$$OH^2 = 9\frac{a^2b^2c^2}{16\Delta^2} - \left(a^2+b^2+c^2\right) = \frac{1}{16\Delta^2}\left[9a^2b^2c^2 - 16\Delta^2\left(a^2+b^2+c^2\right)\right],$$

and, by (1.11), $OH^2 = \frac{f}{16\Delta^2}$. Hence,

$$OH = \frac{\sqrt{f}}{4\Delta} \quad \text{and} \quad HG = \frac{\sqrt{f}}{6\Delta}. \tag{2.10}$$

Applying the median theorem to the triangle F_+HG, we have:

$$4F_+M^2 = 2F_+H^2 + 2F_+G^2 - HG^2.$$

By (2.10) and Propositions 2.1 and 2.2, we obtain:

$$\begin{aligned} 4F_+M^2 &= \frac{g}{24\Delta^2 l_+^2} + \frac{2}{9}l_-^2 - \frac{f}{36\Delta^2} = \frac{1}{72\Delta^2 l_+^2}\left(3g + 16\Delta^2 l_+^2 l_-^2 - 2l_+^2 f\right) \\ &= \frac{1}{72\Delta^2 l_+^2}\left[2\left(a^2+b^2+c^2\right)f - 2l_+^2 f\right] = \frac{1}{72\Delta^2 l_+^2}\cdot 2l_-^2 f = \frac{l_-^2 f}{36\Delta^2 l_+^2}. \end{aligned}$$

Therefore,

$$F_+M^2 = \frac{f}{144\Delta^2}\cdot\frac{l_-^2}{l_+^2}.$$

So, the formula (2.8) is true. In the same way it is shown (2.9). The proof is complete. □

Proposition 2.5. *The distances between F_+, F_- and the point N are given by*

$$F_+N^2 = \frac{1}{576\Delta^2 l_+^2}\left[32\Delta^2 l_+^2 l_-^2 + \left(2l_-^2 - l_+^2\right)f\right], \tag{2.11}$$

and

$$F_-N^2 = \frac{1}{576\Delta^2 l_-^2}\left[32\Delta^2 l_+^2 l_-^2 + \left(2l_+^2 - l_-^2\right)f\right]. \tag{2.12}$$

Proof. F_+N is median in the triangle F_+GM (Fig. 3). We have:

$$4F_+N^2 = 2F_+G^2 + 2F_+M^2 - GM^2.$$

Taking into account the formulas (2.1), (2.8), (2.10), and $GM = \frac{1}{2}HG$, by a simple calculation we obtain the formula (2.11).
F_-N is calculated similarly. This concludes the proof. □

Next, we need some elementary properties of symmedian AK and Fermat cevian AF_+. Just as in the case of the A-angle bisector, we obtain:

Lemma 2.2. *Let L and X_+ be the feet of the symmedian AK and Fermat cevian AF_+ on the side BC. Then,*

1) $BL = \dfrac{ac^2}{b^2+c^2}$, $AL = \dfrac{2bcm_a}{b^2+c^2}$, $AK = \dfrac{2bcm_a}{a^2+b^2+c^2}$;

2) $BX_+ = a\dfrac{4\Delta + \sqrt{3}\left(c^2+a^2-b^2\right)}{2\left(4\Delta+\sqrt{3}a^2\right)}$, $AX_+ = \dfrac{4\Delta l_+}{4\Delta+\sqrt{3}a^2}$,

$$AF_+ = \frac{1}{2\sqrt{3}}\frac{4\Delta+\sqrt{3}\left(b^2+c^2-a^2\right)}{l_+}.$$

Proof. 1) These equations are well-known (see, for example, [12]).
2) Applying twice the sine law to triangle ABX_+, we have:

$$AX_+ = \frac{c\sin B}{\sin\left(B+\alpha_+\right)} \quad \text{and} \quad BX_+ = \frac{c\sin\alpha_+}{\sin\left(B+\alpha_+\right)},$$

and, taking into account (1.5), we find the first and second formula. The third formula was demonstrated in Lemma 1.2. □

Lemma 2.3. *If $a > b > c$, then the order of the feet of the symmedian AK and Fermat cevian AF_+ on the side BC is $B - L - X_+ - C$.*

Proof. We have to prove that $BL < BX_+$ or, equivalently,

$$\frac{ac^2}{b^2+c^2} < a\frac{4\Delta+\sqrt{3}\left(c^2+a^2-b^2\right)}{2\left(4\Delta+\sqrt{3}a^2\right)}.$$

We rewrite this inequality in the form of

$$4\Delta > \sqrt{3}\left(b^2+c^2-a^2\right).$$

Then, $bc\sin A > \sqrt{3}bc\cos A$. Hence, $\tan A > \sqrt{3}$, what is true in condition $a > b > c$. □

Proposition 2.6. *The distances between F_+, F_- and the symmedian point K are given by*

$$F_+K = \frac{\sqrt{f}}{\sqrt{3}\left(l_+^2+l_-^2\right)}\frac{l_-}{l_+}, \tag{2.13}$$

and

$$F_-K = \frac{\sqrt{f}}{\sqrt{3}\left(l_+^2+l_-^2\right)}\frac{l_+}{l_-}. \tag{2.14}$$

Proof. Consider the triangle AKF_+. The lengths of the sides AK and AF_+ are known (Lemma 2.2). Taking into account Lemma 2.3, we have

$$\begin{aligned}\cos\widehat{F_+AK} &= \cos\left(\widehat{BAF_+}-\widehat{BAK}\right) = \cos\left(\widehat{BAF_+}-\widehat{CAA'}\right)\\ &= \cos\left[\alpha_+ - \left(\left(\frac{\pi}{2}-\varphi_A\right)-C\right)\right],\end{aligned}$$

hence $\cos\widehat{F_+AK} = \sin\left(\alpha_+ + C + \varphi_A\right)$. Using (1.5), finally we get:

$$\cos\widehat{F_+AK} = \frac{1}{8bcm_al_+}\left[b^4+c^4+6b^2c^2-a^2b^2-a^2c^2+4\sqrt{3}\Delta\left(b^2+c^2\right)\right].$$

Now, we are ready to apply the cosine law to considered triangle. We have:

$$F_+K^2 = F_+A^2 + AK^2 - 2F_+A\cdot AK\cdot\cos\widehat{F_+AK},$$

Substituting the terms on the right by their expressions found above and performing the calculations, we are led to the formula (2.13).
The formula (2.14) is demonstrated the same. The proof is complete. □

Remark 2.3. Above, I used tacitly the fact that points G, H, O, M, N are collinear and some equalities in the sequence $2OH = 3HG = 6OG = 4ON = 12GN$.

Below, we need the well known property that points F_+, F_-, K, M *are collinear and lie on the Fermat axis in the order* $F_- - K - F_+ - M$. Let's give a simple justification for this statement. We use the formula (4.1), Section 4. Then, taking into account (2.13), (2.14), (2.8), (2.9), it is easy to verify that $F_-F_+ = F_-K + KF_+$ and $F_-M = F_-F_+ + F_+M$. The desired claims follow.

3. Distances between J_+, J_- and G, H, O, K, M, N

We start the section with the mention that the angles $\varphi_A, \alpha_+, \alpha_-$ do not have an obvious utility for calculating the distances of the isodynamic points J_+, J_- to the points G, H, O, K, M, N. Below, some results related to the orthocentroidal triangle will be useful. The *orthocentroidal circle* of triangle ABC is the circle on HG as diameter. Obviously, this circle contains the orthogonal projections A_1, B_1, C_1 of G on the altitudes AD, BE, and respectively CF. The triangle $A_1B_1C_1$ is called the *orthocetroidal triangle* (Fig. 3).

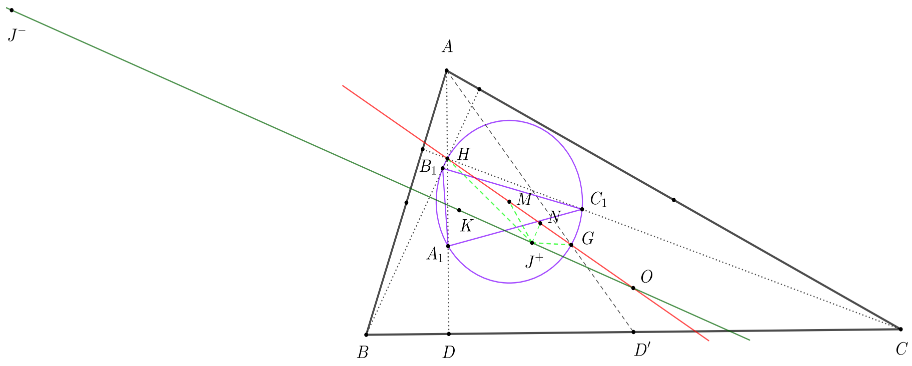

Figure 3

Lemma 3.1. *Triangles ABC and $A_1B_1C_1$ have the properties:*

(i) *they are (inversely) similar;*

(ii) *K is the symmedian point for both triangles;*

(iii) *the Fermat points of the triangle ABC are the isodynamic points of the orthocentroidal triangle $A_1B_1C_1$, i.e. $J_1^+ = F_+$ and $J_1^- = F_-$ (J_1^+, J_1^- denote the isodynamic points of the triangle $A_1B_1C_1$.*

See [4, Prop.5 and Th.10] for an elementary proof.

Remark 3.1. We need the following result: *the points J_+, J_-, O, K are collinear and lie on the Brocard axis in the order $O - J_+ - K - J_-$.* Indeed, according to the previous remark, we have the order $F_- - K - F_+ - M$ on the line F_+F_-. Also, according to the previous lemma, the correspondence: $O \longleftrightarrow M$, $K \longleftrightarrow K$, $J_+ \longleftrightarrow F_+$, $J_- \longleftrightarrow F_-$ preserves the order between homologous points. So, on the line OK we have $J_- - K - J_+ - O$.

As in the case of OH and HG (see (2.10)), we will give formulas for OK and KM that are convenient in the calculations below.

Lemma 3.2. *We have:*

$$OK = \frac{abcl_+l_-}{2\Delta\left(a^2+b^2+c^2\right)}, \quad KM = \frac{l_+l_-\sqrt{f}}{6\Delta\left(a^2+b^2+c^2\right)}. \tag{3.1}$$

Proof. Indeed,

$$\begin{aligned} OK^2 &= R^2 - \frac{3a^2b^2c^2}{\left(a^2+b^2+c^2\right)^2} = \frac{a^2b^2c^2}{16\Delta^2} - \frac{3a^2b^2c^2}{\left(a^2+b^2+c^2\right)^2} \\ &= \frac{a^2b^2c^2\left[\left(a^2+b^2+c^2\right)^2 - 3\cdot 16\Delta^2\right]}{16\Delta^2\left(a^2+b^2+c^2\right)^2} = \frac{a^2b^2c^2\cdot 2l_+^2\cdot 2l_-^2}{16\Delta^2\left(a^2+b^2+c^2\right)^2}, \end{aligned}$$

hence the first formula in (3.1).

On the other hand, the ratio of the similitude of the triangles ABC and $A_1B_1C_1$ is given by the ratio of the diameters of their circumcircles. So it is equal to

$$\frac{2R}{HG} = \frac{abc}{2\Delta}\cdot\frac{6\Delta}{\sqrt{f}} = \frac{3abc}{\sqrt{f}},$$

i.e.

$$\frac{2R}{HG} = \frac{3abc}{\sqrt{f}}. \tag{3.2}$$

According to Lemma 3.1, (i) and (ii), we have the following correspondence of points: $K \longleftrightarrow K$, $O \longleftrightarrow M$. Then,

$$\frac{KO}{KM} = \frac{2R}{HG}, \quad \text{hence} \quad KM = \frac{HG}{2R} \cdot OK = \frac{\sqrt{f}}{3abc} \cdot \frac{abcl_+l_-}{2\Delta\left(a^2+b^2+c^2\right)},$$

end the second formula of the cross reference (3.1) follows. □

Proposition 3.1. *The distances between J_+, J_- and O are given by*

$$J_+O = R\frac{l_-}{l_+} = \frac{abc}{4\Delta} \cdot \frac{l_-}{l_+}, \tag{3.3}$$

and

$$J_-O = R\frac{l_+}{l_-} = \frac{abc}{4\Delta} \cdot \frac{l_+}{l_-}. \tag{3.4}$$

Proof. Since triangles ABC and $A_1B_1C_1$ are similar, we have the correspondences:

$$J_+ \longleftrightarrow J_1^+, \quad O \longleftrightarrow M.$$

Hence, $\dfrac{J_+O}{J_1^+M} = \dfrac{2R}{HG}$. By Lemma 3.1, (iii), $J_1^+ = F_+$. Therefore,

$$J_+O = F_+M \cdot \frac{2R}{HG},$$

and taking into account (2.8), (3.2), we obtain the formula (3.3). Analogously, the formula (3.4) is obtained. The proof is complete. □

Proposition 3.2. *The distances between J_+, J_- and K are given by*

$$J_+K = \frac{\sqrt{3}abc}{a^2+b^2+c^2} \cdot \frac{l_-}{l_+}, \tag{3.5}$$

and

$$J_-K = \frac{\sqrt{3}abc}{a^2+b^2+c^2} \cdot \frac{l_+}{l_-}. \tag{3.6}$$

Proof. Due to the similarity of the triangles ABC and $A_1B_1C_1$, and Lemma 3.1, (ii) and (iii), we have the correspondences: $J_+ \longleftrightarrow F_+$, $J_- \longleftrightarrow F_-$, $K \longleftrightarrow K$. Therefore,

$$J_+K = F_+K \cdot \frac{2R}{HG}, \quad \text{and} \quad J_-K = F_-K \cdot \frac{2R}{HG}.$$

By (2.13), (2.14) and (3.2), we obtain the required formulas. The proof is complete. □

To find the distances from J_+, J_- to G, H, M, N we will use Stewart's theorem (in particular, the median theorem).

Proposition 3.3. *The distances between J_+, J_- and M are given by*

$$J_+M^2 = \frac{1}{144\Delta^2 l_+^2}\left(32\Delta^2 l_-^4 + 2l_+^2 f - 9a^2b^2c^2 l_-^2\right), \tag{3.7}$$

and

$$J_-M^2 = \frac{1}{144\Delta^2 l_-^2}\left(32\Delta^2 l_+^4 + 2l_-^2 f - 9a^2b^2c^2 l_+^2\right). \tag{3.8}$$

Proof. Consider the triangle MKO and apply Stewart's theorem to this triangle and the cevians MJ_+ and MJ_- (Fig. 3). We have:

$$J_+M^2 \cdot OK = MK^2 \cdot J_+O + OM^2 \cdot J_+K - OK \cdot J_+O \cdot J_+K,$$

$$J_-M^2 \cdot OK = MK^2 \cdot J_-O - OM^2 \cdot J_-K + OK \cdot J_-O \cdot J_-K.$$

Substituting OK, MK, $OM = HG$, J_+O, J_-O, J_+K, J_-K by the expressions given by (3.1), (2.10), (3.3), (3.4), (3.5), (3.6), we will finally obtain the formulas (3.7) and (3.8). Thus achieves the proof. □

Proposition 3.4. *The distances between J_+, J_- and G are given by*

$$J_+G = \frac{1}{3}\frac{l_-^2}{l_+}, \tag{3.9}$$

and

$$J_-G = \frac{1}{3}\frac{l_+^2}{l_-}. \tag{3.10}$$

Proof. The centroid G is the midpoint of OM. Then, J_+G, J_-G are medians in the triangles J_+OM and J_-OM, respectively. By the median theorem applied to the triangle J_+OM (Fig. 3), we have:

$$4J_+G^2 = 2J_+O^2 + 2J_+M^2 - OM^2.$$

Using the formulas (3.3), (3.7), (2.10), and performing the calculations, we obtain (3.9). The formula (3.10) is shown in the same way. This completes the proof. □

Proposition 3.5. *The distances between J_+, J_- and H are given by*

$$J_+H^2 = \frac{1}{24\Delta^2 l_+^2}\left(8\Delta^2 l_-^4 + l_+^2 f - 3a^2b^2c^2 l_-^2\right), \tag{3.11}$$

and

$$J_-H^2 = \frac{1}{24\Delta^2 l_-^2}\left(8\Delta^2 l_+^4 + l_-^2 f - 3a^2b^2c^2 l_+^2\right). \tag{3.12}$$

Proof. M is the midpoint of HG. Then, we have: $4J_+M^2 = 2J_+H^2 + 2J_+G^2 - HG^2$. Hence, we have

$$2J_+H^2 = 4J_+M^2 + HG^2 - 2J_+G^2,$$

and, similarly,

$$2J_-H^2 = 4J_-M^2 + HG^2 - 2J_-G^2.$$

It remains to perform routine calculations to obtain the required formulas. The proof is complete. □

Proposition 3.6. *The distances between J_+, J_- and N are given by*

$$J_+N^2 = \frac{1}{192\Delta^2 l_+^2}\left(32\Delta^2 l_-^4 + l_+^2 f - 6a^2b^2c^2 l_-^2\right), \tag{3.13}$$

and

$$J_-N^2 = \frac{1}{192\Delta^2 l_-^2}\left(32\Delta^2 l_+^4 + l_-^2 f - 6a^2b^2c^2 l_+^2\right). \tag{3.14}$$

Proof. Applying the median theorem to the triangle J_+HO, we have:

$$4J_+N^2 = 2J_+H^2 + 2J_+O^2 - OH^2.$$

Taking into account (3.11), (3.3), (2.10), by a simple calculation we get the formula (3.13). The formula (3.14) is similarly obtained, which concludes the proof. □

4. Distances between F_+, F_-, J_+, J_-

In this section, we consider the following six distances: F_+F_-, J_+J_-, F_+J_+, F_-J_-, F_+J_-, F_-J_+.

Proposition 4.1. *For the distances F_+F_- and J_+J_- we have the formulas:*

$$F_+F_- = \frac{1}{\sqrt{3}}\frac{\sqrt{f}}{l_+l_-}, \tag{4.1}$$

and

$$J_+J_- = \sqrt{3}\frac{abc}{l_+l_-}. \tag{4.2}$$

Proof. Applying the cosine law to triangle AF_+F_-, we get (Fig. 1):

$$F_+F_-^2 = F_+A^2 + F_-A^2 - 2F_+A \cdot F_-A \cdot \cos\widehat{F_+AF_-}.$$

In the case of $B > \dfrac{\pi}{3}$,

$$\cos\widehat{F_+AF_-} = \cos\left(\widehat{BAF_+} + \widehat{BAF_-}\right) = \cos\left(\alpha_+ + \pi - \alpha_-\right) = -\cos\left(\alpha_+ - \alpha_-\right).$$

If $B < \dfrac{\pi}{3}$, then

$$\cos\widehat{F_+AF_-} = \cos\left(\widehat{BAF_+} - \widehat{BAF_-}\right) = \cos\left[\alpha_+ - (\alpha_- - \pi)\right] = -\cos\left(\alpha_+ - \alpha_-\right).$$

In both cases, we have:

$$\cos\widehat{F_+AF_-} = -\cos\left(\alpha_+ - \alpha_-\right) = -\left(\cos\alpha_+\cos\alpha_- + \sin\alpha_+\sin\alpha_-\right).$$

Taking into account (1.5), (1.6), by a simple calculation we get

$$\cos\widehat{F_+AF_-} = -\frac{-2a^2+b^2+c^2}{l_+l_-}.$$

Using this and the formulas (1.7), the preceding equation leads after calculations to (4.1). On the other hand, from $ABC \sim A_1B_1C_1$ (Lemma 3.1) we have:

$$\frac{J_+J_-}{J_1^+J_1^-} = \frac{2R}{HG} \quad \text{and} \quad J_+J_- = F_+F_- \cdot \frac{2R}{HG}.$$

By (4.1) and (3.2), it follows that

$$J_+J_- = \frac{1}{\sqrt{3}}\frac{\sqrt{f}}{l_+l_-} \cdot \frac{3abc}{\sqrt{f}} = \sqrt{3}\frac{abc}{l_+l_-},$$

that is (4.2) is true. The proof is complete. □

Remark 4.1. The formulae (4.1) and (4.2) for the distances F_+F_- and J_+J_- can also be written as follows:

$$F_+F_-^2 = \frac{2}{3}\frac{a^6+b^6+c^6+3a^2b^2c^2-a^4b^2-a^2b^4-a^4c^2-a^2c^4-b^4c^2-b^2c^4}{\left(a^2-b^2\right)^2+\left(a^2-c^2\right)^2+\left(b^2-c^2\right)^2},$$

$$J_+J_-^2 = \frac{3}{2}\frac{a^2b^2c^2}{\left(a^2-b^2\right)^2+\left(a^2-c^2\right)^2+\left(b^2-c^2\right)^2},$$

or, using Conway triangle notations,

$$F_+F_-^2 = \frac{2}{3}\frac{a^2S_A^2+b^2S_B^2+c^2S_C^2-6S_AS_BS_C}{\left(S_B-S_C\right)^2+\left(S_C-S_A\right)^2+\left(S_A-S_B\right)^2},$$

$$J_+J_-^2 = \frac{3}{2}\frac{\left(S_B+S_C\right)\left(S_C+S_A\right)\left(S_A+S_B\right)}{\left(S_B-S_C\right)^2+\left(S_C-S_A\right)^2+\left(S_A-S_B\right)^2}.$$

In the next step we will use the following result:

Lemma 4.1. *The lines F_+J_+, F_-J_- are parallel to each other and to Euler line.*

Proof. We only detail that $F_+J_+ \parallel OH$. For this, we consider the triangles KF_+J_+ and KMO (Fig. 2). We have: $\widehat{K}=\widehat{K}$, and $\dfrac{KF_+}{KM}=\dfrac{KJ_+}{KO}$ (by calculation and using (2.13), (3.1), (3.5)). These triangles are similar, and it follows that F_+J_+ is parallel to MO (or OH). For $F_-J_- \parallel OH$, the triangles KF_-J_- and KMO are considered. □

Proposition 4.2. *For the distances F_+J_+ and F_-J_- we have the formulae:*

$$F_+J_+ = \frac{1}{\sqrt{3}}\frac{\sqrt{f}}{l_+^2}, \tag{4.3}$$

and

$$F_-J_- = \frac{1}{\sqrt{3}}\frac{\sqrt{f}}{l_-^2}. \tag{4.4}$$

Proof. From $F_+J_+ \parallel OH$, it follows that $\dfrac{F_+J_+}{MO}=\dfrac{KJ_+}{KO}$. Then, we have:

$$F_+J_+ = MO\cdot\frac{KJ_+}{KO} = \frac{\sqrt{f}}{6\Delta}\cdot\frac{\sqrt{3}abc}{a^2+b^2+c^2}\frac{l_-}{l_+}\cdot\frac{2\Delta\left(a^2+b^2+c^2\right)}{abcl_+l_-} = \frac{1}{\sqrt{3}}\frac{\sqrt{f}}{l_+^2}.$$

F_-J_- is calculated similarly. This concludes the proof. □

Proposition 4.3. *For the distances F_+J_- and F_-J_+ we have the formulas:*

$$F_+J_- = \frac{4\sqrt{3}\Delta}{3l_-} = \frac{l_+^2-l_-^2}{3l_-}, \tag{4.5}$$

and

$$F_-J_+ = \frac{4\sqrt{3}\Delta}{3l_+} = \frac{l_-^2-l_+^2}{3l_-}. \tag{4.6}$$

Proof. Consider the triangle $F_+J_+J_-$ and the cevian F_+K. Using Stewart's theorem, we get:

$$\left(F_+J_-\right)^2\cdot KJ_+ = F_+K^2\cdot J_+J_- + J_+J_-\cdot KJ_+\cdot KJ_- - \left(F_+J_+\right)^2\cdot KJ_-.$$

Substituting KJ_+, KJ_-, F_+K, J_+J_-, F_+J_+ by their expressions given by (3.5), (3.6), (2.13), (4.2), respectively (4.3) and by performing the calculations, we get the formula

(4.5). Then, considering the triangle $F_-J_+J_-$ and the cevian F_-K, by applying the same theorem we obtain the formula (4.6). The proof is complete. □

5. Two applications

I. *Remarks on Evans conic.* In the paper [5], *L.S. Evans* demonstrates that there is a conic which passes through the following notable points: F_+, F_-, J_+, J_-, N_+, and N_- (N_+, N_- are inner and outer Napoleon points of the triangle ABC). More, he informs the reader that *Peter Yff* has calculated the equation of this conic, and *Paul Yiu* has found criteria for it to be an ellipse, parrabola, or a hyperbola. Next, this conic will be called *Evans conic* and will be denoted $\mathcal{E}$.

Using Lemma 4.1 and Proposition 4.3 we will easily show the following result:

Proposition 5.1. *The statements*

(i) *$\mathcal{E}$ can not be circle, and*

(ii) *if $\mathcal{E}$ is ellipse or hyperbole, then its center lies on the line GK*

are true.

Proof. (i) If $\mathcal{E}$ were a circle, then the points F_+, F_-, J_+, J_- would be concyclic (Fig. 4). By Lemma 4.1, the cyclic quadrilateral $F_+J_+F_-J_-$ would be isosceles trapezium, that is $F_+J_- = F_-J_+$. So, according to (4.5) and (4.6), we would have:

$$\frac{4\sqrt{3}\Delta}{3l_-} = \frac{4\sqrt{3}\Delta}{3l_+},$$

i.e. $l_+ = l_-$. Absurd.

(ii) According to Lemma 4.1, the cords F_+J_+ and F_-J_- of the ellipse $\mathcal{E}$ are parallel to the Euler line (Fig. 4). Since the triangles KMO, KF_+J_+ and KF_-J_- are similar, we deduce that the midpoints of the cords F_+J_+ and F_-J_- lie on the line KG. Therefore, KG passes through the center of $\mathcal{E}$. The proof is complete. □

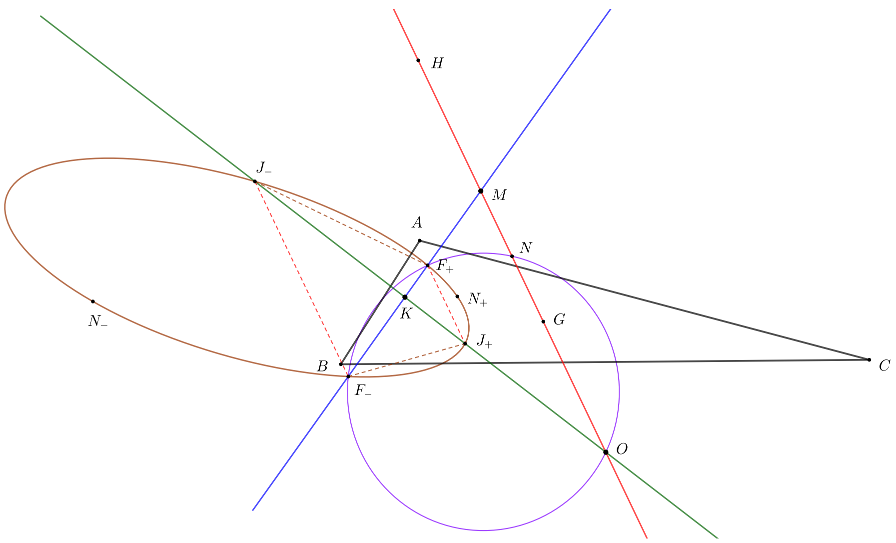

Figure 4

II. *Triangle OMK*. This triangle is determined by the Brocard and Fermat axes and Euler line. Like the triangle OHI, it plays an important role in the geometry of the triangle. We will highlight some properties of the triangle OMK, using the results from the previous sections.

Proposition 5.2. *Triangle OMK is isosceles, with $OK = OM$, if and only if*

$$R^2 = \frac{\left(a^2+b^2+c^2\right)^3}{27\left(a^2b^2+a^2c^2+b^2c^2\right)}. \tag{5.1}$$

Proof. We have:

$$\begin{aligned} OK = OM &\iff \frac{abcl_+l_-}{2\Delta\left(a^2+b^2+c^2\right)} = \frac{\sqrt{f}}{6\Delta} \iff 3abcl_+l_- = \sqrt{f}\left(a^2+b^2+c^2\right) \\ &\iff 9a^2b^2c^2l_+^2l_-^2 = \left[9a^2b^2c^2 - 16\Delta^2\left(a^2+b^2+c^2\right)\right]\left(a^2+b^2+c^2\right)^2 \\ &\iff 9a^2b^2c^2\left[l_+^2l_-^2 - \left(a^2+b^2+c^2\right)^2\right] = -16\Delta^2\left(a^2+b^2+c^2\right)^3 \\ &\iff 9R^2\left[l_+^2l_-^2 - \left(a^2+b^2+c^2\right)^2\right] = -\left(a^2+b^2+c^2\right)^3 \\ &\iff R^2 = \frac{\left(a^2+b^2+c^2\right)^3}{27\left(a^2b^2+a^2c^2+b^2c^2\right)}, \end{aligned}$$

and the statement is proved. □

Remark 5.1. Two other sides of the triangle OMK cannot be equal. Indeed, according to the formulas (2.10) and (3.1) for the lenghts of OM, OK, and KM, we have: $OM = KM \iff l_+l_- = a^2+b^2+c^2$, absurd, and, on the other hand, $OK = KM \iff abc = \sqrt{f} \iff a^2b^2c^2 - f = 0 \iff 16\Delta^2\left(a^2+b^2+c^2\right) = 0$, absurd.

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