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PROPERTIES OF HARMONIC QUADRUPLES THAT TRANSFORM ONE INTO THE OTHER BY PERSPECTIVE PROJECTION WHOSE CENTER LIES AT A POINT ON A CIRCLE

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ABSTRACT. The present paper investigates (1) the properties of harmonic quadruples on a straight line and on a circle that transform one into the other by perspective projection, the center of which lies at a point on the circle; and (2) the collinear properties of points.

1. INTRODUCTION

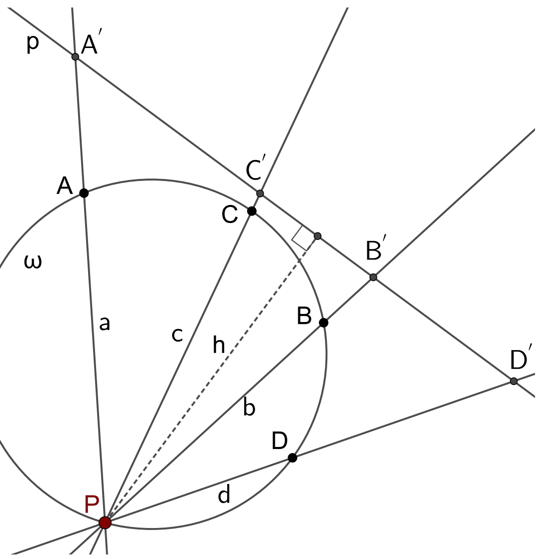

Figure 1: Perspective projection from point P on circle ω between points on ω and points on line p.

In the geometrical state shown in Figure 1:
(i) points A, B, C, D, and P lie on circle ω,

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(ii) $a = PA$, $b = PB$, $c = PC$ and $d = PD$ are four straight lines that pass through point P and intersect an arbitrary straight line p, at points A', B', C', and D', respectively.

With the points arranged as shown in Figure 1, the cross ratio, λ, is defined as follows:

$$\lambda = \left(A', B'; C', D'\right) = \frac{\overline{C'A'}}{\overline{C'B'}} : \frac{\overline{D'A'}}{\overline{D'B'}} \tag{1.1}$$

where directed lengths appear on the right-hand side of 1.1 (see [2, Section 185]).
The cross ratio of lines a, b, c and d that connect point $P \notin l$ with points A', B', C' and D' is defined as the cross ratio of points A', B', C' and D' . In other words,

$$\lambda = (a, b; c, d) = \frac{\overline{C'A'}}{\overline{C'B'}} : \frac{\overline{D'A'}}{\overline{D'B'}} \tag{1.2}$$

The cross ratio of the four lines, a, b, c and d, can be calculated using the formula $(a, b; c, d) = \dfrac{\sin(\widehat{c, a})}{\sin\left(\widehat{c, b}\right)} : \dfrac{\sin\left(\widehat{d, a}\right)}{\sin\left(\widehat{d, b}\right)}$ (see [3]). This may be proven by calculating the area of each of the triangles $C'PA'$ and $C'PB'$ in two ways. The following is thus obtained: $\frac{1}{2}C'A' \cdot h = \frac{1}{2}PC' \cdot PA' \sin(\widehat{c, a})$ (see Figure 1) and $\frac{1}{2}C'B' \cdot h = \frac{1}{2}PC' \cdot PB' \sin\left(\widehat{c, b}\right)$.
Hence:

$$\frac{C'A'}{C'B'} = \frac{PA'}{PB'} \cdot \frac{\sin(\widehat{c, a})}{\sin\left(\widehat{c, b}\right)} \tag{1.3}$$

Similarly, using triangles $D'PA'$ and $D'PB'$, one obtains

$$\frac{D'A'}{D'B'} = \frac{PA'}{PB'} \cdot \frac{\sin\left(\widehat{d, a}\right)}{\sin\left(\widehat{d, b}\right)} \tag{1.4}$$

From equalities 1.3 and 1.4, we obtain that for the cross ratio of the directed distances between points A', B', C' and D' , there holds:

$$\frac{\overline{C'A'}}{\overline{C'B'}} : \frac{\overline{D'A'}}{\overline{D'B'}} = \frac{\sin(\widehat{c, a})}{\sin\left(\widehat{c, b}\right)} : \frac{\sin\left(\widehat{d, a}\right)}{\sin\left(\widehat{d, b}\right)} \tag{1.5}$$

The angles appearing on the right-hand side of equality 1.5 are directed angles. Clockwise angles are negative and counterclockwise angles are positive. It is easy to see (see Figure 2 and Figure 3) that when the location of points A, B, C, and D on the circle is as shown in Figure 1, of the four angles that appear in equality 1.5, three have one sign and the fourth has the opposite sign. Therefore the sign of the right-hand side in equality 1.5 is negative.
Now, let us consider the four points A, B, C, and D on circle ω (whose radius is R) (see Figure 1). Each of the triangles FAC, FCB, FDA, and FDB is inscribed in circle ω.
Therefore, according to the Law of Sines, there holds that
$\dfrac{CA}{\sin(\widehat{c, a})} = \dfrac{CB}{\sin\left(\widehat{c, b}\right)} = \dfrac{DA}{\sin\left(\widehat{d, a}\right)} = \dfrac{DB}{\sin\left(\widehat{d, b}\right)} = 2R.$

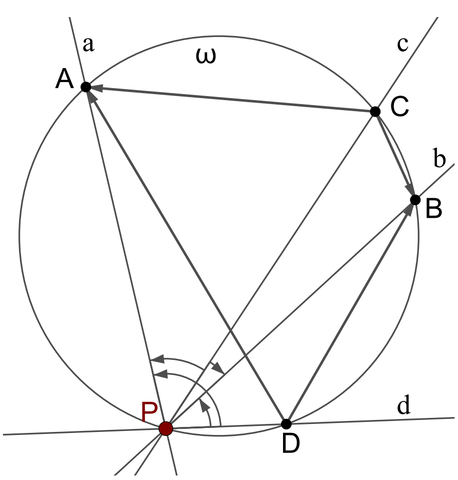

Figure 2: Angle $\left(\widehat{c,b}\right)$ and chord CB are negative; angles $(\widehat{c,a})$, $\left(\widehat{d,a}\right)$, and $\left(\widehat{d,b}\right)$ and corresponding chords CA, DA, and DB are positive.

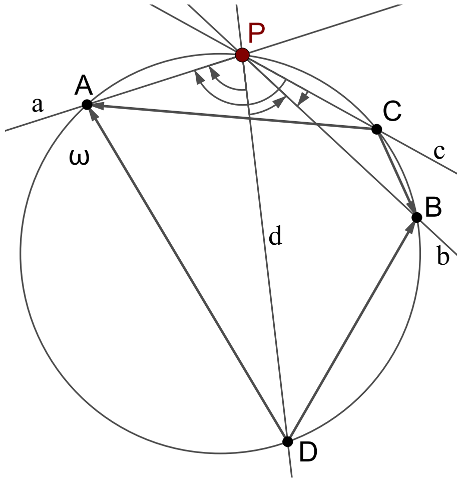

Figure 3: Angle $\left(\widehat{d,b}\right)$ and chord DB are positive; $(\widehat{c,a})$, $\left(\widehat{c,b}\right)$, and $\left(\widehat{d,a}\right)$ and corresponding chords CA, CB, and DA are negative.

Hence it follows that $\dfrac{CA}{CB} = \dfrac{\sin(\widehat{c,a})}{\sin\left(\widehat{c,b}\right)}$, $\dfrac{DA}{DB} = \dfrac{\sin\left(\widehat{d,a}\right)}{\sin\left(\widehat{d,b}\right)}$, and finally

$$\frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}} = \frac{\sin(\widehat{c,a})}{\sin\left(\widehat{c,b}\right)} : \frac{\sin\left(\widehat{d,a}\right)}{\sin\left(\widehat{d,b}\right)} \tag{1.6}$$

where each chord appearing on the left-hand side of equality 1.6 obtains the sign of the angle subtending it. From equalities 1.5 and 1.6 it follows that

$$\frac{\overline{C'A'}}{\overline{C'B'}} : \frac{\overline{D'A'}}{\overline{D'B'}} = \frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}} \tag{1.7}$$

which means that with respect to points A, B, C, and D on circle ω, it is natural to consider the expression $\dfrac{\overline{CA}}{\overline{CB}} : \dfrac{\overline{DA}}{\overline{DB}}$ as a cross ratio of the four points on the circle (see [4, p 243]). From equality 1.7, it follows that in a perspective projection whose center lies at a point on circle ω, the cross ratio of the four points on ω and of the four corresponding points on the straight line will be conserved. In particular, if four points on the straight line constitute a harmonic quadruple, then the four corresponding points on ω (in the projection) also constitute a harmonic quadruple, and vice versa.

In section 2, we shall investigate collinearity properties pertaining to harmonic quadruples transformed one into the other in a perspective projection whose center is on the circle, and in section 3, we shall investigate collinearity properties pertaining to harmonic quadruples on two intersecting circles.

2. Properties of harmonic quadruples on the circle and on the straight line defined using the circle

Property 1. *Let ω be a circle in which FI is a diameter; f is the tangent to the circle at point F; and $M, N \in \omega$ so that there holds $\measuredangle NFI = \measuredangle MFI$. Then:*

(a) N, I, M and F constitute a harmonic quadruple of points on circle ω; and

(b) straight lines FN, FI, FM and f constitute a harmonic quadruple of lines.

Proof.

(a) From the data of this property, it follows that $MI = IN$ and $MF = NF$. Therefore, the cross ratio of the directed chords is $\dfrac{\overline{MI}}{\overline{NI}} : \dfrac{\overline{MF}}{\overline{NF}} = -1$. In other words N, I, M and F constitute a harmonic quadruple of points on circle ω.

(b) From the data, it follows that diameter FI bisects chord MN (at point A in Figure 4). Therefore, on straight line MN, points M, N, A, and the point at infinity constitute a harmonic quadruple. Together with (a), it thus follows that by perspective projection using point F, the four straight lines, FM, $FI(= FA)$, FN and f, constitute a harmonic quadruple.

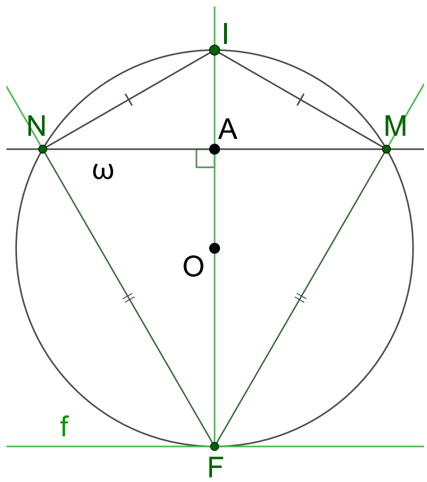

Figure 4: N, I, M and F constitute a harmonic quadruple of points on circle ω. Straight lines FN, FI, FM and f constitute a harmonic quadruple of lines.

□

Property 2. *In addition to the data of Property 1, let $E \in \omega$, $MN \cap IE = G$, $FM \cap IE = B$, $FN \cap IE = C$* (see Figure 5). *Here, the four points G, B, E, and C constitute a harmonic quadruple on straight line IE.*

Proof. We use the method of analytic geometry. We choose a system of coordinates in which the y-axis coincides with straight line FI, the origin is at center O of circle ω, and the unit magnitude equals the radius of ω (see Figure 5). In other words, the equation of the circle is $x^2 + y^2 = 1$.

In this system, the coordinates of points I and F are $I\,(0; 1)$ and $F\,(0; -1)$. For the points E, M, and N which are located on the circle ω we denote: $x_E = e$ and $x_M = m$. Hence

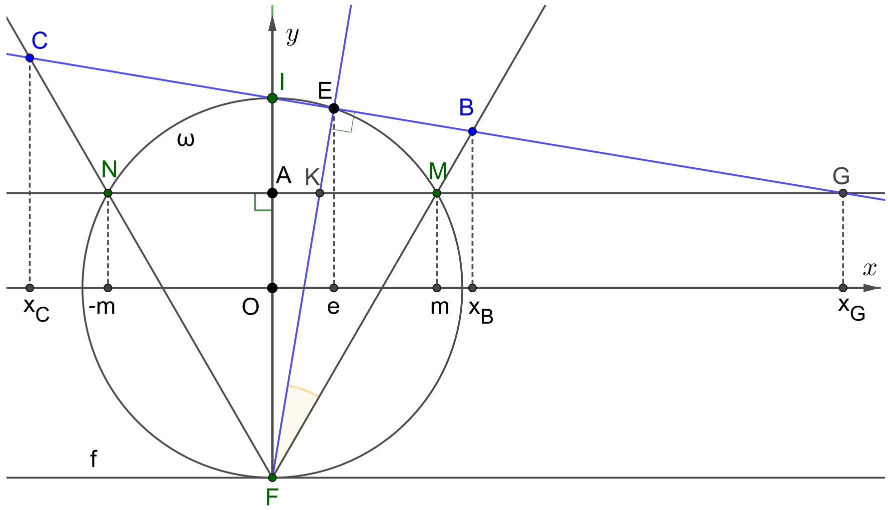

Figure 5: Points G, B, E, and C constitute a harmonic quadruple on straight line IE.

$E\left(e; \sqrt{1-e^2}\right)$, $M\left(m; \sqrt{1-m^2}\right)$ and $N\left(-m; \sqrt{1-m^2}\right)$.
G is the point of intersection of straight lines MN and IE. Let us find the coordinates of G.
The equation of straight line MN is

$$y = y_M = \sqrt{1-m^2} \tag{2.1}$$

First, let us find the equation of straight line IE. Since the slope of this line is $m_{IE} = \dfrac{\sqrt{1-e^2}-1}{e}$, its equation is $y - 1 = \dfrac{\sqrt{1-e^2}-1}{e}(x-0)$, or

$$y = \frac{\sqrt{1-e^2}-1}{e}x + 1 \tag{2.2}$$

From equations 2.1 and 2.2 we obtain the equation $\dfrac{\sqrt{1-e^2}-1}{e}x + 1 = \sqrt{1-m^2}$, from which we obtain the x coordinate of point G: $x_G = \dfrac{\left(\sqrt{1-m^2}-1\right)e}{\sqrt{1-e^2}-1}$.

Therefore, the coordinates for point G are $G\left(\dfrac{\left(\sqrt{1-m^2}-1\right)e}{\sqrt{1-e^2}-1}; \sqrt{1-m^2}\right)$.

If we denote $\sqrt{1-e^2}-1 = a$ and $\sqrt{1-m^2}+1 = b$, the coordinates for point G may be written $G\left(\dfrac{(b-2)e}{a}; b-1\right)$.

Now let us find the coordinates of points B and C. We find the equation of straight line FM: $y - y_F = m_{FM}(x - x_F) \Rightarrow y + 1 = \dfrac{\sqrt{1+m^2}+1}{m-0}(x-0)$, which can be rewritten as

$$y = \frac{b}{m}x - 1 \tag{2.3}$$

Similarly, the equation of straight line FN will be

$$y = -\frac{b}{m}x - 1 \tag{2.4}$$

Finally, the equation of straight line IE is

$y - y_I = m_{IE}(x - x_I) \Rightarrow y - 1 = \dfrac{\sqrt{1-e^2}-1}{e-0}(x-0)$, and hence

$$y = \frac{a}{e}x + 1 \tag{2.5}$$

To find the coordinates of point B, we solve the set of equations 2.3 and 2.5:

$\dfrac{b}{m}x - 1 = \dfrac{a}{e}x + 1 \Rightarrow \dfrac{eb-am}{em}x = 2$, and hence $x_B = \dfrac{2em}{eb-ma}$ and also

$y_B = \dfrac{a}{e}\cdot\dfrac{2em}{eb-ma} + 1 = \dfrac{eb+ma}{eb-ma}$ and therefore $B\left(\dfrac{2em}{eb-ma}; \dfrac{eb+ma}{eb-ma}\right)$.

Similarly, to find the coordinates of point C, we solve the set of equations 2.4 and 2.5 and obtain $C\left(\dfrac{-2em}{eb+ma}; \dfrac{eb-ma}{eb+ma}\right)$.

Since the cross ratio of the points on straight line IE is equal to the cross ratio of the projections of these points on the x-axis (i.e, the cross ratio of the x-coordinates of the points), it is sufficient to find cross ratio $(x_B, x_C; x_E, x_G)$.

$$\begin{aligned}
(x_B, x_C; x_E, x_G) &= \frac{\overline{x_E x_B}}{\overline{x_E x_C}} : \frac{\overline{x_G x_B}}{\overline{x_G x_C}} = \frac{x_E - x_B}{x_E - x_C} : \frac{x_G - x_B}{x_G - x_C} \\
&= \frac{e - \dfrac{2em}{eb-ma}}{e + \dfrac{2em}{eb+ma}} : \frac{\dfrac{(b-2)\,e}{a} - \dfrac{2em}{eb-ma}}{\dfrac{(b-2)\,e}{a} + \dfrac{2em}{eb+ma}} \\
&= \frac{\dfrac{e\,(eb-ma-2m)}{eb-ma}}{\dfrac{e\,(eb+ma+2m)}{eb+ma}} \times \frac{\dfrac{eb\,(eb-2e+ma)}{a\,(eb+ma)}}{\dfrac{eb\,(eb-2e-ma)}{a\,(eb-ma)}} \\
&= \frac{(eb-ma-2m)\,(eb-2e+ma)}{(eb+ma+2m)\,(eb-2e-ma)} \\
&= \frac{e^2b^2 - 2e^2b - 2m^2a - m^2a^2 + 2ema + 4em - 2emb}{e^2b^2 - 2e^2b - 2m^2a - m^2a^2 - 2ema - 4em + 2emb}
\end{aligned}$$

Let us prove that the expression $e^2b^2 - 2e^2b - 2m^2a - m^2a^2$, which appears in the numerator and in the denominator of the last fraction, is equal to 0.

$$\begin{aligned}
&e^2b^2 - 2e^2b - 2m^2a - m^2a^2 \\
&\quad = e^2\left(\sqrt{1-m^2}+1\right)^2 - 2e^2\left(\sqrt{1-m^2}+1\right) - 2m^2\left(\sqrt{1-e^2}-1\right) - m^2\left(\sqrt{1-e^2}-1\right)^2 \\
&\quad = 2e^2 - e^2m^2 + 2e^2\sqrt{1-m^2} - 2e^2\sqrt{1-m^2} - 2e^2 - 2m^2\sqrt{1-e^2} + 2m^2 - 2m^2 + e^2m^2 \\
&\qquad + 2m^2\sqrt{1-e^2} = 0
\end{aligned}$$

We obtained $(x_B, x_C; x_E, x_G) = \dfrac{0 + 2ema + 4em - 2emb}{0 - 2ema - 4em + 2emb} = -1$, and therefore $(B, C; E, G) = -1$, in other words points G, B, E, and C constitute a harmonic quadruple $\square$

Property 3. *In addition to the data of Property 2, let $FG \cap \omega = H$ and $FE \cap MN = K$. Then points H, K, and I are collinear* (see Figure 6).

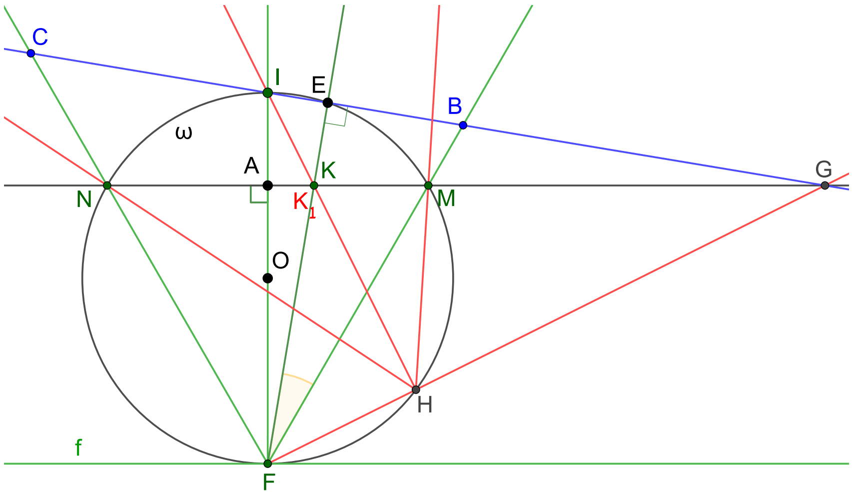

Figure 6: Straight lines HM, HI, HN, and HF constitute a harmonic quadruple. Points H, K, and I are collinear.

Proof. Since points M, I, N, and F are a harmonic quadruple on circle ω and also $H \in \omega$, it follows that the four straight lines, HM, HI, HN, and HF, constitute a harmonic quadruple. From the data of Property 3, it follows that straight line HF intersects straight line MN at point G. We let K_1 denote the point of intersection of straight lines HI and MN (see Figure 6). Therefore, the above-mentioned four lines intersect straight line MN at the harmonic quadruple points M, K_1, N, and G. On the other hand, the harmonic quadruple of straight lines FM, FK, FN, and FG intersect straight line MN at harmonic quadruple points M, K, N, and G. Therefore, points K and K_1 coincide, and hence points H, K, and I are collinear. □

3. Properties of the collinearity of points formed by two intersecting circles

Property 4. *In addition to the data of Property 2, let $P \in FM$ and also $P \neq B$; $GP \cap FE = J$, $GP \cap FN = Q$; circle ε passes through points F, P, and Q and intersects circle ω and straight lines f and FI at the additional points H_1, F_1, and U, respectively* (see Figure 7). *Then:*

(a) Points G, H_1 and F_1 are collinear;
(b) Points H_1, J and U are collinear.

Proof.
(a) We use the method of analytic geometry. Let us choose the system of coordinates as shown in the proof of Property 2 (see Figure 7). Then some of the results we obtained above also hold in this proof, and in particular: $I\,(0;1)$, $F\,(0;-1)$, $M\,(m;b-1)$, $E\,(e;a+1)$, $N\,(-m;b-1)$, $G\left(\dfrac{(b-2)\,e}{a};b-1\right)$ (where $a = \sqrt{1-e^2} - 1$ and $b = \sqrt{1-m^2} + 1$), and also, equalities 2.3 and 2.4 that are the equations of straight lines FM and FN, respectively.
We denote by c the x-coordinate of point P (where $c \neq m$). Point P lies on a straight line

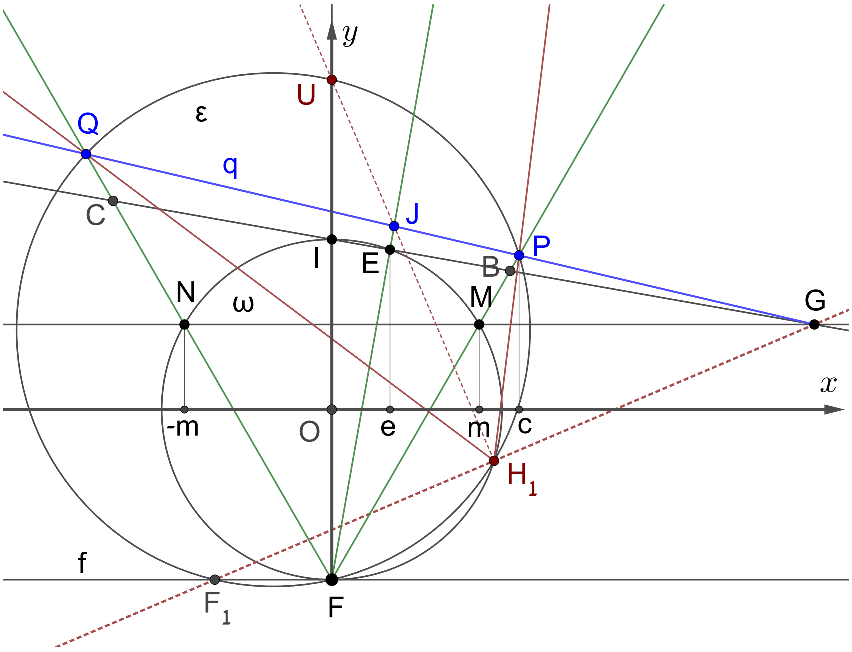

Figure 7: Straight line g passes through point G and intersects FM and FN at points P and Q, respectively. Circle ε passes through F, P, and Q and intersects ω at the additional point H_1. Also, f is tangent to circle ω at point F and intersects ε at point F_1

FM. Thus, according to equation 2.3, the y coordinate of point P is $y_P = \dfrac{bc}{m} - 1$. In other words, $P\left(c; \dfrac{bc}{m} - 1\right)$.

Q is the point of intersection of straight lines GP and FN. Let us find the coordinates of Q.

The slope of straight line GP is $m_{GP} = \dfrac{\dfrac{bc}{m} - 1 - b + 1}{c - \dfrac{(b-2)\,e}{a}} = \dfrac{b\,(am - ac)}{m\,(be - 2e - ac)}$. Therefore, the equation of GP is $y - b + 1 = \dfrac{b\,(am - ac)}{m\,(be - 2e - ac)}\left(x - \dfrac{(b-2)\,e}{a}\right)$ or

$y = \dfrac{b\,(am - ac)}{m\,(be - 2e - ac)}x - \dfrac{cb\,(be - 2e - am)}{m\,(be - 2e - ac)} - 1.$

We denote $v = am–ac$ and $w = be - 2e - ac$, from which we obtain $v - w = am - be + 2e$. Therefore, the equation of straight line GP acquires the form:

$$y = \frac{bv}{mw}x - \frac{cb\,(v - w)}{mw} - 1 \tag{3.1}$$

From equations 2.4 and 3.1 we obtain the equation $\dfrac{bv}{mw}x - \dfrac{cb\,(v-w)}{mw} - 1 = -\dfrac{b}{m}x - 1$.

The solution of this equation gives us the x-coordinate of point Q:

$\dfrac{bv + bw}{mw}x = \dfrac{cb\,(v - w)}{mw} \Rightarrow x_Q = \dfrac{c\,(v - w)}{v + w}$.

Therefore the y-coordinate of point Q is: $y_Q = -\frac{b}{m} \cdot \frac{c\left(v-w\right)}{v+w} - 1 = -\frac{cb\left(v-w\right)}{m\left(v+w\right)} - 1$, and finally $Q\left(\frac{c\left(v-w\right)}{v+w}; -\frac{cb\left(v-w\right)}{m\left(v+w\right)} - 1\right)$.

We let O_1 denote the center of the circle, ε, that passes through points F, P, and Q. O_1 is the point of intersection of the mid-perpendiculars to segments FP and FQ. Let us find the coordinates of O_1.

Let Y be the midpoint of segment FP. Therefore, $Y\left(\frac{c}{2}; \frac{cb}{2m} - 1\right)$.

The slope of straight line FP is $m_{FP} = \frac{b}{m}$, therefore the slope of the line perpendicular to FP is $m_{\perp FP} = -\frac{m}{b}$.

Hence, the equation of the mid-perpendicular to segment FP is $y - \frac{cb}{2m} + 1 = -\frac{m}{b}\left(x - \frac{c}{2}\right)$ or

$$y = -\frac{m}{b}x + \frac{c\left(m^2+b^2\right)}{2bm} - 1 \tag{3.2}$$

Similarly, for Z, which is the midpoint of segment FQ, we obtain $Z\left(\frac{cb\left(v-w\right)}{2m\left(v+w\right)}; -\frac{cb\left(v-w\right)}{m\left(v+w\right)} - 1\right)$.

The slope of the straight line that is perpendicular to FQ is $m_{\perp FQ} = \frac{m}{b}$.

Therefore, the equation of the mid-perpendicular to segment FQ is $y + \frac{cb\left(v-w\right)}{2m\left(v+w\right)} + 1 = \frac{m}{b}\left(x - \frac{c\left(v-w\right)}{2\left(v+w\right)}\right)$ or

$$y = \frac{m}{b}x - \frac{c\left(v-w\right)\left(m^2+b^2\right)}{2bm\left(v+w\right)} - 1 \tag{3.3}$$

From equations 3.2 and 3.3 we obtain the equation $\frac{m}{b}x - \frac{c\left(v-w\right)\left(m^2+b^2\right)}{2bm\left(v+w\right)} - 1 = -\frac{m}{b}x + \frac{c\left(m^2+b^2\right)}{2bm} - 1$. The solution of this equation will give us the x-coordinate of point O_1:

$\frac{2m}{b}x = \frac{c\left(m^2+b^2\right)}{2bm} + \frac{c\left(v-w\right)\left(m^2+b^2\right)}{2bm\left(v+w\right)} \Rightarrow x_{O_1} = \frac{cv\left(m^2+b^2\right)}{2m^2\left(v+w\right)}$.

Since $b = \sqrt{1-m^2} + 1$, it follows that $m^2 + b^2 = 2b$ and therefore $x_{O_1} = \frac{cvb}{m^2\left(v+w\right)}$ and $y_{O_1} = -\frac{m}{b} \cdot \frac{cvb}{m^2\left(v+w\right)} + \frac{c}{m} - 1 = \frac{cw}{m\left(v+w\right)} - 1$.

In summary, for point O_1 we obtain $O_1\left(\frac{cvb}{m^2\left(v+w\right)}; \frac{cw}{m\left(v+w\right)} - 1\right)$.

Now let us find the radius of circle ε as the length of segment O_1F:

$$O_1F = \sqrt{\left(\frac{cvb}{m^2\left(v+w\right)} - 0\right)^2 + \left(\frac{cw}{m\left(v+w\right)} - 1 + 1\right)^2}$$
$$= \sqrt{\frac{c^2v^2b^2}{m^4\left(v+w\right)^2} + \frac{c^2w^2}{m^2\left(v+w\right)^2}} = \sqrt{\frac{c^2\left(b^2v^2+m^2w^2\right)}{m^4\left(v+w\right)^2}}.$$

It follows that the equation of circle ε is

$$\left(x-\frac{cvb}{m^2\left(v+w\right)}\right)^2+\left(y-\frac{cw}{m\left(v+w\right)}+1\right)^2=\frac{c^2\left(b^2v^2+m^2w^2\right)}{m^4\left(v+w\right)^2}.$$

Circles ε and ω intersect at the two points, F_1 and H_1. Let us find the coordinates of point H_1 as the solution of the system of equations of circles ε (see equation above) and ω (whose equation is $x^2+y^2=1$).

From the equation of circle ε we obtain:

$$x^2-\frac{2cvb}{m^2\left(v+w\right)}x+\frac{c^2b^2v^2}{m^4\left(v+w\right)^2}+y^2+\frac{c^2w^2}{m^2\left(v+w\right)^2}+1-\frac{2cw}{m\left(v+w\right)}y+2y-\frac{2cw}{m\left(v+w\right)}$$
$$=\frac{c^2\left(b^2v^2+m^2w^2\right)}{m^4\left(v+w\right)^2}.$$

We replace the sum $x^2+y^2=1$ by 1 and obtain:

$$-\frac{2cvb}{m^2\left(v+w\right)}x+\frac{c^2\left(b^2v^2+m^2w^2\right)}{m^4\left(v+w\right)^2}+2-\frac{2cw}{m\left(v+w\right)}y+2y-\frac{2cw}{m\left(v+w\right)}$$
$$=\frac{c^2\left(b^2v^2+m^2w^2\right)}{m^4\left(v+w\right)^2}$$

or $\left(2-\frac{2cw}{m\left(v+w\right)}\right)y=\frac{2cvb}{m^2\left(v+w\right)}x+\frac{2cw}{m\left(v+w\right)}-2.$

We can then divide both sides of the last equation by the expression $2-\frac{2cw}{m\left(v+w\right)}$, and obtain $y=\frac{cbv}{m\left(mv+mw-cw\right)}x-1.$

If we denote $vm+mw-cw=s$, we obtain the following equation: $y=\frac{cbv}{ms}x-1.$

We substitute this expression for y in the equation for circle ω.

This gives us:

$$x^2+\left(\frac{cbv}{ms}x-1\right)^2=1\Rightarrow x^2+\frac{c^2b^2v^2}{m^2s^2}x^2-\frac{2cbv}{ms}x+1=1$$
$$\Rightarrow x\left(x+\frac{c^2b^2v^2}{m^2s^2}x-\frac{2cbv}{ms}\right)=0$$

The solutions of this equation are $x=0$ (the x-coordinate of point F) and $x=\frac{2cbvms}{m^2s^2+c^2b^2v^2}$ (the x-coordinate of point H_1).

Therefore, the y-coordinate of point H_1 is $y=\frac{cbv}{ms}\cdot\frac{2cbvms}{m^2s^2+c^2b^2v^2}-1=\frac{2c^2b^2v^2}{m^2s^2+c^2b^2v^2}-1.$

Therefore, for point H_1, we obtain $H_1\left(\frac{2cbvms}{m^2s^2+c^2b^2v^2};\frac{2c^2b^2v^2}{m^2s^2+c^2b^2v^2}-1\right).$

Let us now find the coordinates of point F_1. The tangent to circle ω at point F is perpendicular to the x-axis at point $x=-1$, therefore, the equation of the tangent is $y=-1$. Therefore, the x-coordinate of point F_1 is the solution of the equation:

$$\left(x-\frac{cvb}{m^2\left(v+w\right)}\right)^2+\left(-1-\frac{cw}{m\left(v+w\right)}+1\right)^2=\frac{c^2\left(d^2v^2+m^2w^2\right)}{m^4\left(v+w\right)^2}$$

$$x^2-\frac{2cvb}{m^2\left(v+w\right)}x+\frac{c^2b^2v^2}{m^4\left(v+w\right)^2}+\frac{c^2w^2}{m^2\left(v+w\right)^2}=\frac{c^2\left(d^2v^2+m^2w^2\right)}{m^4\left(v+w\right)^2},$$

and finally $x\left(x-\dfrac{2cvb}{m^2(v+w)}\right)=0 \Rightarrow x=\dfrac{2cvb}{m^2(v+w)}$.
Thus, for point F_1, we obtain $F_1\left(\dfrac{2cvb}{m^2(v+w)};-1\right)$.
Let us prove that points G, H_1, and F_1 are collinear. To do that we find slopes m_{GF_1} and $m_{H_1F_1}$.

$$\begin{aligned} m_{GF_1} &= \frac{b-1+1}{\dfrac{(b-2)e}{a}-\dfrac{2cvb}{m^2(v+w)}} = \frac{bam^2(v+w)}{(b-2)em^2(v+w)-2acvb} \\ &= \frac{abm^2(v+w)}{(b-2)em^2(v+w)-2acvb}. \end{aligned}$$

$$m_{H_1F_1} = \frac{\dfrac{2c^2b^2v^2}{m^2s^2+c^2b^2v^2}-1+1}{\dfrac{2cbvms}{m^2s^2+c^2b^2v^2}-\dfrac{2cvb}{m^2(v+w)}} = \frac{cbvm^2(v+w)}{m^3s(v+w)-m^2s^2-c^2b^2v^2}.$$

Consider the difference

$$\begin{aligned} m_{H_1F_1}-m_{GF_1} &= \frac{cbvm^2(v+w)}{m^3s(v+w)-m^2s^2-c^2b^2v^2}-\frac{abm^2(v+w)}{(b-2)em^2(v+w)-2acvb} \\ &= bm^2(v+w)\cdot\frac{(b-2)ecvm^2(v+w)-2ac^2v^2b-am^3s(v+w)+am^2s^2+ac^2b^2v^2}{[m^3s(v+w)-m^2s^2-c^2b^2v^2]\cdot[(b-2)em^2(v+w)-2acvb]}. \end{aligned}$$

In this expression, the last multiplier is a fraction. We denote the numerator of this fraction by A. We substitute the appropriate expressions for s and s^2, and then simplify. Thus we obtain:

$$\begin{aligned} A &= (b-2)ecvm^2(v+w)-2bac^2v^2-am^3(vm+mw-cw)(v+w) \\ &\quad +am^2\left(m^2v^2+m^2w^2+c^2w^2+2m^2vw-2cmvw-2cmw^2\right)+ac^2b^2v^2 \end{aligned}$$

After replacing $2b$ with m^2+b^2, opening the parentheses, and collecting the terms, we will obtain

$$\begin{aligned} A &= (b-2)ecvm^2(v+w)-ac^2v^2m^2-am^3cvw-am^3cw^2+am^2c^2w^2 \\ &= (b-2)ecvm^2(v+w)-am^3cw(v+w)+am^2c^2\left(w^2-v^2\right) \\ &= cm^2(v+w)\left((b-2)ev-amw+ac(w-v)\right) \\ &= cm^2(v+w)\left(((b-2)e-ac)v+(ac-am)w\right) \\ &= cm^2(v+w)(wv-vw)=0. \end{aligned}$$

To summarize, we have obtained that $A=0$; therefore, there holds that $m_{H_1F_1}-m_{GF_1}=0$ or $m_{H_1F_1}=m_{GF_1}$. Therefore, we have shown that points G, H_1, and F_1 are collinear.

(b) The perspective projection between straight lines IE and GP from point F transforms points G, B, E, and C to points G, P, J, and Q, respectively. Therefore, points G, P, J, and Q also constitute a harmonic quadruple. It follows that straight lines H_1G, H_1P, H_1J, and H_1Q will also constitute a harmonic quadruple.
On the other hand, in circle ε, there holds that point U is the midpoint of arc PQ and that segment UF_1 is a diameter of ε (because $\measuredangle UFF_1=90^\circ$, see Figure 8). Therefore $PU=UQ$ and $PF_1=F_1Q$, and it follows that the cross ratio of these directed chords is $\dfrac{\overline{PU}}{\overline{QU}}:\dfrac{\overline{PF_1}}{\overline{QF_1}}=-1$, in other words, P, U, Q, and F_1 are a harmonic quadruple of points on circle ε. Therefore, in the perspective projection from point H_1 of these points, we obtain that straight lines H_1P, H_1U, H_1Q, and H_1F_1 constitute a harmonic quadruple.

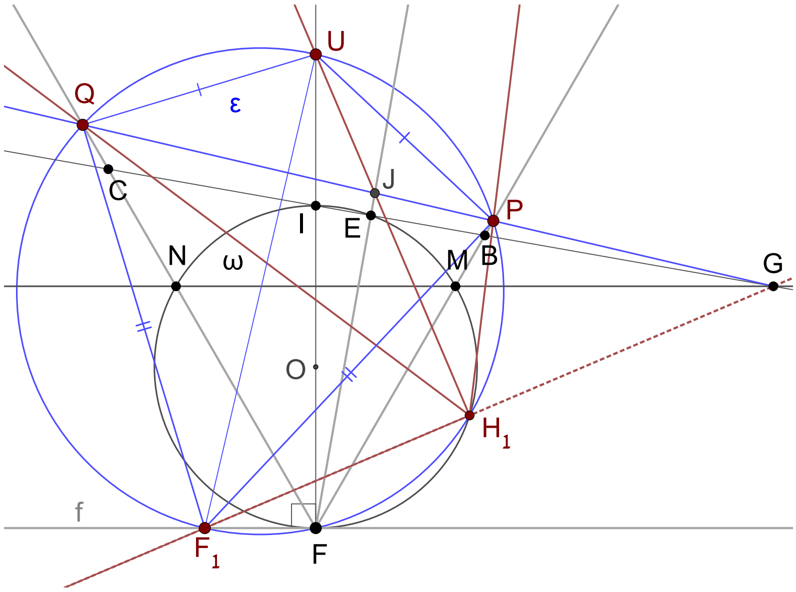

Figure 8: Points G, H_1 and F_1 are collinear. Points H_1, J and U are collinear.

Because points G, H_1, and F_1 are collinear, in the harmonic quadruple H_1G, H_1P, H_1J, H_1Q and in the harmonic quadruple H_1F_1, H_1P, H_1U, H_1Q, there are three coinciding straight lines: $H_1G(= H_1F_1)$, H_1P, and H_1Q . Therefore, the remaining straight lines, H_1J and H_1U, also coincide. In other words, $J \in H_1U$. □

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