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CIRCLE PENCILS AND UNBOUNDED CONICS

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ABSTRACT. In this article we study a method to produce unbounded conics represented as images of lines via a quadratic transformation. The transformation is defined from a couple consisting of a pencil of circles and a point at infinity, through which pass all the conics considered. In addition we show how an arbitrary chord of such a conic determines appropriate pencils and corresponding lines representing the conic through the related quadratic transformation.

1. TRANSFORMATION FROM A PENCIL AND A LINE

Our starting point is a *"coaxal pencil"* of circles $\mathcal{P}$ and a line, from which we use only its *"point at infinity"*, equivalently its *"direction,"* assumed to be different from that of the *"radical axis"* of the pencil ([1, p.201], [2, p.106]). We represent this direction by the unit vector e. Without loss of generality we may also assume that the pencil is generated by its *radical axis* coinciding with the y-axis and the circle $\kappa(A,r)$, which has its center $A(a,0)$ on the x-axis and is represented by an equation of the form (See Figure 1):

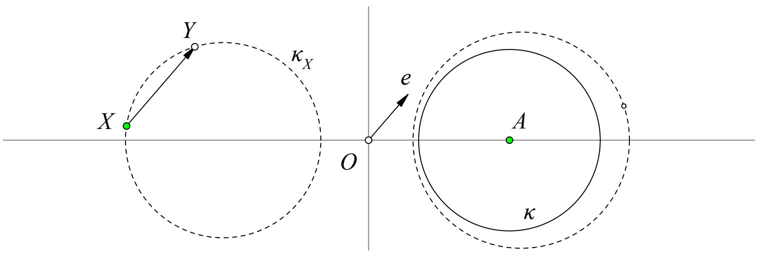

Figure 1. Pencil generated by κ and the y-axis and transformation $f : X \mapsto Y$

$$g(x,y) = (X-A)^2 - r^2 = (x-a)^2 + y^2 - r^2 = 0.$$

The pencil of circles is then described by a real parameter k and the equation :

$$(1-k)\cdot g(x,y) + k\cdot h(x,y) = (1-k)\cdot((x-a)^2+y^2-r^2) + k\cdot x = 0, \qquad (1.1)$$

where we have set $h(x,y) = x$. Using the pencil $\mathcal{P}$ and the direction e we define the transformation f depending on these two elements as follows:

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(1) For each point $X(x, y)$ not lying on the y-axis consider the unique circle-member κ_X of the pencil passing through X.
(2) Construct $Y = f(X)$ as the other intersection point of κ_X with the line ε_X, passing through X, which is parallel to e.

Obviously the member circle κ_{X_0} of the pencil passing through X_0 is represented by

$$\kappa_{X_0} \;:\; g(X_0)h(X) - h(X_0)g(X) \;=\; 0. \tag{1.2}$$

Representing ε_{X_0} in parametric form $\{X_0 + te\}$ with the unit vector $e(e_1, e_2)$ and denoting the inner product by $X \cdot Y$, we have:

$$\begin{aligned}
g(X_0 + te) &= g(X_0) + 2t(X_0 - A) \cdot e + t^2, \\
h(X_0 + te) &= h(X_0) + te_1, \quad \Rightarrow \\
0 &= g(X_0)h(X_0 + te) - h(X_0)g(X_0 + te) \\
&= g(X_0)[h(X_0) + te_1] - h(X_0)[g(X_0) + 2t(X_0 - A) \cdot e + t^2] \\
&= g(X_0)e_1 - 2h(X_0)(X_0 - A) \cdot e - h(X_0)t \quad \Rightarrow \\
t &= \frac{g(X_0)e_1 - 2h(X_0)(X_0 - A) \cdot e}{h(X_0)}.
\end{aligned}$$

Replacing this into the equation $Y(x', y') = X + te$ leads to the expression of the transformation:

$$f(X) \;=\; Y \;=\; X \;+\; \frac{g(X)e_1 - 2h(X)(X - A) \cdot e}{h(X)} e \quad \Leftrightarrow \tag{1.3}$$

$$\left.\begin{aligned}
x' &= \frac{(-e_2 x + e_1 y)^2 - e_1^2(r^2 - a^2)}{x}, \\
y' &= \frac{e_1 e_2(y^2 - x^2) + (e_1^2 - e_2^2)xy - e_1 e_2(r^2 - a^2)}{x}.
\end{aligned}\right\} \tag{1.4}$$

Following properties are obvious consequences of the definition of f :

(3) The transformation $Y = f(X)$ is well defined for every point of the plane not lying on the y-axis, i.e. the radical axis of the pencil.
(4) f is involutive satisfying $f^2 = 1$.
(5) f restricted, on a member κ_X of the circle pencil, leaves it invariant and coincides there with the reflection along the line passing through the center of κ_X and being orthogonal to the direction of e.
(6) f leaves also invariant every line having the direction e, acting on it, in general, as a *"line homography"*.

Remark 1.1. *Last property results as a special case of the more general "Desargues involution" theorem, stating that a pencil of conics defines, by the intersections of its members with a line ε, an "involution" of the points of ε (*[3, I,p.128], [4, p.301]*).*

2. Fixed points

From equation (1.3) follows that the fixed points of the transformation satisfy the equation

$$g(X) \cdot e_1 - 2h(X)(X - A) \cdot e \;=\; 0 \quad \Leftrightarrow \quad e_1 y^2 - 2e_2 xy - e_1 x^2 - e_1(r^2 - a^2) = 0. \tag{2.1}$$

From standard computations of the kind, the center and the asymptotes of conics in terms of the coefficients of its equation (2.1) in Cartesian coordinates ([5, p.327], [6]), it is readily seen that this equation represents a *rectangular hyperbola* μ with center at the origin and asymptotic directions bisecting the angle between the y-axis and e. For a non-intersecting pencil, figure 2 shows this hyperbola, which passes through the *"limit points"* of the pencil. From the geometric construction of f follows that the points X of this hyperbola have the characteristic property that the tangent there of the corresponding member circle κ_X of the pencil is parallel to e, consequently the circle κ_X intersects the hyperbola at a second point X', such that XX' is orthogonal to e.

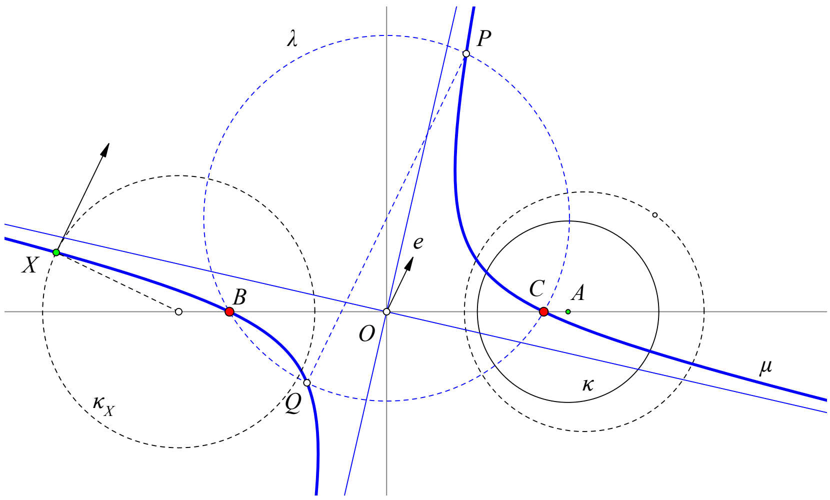

Figure 2. Fixed points of f lie on the rectangular hyperbola μ

Figure 3 shows the hyperbola carrying the fixed points of f for the case of a coaxal pencil of intersecting type. The properties are the same as before, but this time the hyperbola passes through the base points $\{D, E\}$, which are common to all members of the pencil. In the case of a tangential pencil, whose members are all circles tangent to the y-axis at the origin, the corresponding fixed point locus degenerates to two orthogonal lines since equation (2.1) takes the form of the product of lines:

$$e_1y^2 - 2e_2xy - e_1x^2 = 0 \quad \Leftrightarrow \quad (e_1y - (e_2+1)x)(e_1y - (e_2-1)x) = 0 \quad \text{for} \quad e_1 \neq 0.$$

Remark 2.1. *Using equation (2.1) it is easy to see the following properties satisfied by these rectangular hyperbolas:*

(1) *In the case of non-intersecting pencil (see Figure 2), the tangents at the limit points* $\{B, C\}$ *are orthogonal to* e *and point in the conjugate direction to that of the* $x-$*axis.*
(2) *In the case of intersecting pencil (see Figure 3), the tangents at the base points* $\{D, E\}$ *are parallel to* e *and point in the conjugate direction to that of the* $y-$*axis.*
(3) *In both cases, a circle* λ *orthogonal to the pencil, intersects the hyperbola at two other points* $\{P, Q\}$ *such that line* PQ *is parallel to* e *(see Figure 2).*
(4) *These properties formulate a sort of converse of the following well known general property of rectangular hyperbolas (*[7, p.154]*) leading to generations of coaxal pencils: " The circles having as diameters chords of a rectangular hyperbola* μ *which are parallel to the fixed direction* e*, define a coaxal pencil* $\mathcal{P}$*. Their centers lie on the conjugate direction* ε

of e and their radical axis ζ is the orthogonal to ε through their center O. The orthogonal to this, pencil $\mathcal{P}'$, is created analogously by the orthogonal to e direction e' of parallel chords".

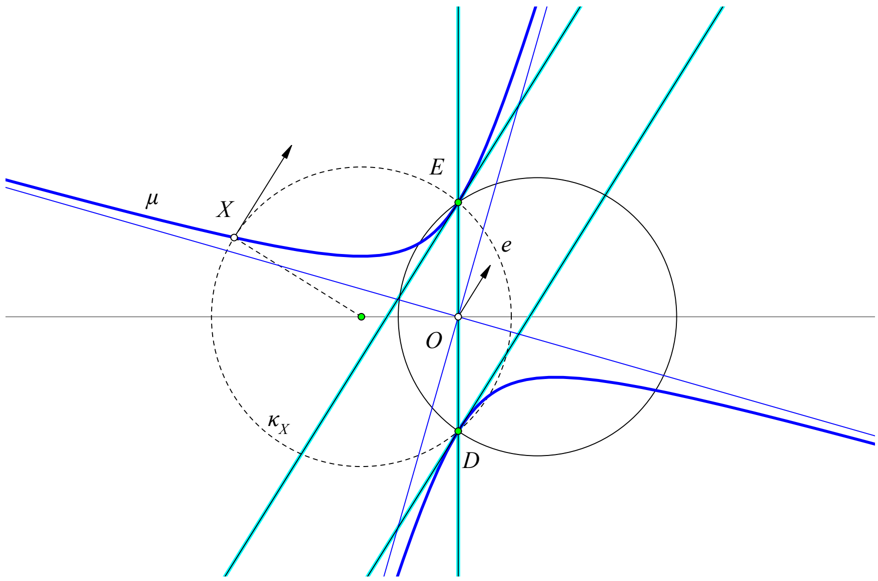

Figure 3. Fixed points of f for a pencil of intersecting type

3. Quadratic transformations

The title *"quadratic"* stems from the expression of f through equation (1.3), which, turning from the cartesian coordinates (x, y) to the corresponding homogeneous coordinates (x, y, z), becomes

$$\left.\begin{array}{rcl} x' &=& (-e_2x+e_1y)^2 - e_1^2(r^2-a^2)z^2, \\ y' &=& e_1e_2(y^2-x^2)+(e_1^2-e_2^2)xy - e_1e_2(r^2-a^2)z^2, \\ z' &=& xz. \end{array}\right\} \tag{3.1}$$

On the right side we have *quadratic polynomials* and the transformation belongs to the more general group of *"Cremona quadratic transformations"*, studied in the context of *"algebraic geometry"* ([8, p.19], [9, p.329]). Property 6) of section 1 even refines the type of the quadratic transformation f to the one of *"de Jonquieres transformations"*, which are Cremona transformations of the plane preserving a pencil of lines ([10, p.51]). All quadratic Cremona transformations define three *"exceptional"* or *"fundamental"* points, which can be different or coinciding, real or complex and are determined by the solutions of the system of equations $x' = y' = z' = 0$. In our case the fundamental points are

$$(0\,,\,\sqrt{r^2-a^2}\,,\,1), \quad (0\,,\,-\sqrt{r^2-a^2}\,,\,1), \quad (e_1\,,\,e_2\,,\,0). \tag{3.2}$$

The last point is at infinity in the direction of e and the other two points are real and distinct only in the case of pencils of intersecting type, for which by assumption $r > a$, coinciding then with the *"base points"* $\{D, E\}$ of the pencil. Figure 3 shows this case together with the *"exceptional lines"*, which are the lines joining the three exceptional points, i.e. the y-axis and the two parallels to e from the base points $\{D,E\}$, coinciding with the tangents of the rectangular hyperbola μ at these points. It is well known ([9, p.331]) that in this case, changing the projective homogeneous coordinates base to these three exceptional points $\{D,E,F\}$ say, the transformation obtains the form of the usual *"isogonal conjugation"*:

$$x' = yz, \quad y' = zx, \quad z' = xy,$$

and the images $f(\zeta)$ of lines ζ of the plane, different from the three exceptional lines, are conics passing through $\{D,E,F\}$. These are the *"triangle conics"* or *"circumconics"* of the triangle DEF ([11, p. 109]), forming the so called *"homaloidal net"* of conics in the context of Cremona quadratic transformations ([12, p.294]).
In our case, besides $\{D,E\}$, the third exceptional point F in (3.2) is at infinity in the direction of e. Hence we conclude that all the images $f(\zeta)$ of lines, are unbounded conics passing through that point, i.e. hyperbolas, parabolas and, in the case of degenerate conics, products of two lines, one of them in the direction of e. Also, in the case the exceptional points $\{D,E\}$ are *real*, which by equation (3.2) means that the pencil is of intersecting or tangential type, all the conics will pass through these points. Next sections discuss some additional peculiarities of the conics defined by the various types of circle pencils.

4. Lines mapping to unbounded conics

Despite the general fact, alluded to at the end of the preceding section and guaranteeing the validity of the next statement, we will prove here our case, going through an explicit calculation, needed also for the subsequent discussion.

Theorem 4.1. *The transformation $Y = f(X)$ maps non-parallel to e lines of the plane to unbounded conics.*

Proof. This can be seen by continuing with the preceding calculations. Without loss of generality, for non-vertical lines (non-parallel to the radical axis) ζ we may assume that they are described by some parametric equation of the form

$$\zeta \ : \ X \ = \ S + tv,$$

where $S(0,s)$ is a point of the y-axis, identified with the radical axis of the pencil, and $v = (v_1, v_2)$ is a unit vector with $v_1 \neq 0$. Denoting by $\{J(x,y) = (-y,x)\}$ the $\frac{\pi}{2}$−rotation and taking inner products of both sides of equation (1.3) with the unit vectors $\{Je, e\}$, we find:

$$Y \cdot Je = X \cdot Je, \qquad\qquad Y \cdot e = X \cdot e + \frac{g(X)e_1 - 2h(X)(X-A)\cdot e}{h(X)}.$$

For points $X = S + tv$, of the line ζ the first equation can be used to express t as a function of Y, provided v is non-parallel to e i.e. $v \cdot Je \neq 0$.

$$Y \cdot Je = (S+tv)\cdot Je \qquad \Rightarrow \qquad t = \frac{Y\cdot Je - S\cdot Je}{v \cdot Je} = \frac{(Y\cdot Je) - se_1}{v_2e_1 - v_1e_2}. \tag{4.1}$$

$$Y \cdot e = e_1(tv_1) + e_2(s+tv_2) \ + \ \frac{g(S+tv)e_1 - 2h(S+tv)(S-A+tv)\cdot e}{h(S+tv)}. \tag{4.2}$$

Setting $\{x' = Y \cdot e,\ y' = Y \cdot Je\}$, and eliminating t from the two last equations, we obtain for $v_1 \neq 0$, the equation:

$$(v_2)y'^2 - (v_1)x'y' + (sv_1e_1)x' - (sv_1e_2)y' - e_1(v \cdot Je)(r^2 - a^2) = 0. \tag{4.3}$$

This is a quadratic equation in cartesian coordinates w.r. to the orthonormal frame $\{e, Je\}$ of the general form

$$Ax'^2 + 2Bx'y' + Cy'^2 + 2Dx' + 2Ey' + F = 0.$$

The theorem results by considering the corresponding *"invariants"* ([13, p.180], [6, p.3]) of the quadratic equation (4.3):

$$\begin{aligned} J_3 &= \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} &&= -\frac{1}{4}e_1(v \cdot Je)(s^2 - (r^2 - a^2))v_1^2, \\ J_2 &= AC - B^2 &&= -\frac{1}{4}v_1^2, \\ J_1 &= A + C &&= v_2. \end{aligned}$$

We notice here that degenerate conics correspond to $J_3 = 0$, which, by our assumptions $\{e_1 \neq 0\,,\ v_1 \neq 0 \quad \text{and} \quad v \cdot Je \neq 0\}$, occur when $s^2 - (r^2 - a^2) = 0$. Thus, under the previous restrictions, we have real solutions only in the case of pencils of intersecting type $(r > a)$. In this case the point S coincides with either of the two *base points* on the y-axis, through which pass all circles of the pencil. A detailed analysis of this situation is given in section 5. We notice also that for $J_1 = v_2 = 0$ i.e. horizontal lines ζ, map to *rectangular* hyperbolas.

An analogous calculation for the excluded case $v_1 = 0$, shows that the corresponding image $f(\zeta)$ of vertical lines $\{\zeta : (s,t) | t \in \mathbb{R}\}$ has the equation in (x', y') coordinates w.r. to the frame $\{e, Je\}$:

$$y'^2 - se_1x' + se_2y' - e_1^2(r^2 - a^2) = 0, \tag{4.4}$$

representing a parabola. □

5. HYPERBOLAS

In this section we examine hyperbolas $f(\zeta)$, which, by the preceding section, result from lines ζ in *general position*, i.e. lines non-parallel to the radical axis of the pencil (y-axis), non-parallel to the line of centers of the pencil (x-axis) and non parallel to the direction e. Next theorems give some additional properties of these hyperbolas (see Figure 4).

Theorem 5.1. *With the notation and conventions adopted so far, the following are valid properties:*

(1) When the line ζ moves parallel to itself the center L of the corresponding hyperbola $f(\zeta)$ moves on a line η through the origin.
(2) One of the asymptotes is parallel to e and passes through the point S.
(3) The direction of the line $\eta = OL$ is conjugate to the direction of the radical axis (y-axis). If the line OL intersects the hyperbola, then the tangents at the intersection points $\{T, T'\}$ are parallel to the y-axis.
(4) The other asymptote passes through the symmetric $S' = -S$ of S w.r. to the origin O and forms with the previous asymptote an angle equal to the angle of the line ζ and the y-axis.

(5) *For parallel lines* $\{\zeta\}$ *corresponding to values* $\{s_1, s_2\}$ *of* $S(0,s)$*, the corresponding hyperbolas* $\{f(\zeta)\}$ *are homothetic w.r.t. to* O *with homothety ratio* s_2/s_1*.*

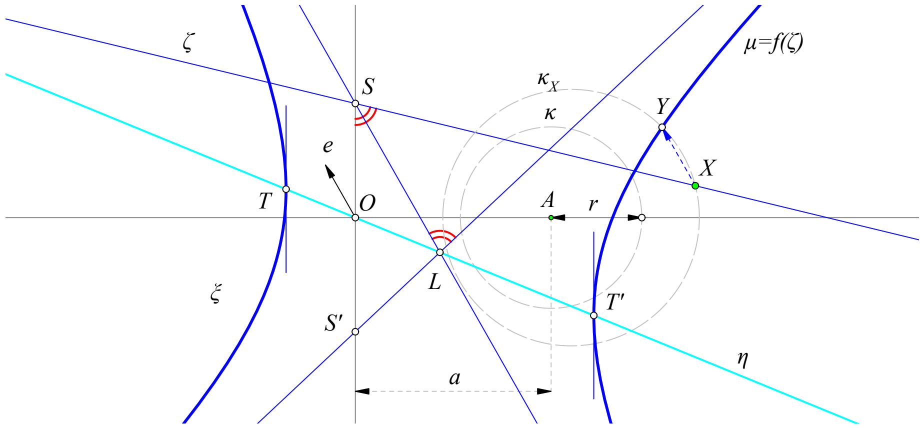

Figure 4. The image $f(\zeta)$ of a generic line ζ is a hyperbola

Proof. *Nr-1* results by applying the well known formulas for the centers of conics ([5, I,p.327], [6, p.8]), calculated in the (x', y')-system:

$$x'_0 = \frac{BE - DC}{AC - B^2} = s\left(\frac{2e_1v_2 - e_2v_1}{v_1}\right), \qquad y'_0 = \frac{DB - AE}{AC - B^2} = s(e_1). \tag{5.1}$$

When ζ moves parallel to itself, then only s varies, whereas $v = (v_1, v_2)$ remains constant, implying the claim.

Nr-2 follows from the involutive property of f and the invariance of lines ε parallel to direction e and properties (4) and (6) of section 1. By these properties, the number of points in the intersection $f(\zeta) \cap \varepsilon'$ is equal to the number of points in the intersection of $f(f(\zeta)) \cap f(\varepsilon') = \zeta \cap \varepsilon'$, which by assumption is precisely 1. This proves that e points to an asymptotic direction. That the corresponding asymptote passes through S, follows from an easy calculation of the intersection point of the y-axis with the line $\{L + te\}$. Alternatively to this all, when X tends to S, the circle κ_X tends to coincide with the radical axis of the pencil (y-axis) and point $Y = f(X)$ tends to the point at infinity in the direction of e.

Nr-3 results by an easy calculation of the polar of the point at infinity $(x_0, y_0, 0)$ using equation (4.3) and showing the coincidence of its direction with that of the y-axis.

Nr-4. The first part is a consequence of the conjugacy of the directions of $\{OL, OS\}$ w.r. to the conic proved in *nr-3*. The second part follows by a standard computation of the angle of asymptotes through the invariants ([6, p.15]):

$$\cos(\theta) = \frac{J_1}{\sqrt{J_1^2 - 4J_2}} = v_2.$$

Nr-5 follows from the similarity of the corresponding triangles $\{SLS'\}$. □

Remark 5.1. *From nr-2 of the theorem follows that the two half-lines of* ζ*, defined by* S*, map to the two different branches of the hyperbola. Thus, points* $\{X, X'\}$ *on the same half-line define* *by their images $\{Y = f(X), Y' = f(X')\}$ chords of the hyperbola with endpoints on the same branch of the curve.*

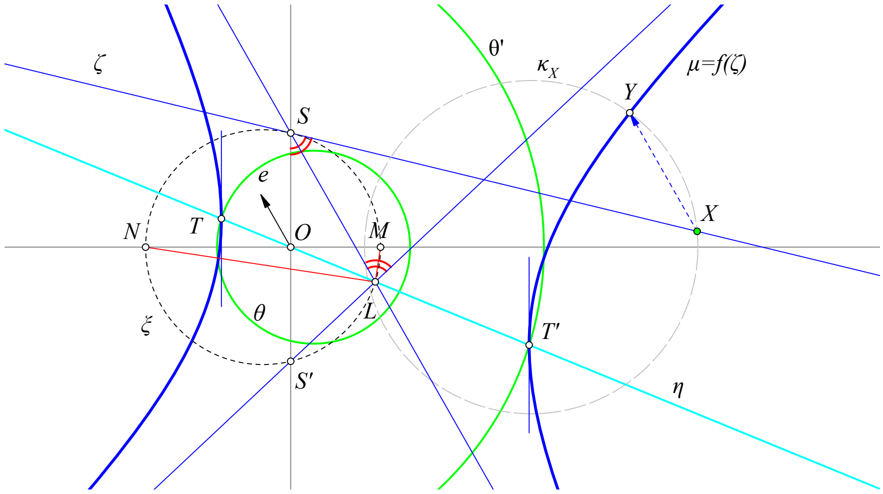

Figure 5. Fixing S and varying line ζ

Theorem 5.2. *With the notation and conventions adopted so far, the following are valid properties (see Figure 5):*

(1) *Fixing S and the line ζ and changing the direction e produces similar hyperbolas with constant angle $\widehat{SLS'}$ of asymptotes, so that L varies on a circle ξ, passing through $\{S, S'\}$ and tangent to line ζ at S.*

(2) *The intersection points $\{M, N\}$ of the circle ξ with the y-axis define the axes $\{LM, LN\}$ of the hyperbola.*

(3) *If the line OL intersects the hyperbola at points $\{T, T'\}$, then the ratios OT/OL and OT'/OL remain constant and points $\{T, T'\}$ move on two circles $\{\theta, \theta'\}$ homothetic to ξ w.r. to O.*

Proof. *Nrs 1-2* are immediate consequences of *nr-4* of theorem 5.1.
Nr-3 results by a calculation of the relative lengths leading to

$$\frac{OT}{OL} = \frac{s - \sqrt{s^2 - (r^2 - a^2)}}{s}, \qquad \frac{OT'}{OL} = \frac{s + \sqrt{s^2 - (r^2 - a^2)}}{s},$$

thereby proving the claim, since these expressions are independent of e. □

Figure 6 displays some worth noticing additional properties of these hyperbolas, which we formulate as a theorem.

Theorem 5.3. *Under the notation and conventions adopted so far, the following are valid properties.*

(1) *The chords YY' of the hyperbola $\mu = f(\zeta)$ resulting from pairs of intersection points (X, X') of ζ with the circles of the pencil, have have length $|YY'| = |XX|$ and point to a fixed direction, which is the reflected of the direction of the line ζ w.r. to the direction Je.*

(2) *Next properties concern the lines* ζ*, for which there are two member circles* $\{\kappa_0, \kappa_1\}$ *of the pencil tangent to it. Then, the contact points* $\{X_0, X_1\}$ *map under* f *onto two points* $\{Y_0, Y_1\}$*. These are also correspondingly contact points of the hyperbola with the two circles.*

(3) *The tangents to the hyperbola at* $\{Y_0, Y_1\}$ *are parallel and line* Y_0Y_1 *passes through the center* L *of the hyperbola and the homothety center* G *of the circles* $\{\kappa_0, \kappa_1\}$*. The tangents at* $\{Y_0, Y_1\}$ *are parallel to the tangent of the circle* ξ *at* L*.*

(4) *The three circles* $\{\kappa_0, \xi, \kappa_1\}$ *are homothetic w.r. to the intersection point* G *of the x-axis with line* ζ*. Points* $\{X_0, X_1\}$ *are symmetric w.r. to* S *and points* $\{Y_0, Y_1\}$ *are symmetric w.r. to* L *and the line* Y_0Y_1 *passes through* G*.*

(5) *Points* $\{S, S'\}$ *and the intersections* $\{Z, Z'\}$ *of line* YY' *with the asymptotes are concyclic on a circle* λ_X *concentric with* κ_X*.*

(6) *The direction of* YY' *is antiparallel w.r. to the triangle* SLS' *and its side* SS'*. The line* Y_0Y_1 *passes through the middle* P *of* YY' *and is a symmedian of the triangle* SLS'*. It is also a direction conjugate to that of* YY' *w.r. to the hyperbola.*

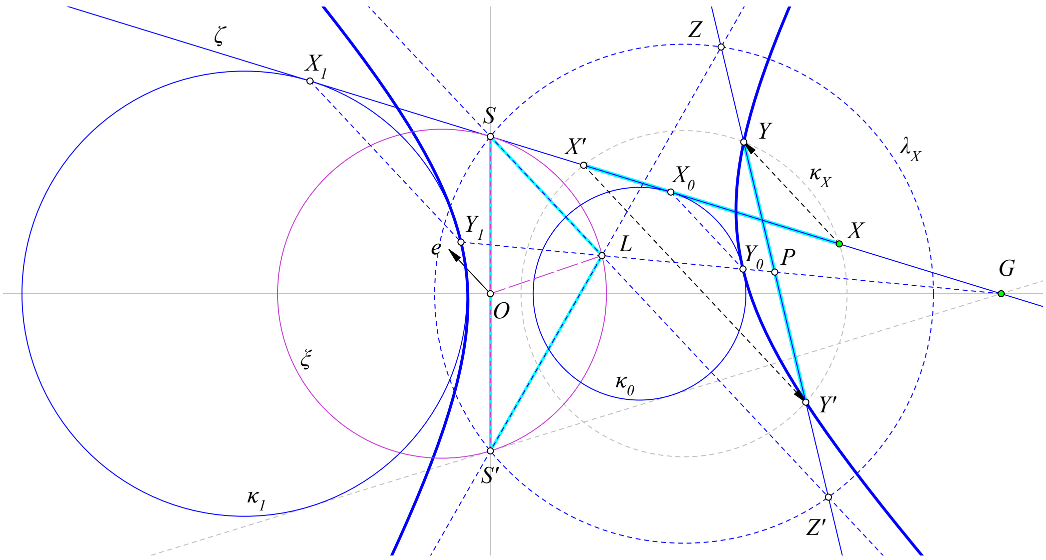

Figure 6. Three homothetic circles w.r. to G and a symmedian line

Proof. *Nr-1* is valid because $\{XY, X'Y'\}$ are parallels and define an isosceles trapezium inscribed in κ_X.

Nr-2. From the invariance of κ_0 under f, the tangent to the circle at X_0, which is line ζ, maps to the tangent at Y_0 to the circle, but also to the hyperbola at this point. An other aspect, is to view the tangent at Y_0 as a limiting position of the parallel lines $\{YY'\}$ while X moves towards X_0. Analogous property is valid also for the circle κ_1 and the tangents at $\{Y_0, Y_1\}$ are parallel.

Nrs 3--4. The property for the homothety center G follows from *nr-2* and the property for point L follows from the parallelity of the two tangents at the points Y_0, Y_1 of the hyperbola. The property of the tangent to ξ at L follows from the homothety of ξ to either of the circles κ_0, κ_1 w.r. to G. The other claims are consequences of the homotheties w.r. to G.

Nr-5 follows from the easy to prove equality of the angles

$$\widehat{X'Y'Y} = \widehat{XX'Y'} = \widehat{LSX'} = \widehat{SS'L},$$

implying that the quadrangle $SS'Y'Y$ is cyclic. On the other side, the centers of $\{\kappa_X, \lambda_X\}$ are both on the x-axis and the coincident medial lines of ZZ' and YY'.
Nr-6. The antiparallel property follows from *nr-5*. The property for the middle P of YY' follows from general properties of the antiparallels ([14]) and the fact that YY' and ZZ' have the same middle. The last claim is a consequence of the previous one combined with *nr-3*. □

Remark 5.2. *The restriction of nr-2 is not valid only in the case of pencils of intersecting type and a line ζ passing through a point S lying between its base points $\{D, E\}$. In this case every circle of the pencil intersects the line ζ in two points $\{X, X'\}$ lying on either side of S, so that the corresponding chord YY' of the hyperbola has its endpoints on different branches (see Figure 7). Also the involution $X \mapsto X'$ defined by the pencil on ζ has no fixed points and there are no members κ_X of the pencil tangent to that line. In all other cases of pencils and generic lines ζ, the involution defined by the pencil on ζ, has two fixed points and there are two corresponding circles κ_0, κ_1 as those appearing in figure 6. In all these cases also the segments XX' intercepted on ζ by the member-circles of the pencil, map via f onto arcs of the hyperbola, having their endpoints $\{Y, Y'\}$ on the same branch.*

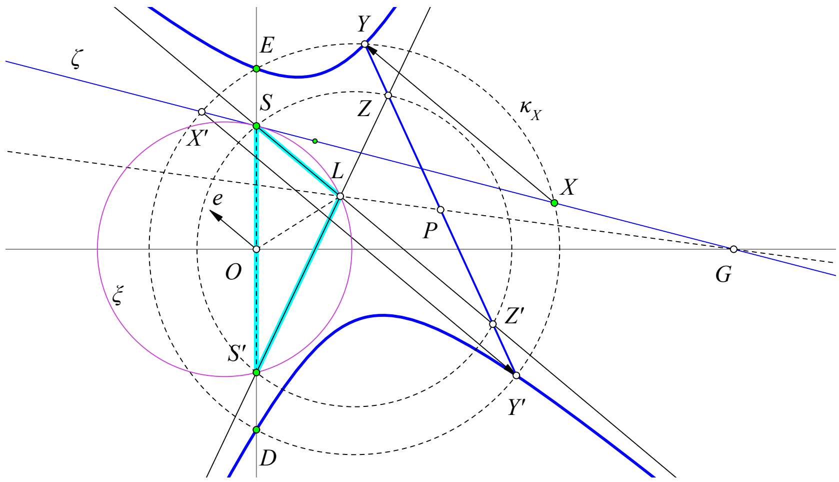

Figure 7. Case of non-existence of circles $\{\kappa_0, \kappa_1\}$

Another characteristic of this special configuration of intersecting pencil and line ζ, having its base points $\{D, E\}$ on different sides of S is the one suggested by figure 8. In this are drawn several hyperbolas resulting from the same line ζ but different directions e. All these hyperbolas have their centers L on the circle ξ, and their asymptotes passing through the points $\{S, S'\}$. For the positions of e along the y-axis, as well as the parallel to ζ, the corresponding hyperbolas are degenerate consisting of two lines: the y-axis and the tangent to ξ at S, correspondingly at S'. For all other positions of e the hyperbolas are non degenerate and pairwise similar or each similar to the conjugate hyperbola of the other. The particular characteristic of this one-parameter-set of hyperbolas is that they do not have an envelope. With the exception of the y-axis, through every other point

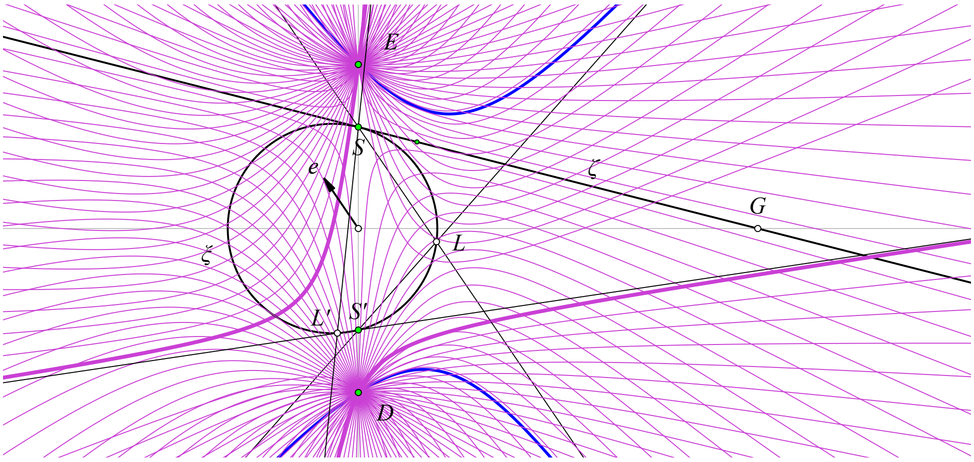

Figure 8. Hyperbolas for fixed ζ and various directions of e, I

of the plane pass two such hyperbolas. Figure 9 shows the behaviour of the analogous configuration for the other kinds of pencils and lines ζ, for which there exist the two circles $\{\kappa_0, \kappa_1\}$. The properties of the one-parameter-set of hyperbolas are the same as in the preceding case with one exception. In this case the two circles are enveloping the hyperbolas and there is no hyperbola passing through the non common inner domain of the two circles.

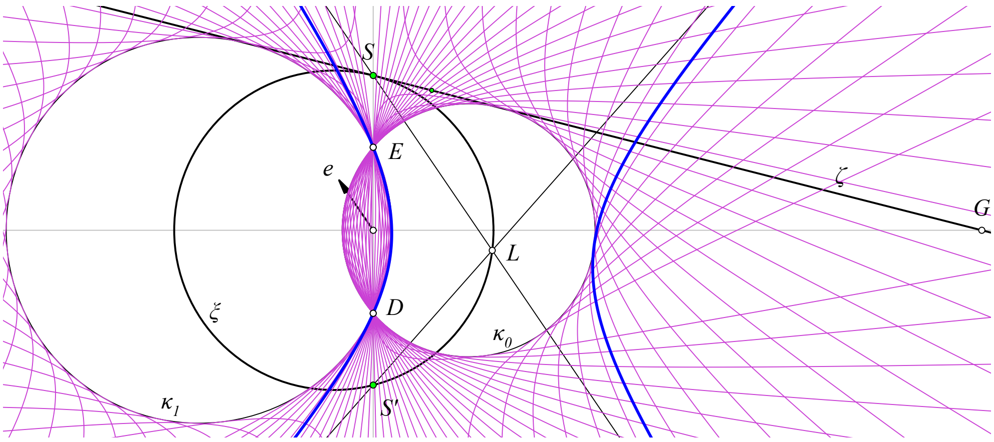

Figure 9. Hyperbolas for fixed ζ and various directions of e, II

6. Parabolas

We saw in section 4, that for lines ζ parallel to the radical axis of the pencil (y-axis) and passing through a point $G(s, 0)$ of the x-axis, we obtain parabolas described by equation (4.4). Working in the (x', y') system of coordinates relative to the frame $\{e, Je\}$, we see that for a point $P(x', y')$ on the parabola, the point $P'(x', -y' - se_2)$ is also on the parabola and PP' is a multiple of Je. From this follow several properties, which we gather in a theorem and leave their proof as easy exercises (see Figure 10).

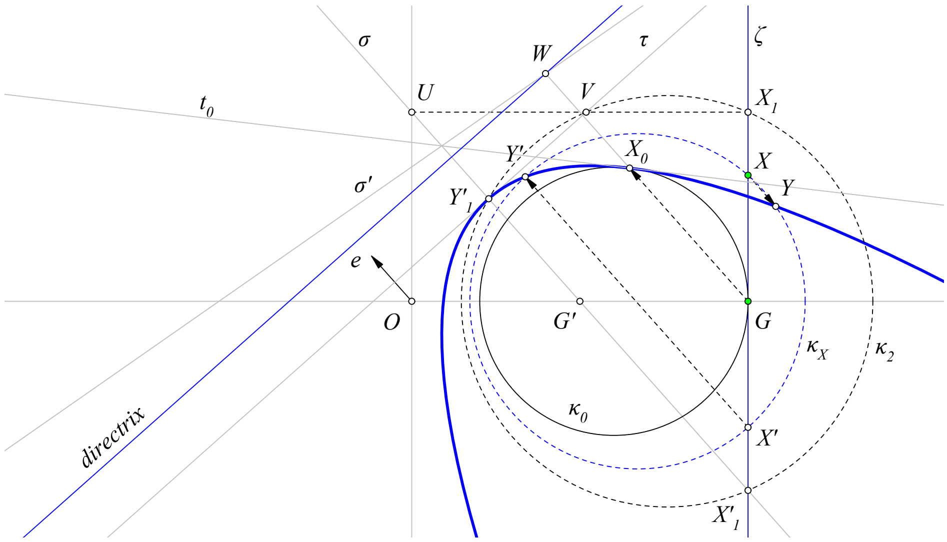

Figure 10. Parabola for a line ζ parallel to the radical axis

Theorem 6.1. *Under the notation and conventions adopted so far, the following are valid properties:*

(1) *The middles of the chords PP' are on a line σ parallel to e, and passing through the middle G' of OG.*
(2) *The axis σ of these parabolas is the parallel to e through G'.*
(3) *The circles κ_X of the pencil intersect line ζ in points $\{X, X'\}$ symmetric w.r. to the x-axis and together with their images $\{Y = f(X), Y' = f(X')\}$ the form an equilateral trapezium $XYX'Y'$ inscribed in κ_X.*
(4) *If U is the intersection of σ and the y-axis and X_1 its projection on ζ, then the vertex Y_1' of the parabola is the image $f(X_1')$ of the symmetric of X_1 w.r. to the x-axis.*
(5) *The member circle κ_2 of the pencil passing through $\{X_1, X_1', Y_1'\}$ passes also through the intersection V of the line UX_1 and the tangent τ at the vertex of the parabola.*
(6) *The reflection w.r. to σ of the tangent to the parabola at X_0 is the tangent parallel to the y-axis.*

Analogous properties to those proved for hyperbolas in section 5 are valid also here for parabolas. Thus, there is one only circle κ_0 of the pencil tangent to line ζ at G. At the point $X_0 = f(G)$ the tangents of the parabola and the circle κ_0 coincide. The direction of this tangent is the same with the fixed direction of all lines YY' resulting from the intersections $\{X, X'\}$ of ζ and κ_X via $f : Y = f(X)$, $Y' = f(X')$. This direction is the symmetric of that of ζ w.r. Je. Notice that knowing the axis σ and the tangent t_0 of the parabola at X_0, we can locate immediately the directrix by taking the reflected of σ' of σ w.r. to t_0 and considering its intersection W with the line GX_0. Point W is on the directrix, which is parallel to Je. Figure 11 shows various parabolas resulting for a fixed line ζ and variable directions $\{e\}$. The case shown is for a pencil of non intersecting type, in which they envelope the circle κ_0.

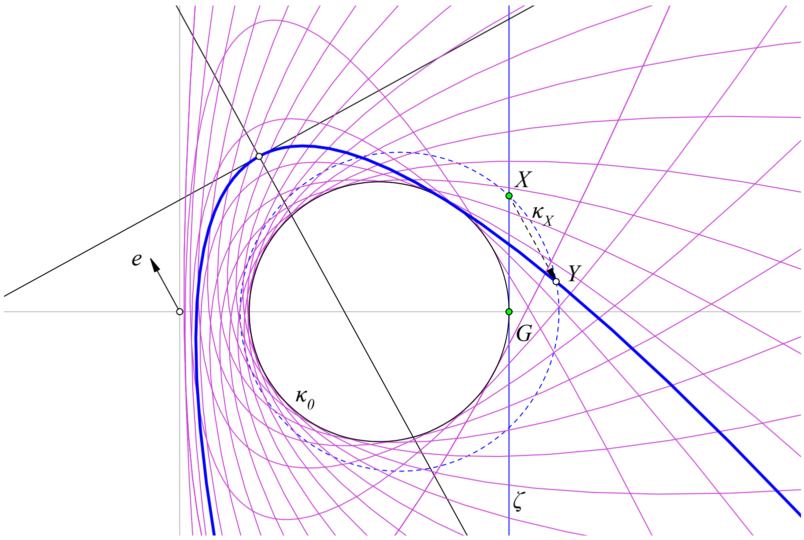

Figure 11. Parabolas for fixed ζ and variable directions $\{e\}$

7. Inverse construction for hyperbolas

In the *"inverse construction problem"* our aim is to produce a *given* hyperbola or parabola by means of an appropriate pencil and a quadratic transformation defined by the recipe discussed in section 1. In the case of hyperbolas, theorem 5.1 implies that the choice of the direction e is mandatory, since it is parallel to one of the asymptotes. Analogously, in the case of parabolas, by theorem 6.1, the direction of e is that of the axis.

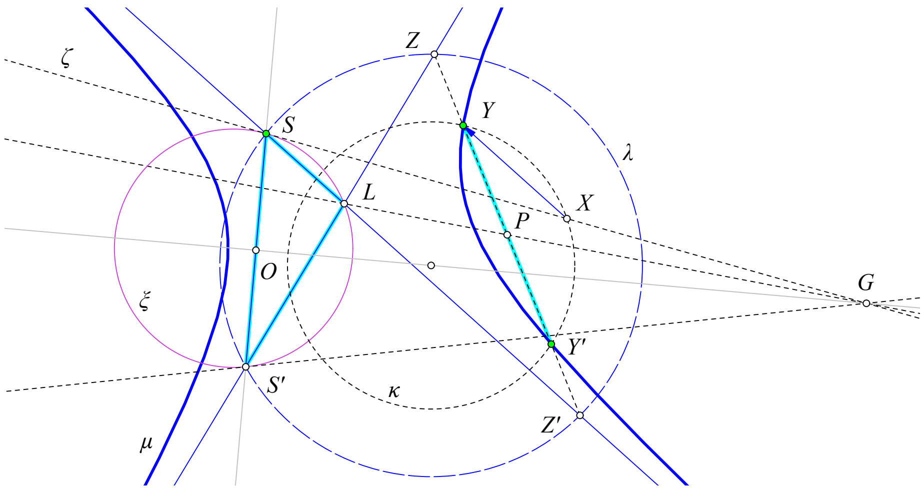

Figure 12. Inverse construction of a pencil generating the given hyperbola

Theorem 7.1. *For every chord YY' of a hyperbola μ, there is a line ζ defining μ as image $\mu = f(\zeta)$ of a quadratic transformation w.r.t. a pencil of circles and the direction e of one of its asymptotes (see Figure 12).*

Proof. Consider first the case of a chord having its endpoints on the same branch of the hyperbola. To construct the required line ζ consider a point S on an asymptote of μ and draw the circle $\lambda = (SZZ')$ through S and the intersections $\{Z, Z'\}$ of YY' with the asymptotes (see Figure 12). The circle λ intersects a second time the other asymptote at a point S', defining the cyclic quadrilateral $SS'Z'Z$ and the *antiparallel* segment SS' to ZZ' w.r. to the triangle LZZ', where L is the center of μ. From standard properties of symmedians and antiparallels ([14]) follows that the tangents to the circumcircle ξ of the triangle $SS'L$ at $\{S, S'\}$ and the line PL, where P is the middle of YY' intersect at the same point G. Then line GO, through the middle of SS', is orthogonal to the latter. We define ζ to be the tangent to ξ at S. The radical axis of the pencil to be defined will be line SS'. An additional circle of the pencil is defined to be the circle κ concentric to λ and passing through Y.
Using the pencil $\mathcal{P}$ generated by the line SS' and the circle κ, the direction e of the asymptote SL and the line ζ, we obtain, by applying the recipe of section 1, a hyperbola. Applying theorem 5.3, we see that this hyperbola has its center at L and its asymptotes are $\{LS, LS'\}$.

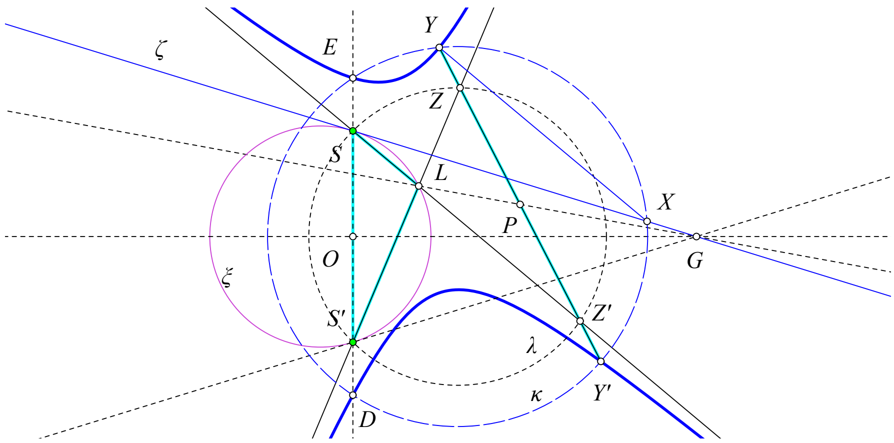

Figure 13. The case of chords YY' with endpoints on different branches

In addition, it is easily seen that the hyperbola passes through the point Y. Thus, the produced hyperbola and the initial one have the same asymptotes and pass both through the same point Y, hence they are identical. Figure 13 illustrates the case of chords with endpoints on different branches. The proof of this case can be carried over verbatim from the previous one. The only difference is that points $\{Z, Z'\}$ are between $\{Y, Y'\}$. $\square$

Remark 7.1. *At this point we should notice that selecting the other asymptotic direction e' of line LS' and repeating the procedure described in the last theorem, defines another quadratic transformation f'. The same hyperbola μ is obtained as image $f'(\zeta')$ by applying f' to the other tangent ζ' to circle ξ at S'. Figure 14 illustrates the case. The cicle pencil $\mathcal{P}$ is the same for the two quadratic transformations $\{f, f'\}$. It is in both cases generated by the line SS' and the circle κ. By theorem 5.3 we know that the points $\{X, X'\}$ mapping via f onto $\{Y, Y'\}$ form an isosceles trapezium $XYX'Y'$. Similarly the points $\{X_1, X_2\}$ mapping via f' onto $\{Y, Y'\}$ form the isosceles trapezium X_1YX_2Y'. By the symmetry w.r. to line OG follows*

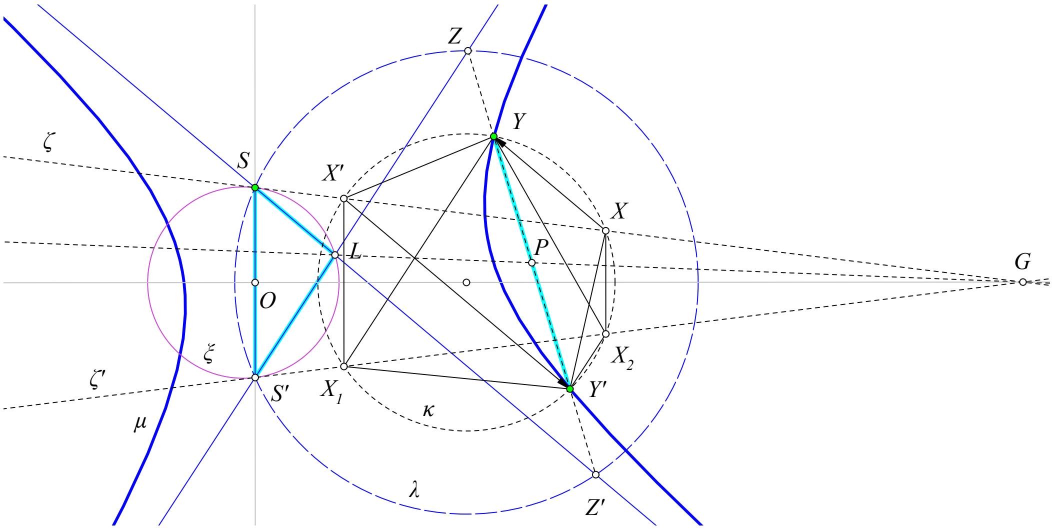

Figure 14. The two lines $\{\zeta, \zeta'\}$ representing the same hyperbola

that $\{(X', X_1), (X, X_2)\}$ *are pairs of reflected points w.r. to* OG*. There results the relation between the two quadratic transformations*

$$f'(X) = (f \circ R_{OG})(X), \tag{7.1}$$

where R_{OG} *is the reflection w.r. to the line* OG *of centers of the pencil.*

Next theorem refers to figure 15 for a particular position of line SS' occurring for some chords of the hyperbola.

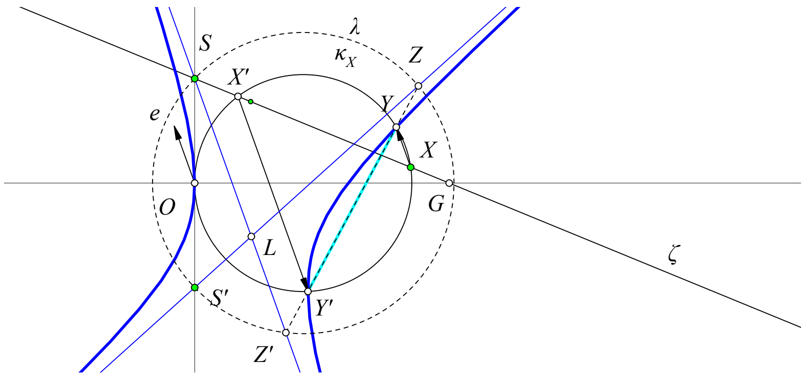

Figure 15. The hyperbola generated by a pencil of tangential type

Theorem 7.2. *For every chord* YY'*, with endpoints on the same branch of a hyperbola, there is one point* O *on the other branch, defining a tangential pencil at* O*, which produces the hyperbola by means of an appropriate quadratic transformation* f*. The pencil is generated by the tangent* SS' *at* O *and the circle* $\lambda = (OYY')$*. Point* O *is the contact point of the tangent which is antiparallel to* YY' *w.r. to the asymptotes. The line* ζ*, representing the hyperbola as* $f(\zeta)$*, is the one defined by the points* $\{X = f(Y), X' = f(Y').\}$

Proof. From theorem 7.1 it is clear that, given the chord YY', there are infinite many choices for the lines SS', which are all antiparallel to the side ZZ' of the triangle LZZ' (See Figure 15). In the case of chords YY' with endpoints on the same branch, a particular such line is tangent to the hyperbola at a point O on the other branch than that of $\{Y, Y'\}$. This can be considered as limiting case of parallel lines intersecting the hyperbola at two distinct points $\{D, E\}$. The corresponding pencil of circles passes through $\{D, E\}$ and in the limiting case the pencil is of the tangential type the two points coinciding with point O and the corresponding line SS' becoming the tangent at O □

For chords YY' of hyperbolas with endpoints on different branches there is no analogous property to the preceding one. There is though a similar property for all kinds of chords YY' resulting also as a limiting position of the parallel translates of line SS' representing the radical axis of the pencil. This results when the radical axis is taken to pass through the center L of the hyperbola. Figure 16 shows such a case, which could be

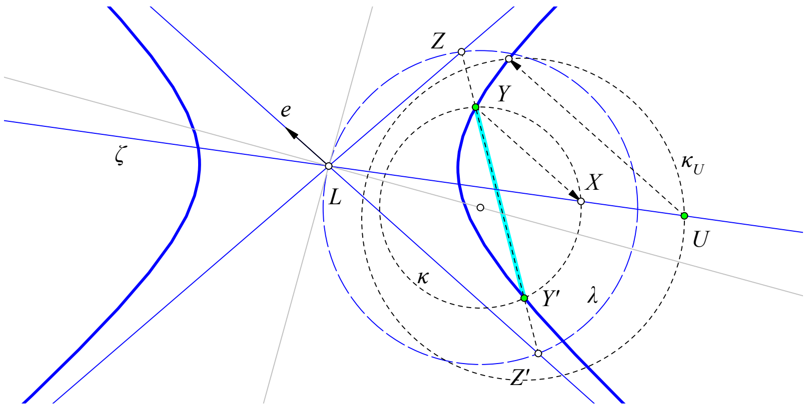

Figure 16. A kind of *"canonical"* generation from YY' (I)

called *"canonical"* generation of the hyperbola through a circle pencil for a given chord YY'. From the preceding discussion follows the next property, which we formulate as a theorem.

Theorem 7.3. *For every chord YY' of a hyperbola μ, there is a pencil of circles, whose radical axis passes through the center L of the hyperbola and which generates μ by means of a quadratic transformation f. The direction e is an asymptotic one and a circle member κ of the pencil is obtained as the concentric passing through Y of the circle $\lambda = (LZZ')$, where $\{Z, Z'\}$ are the intersections of YY' with the asymptotes. The radical axis of the pencil is the tangent at L of λ and the line ζ, representing the conic as the image $f(\zeta)$, is the one passing through L and $X = f(Y)$.*

Figure 17 illustrates the property for a chord YY' with endpoints on different branches of the hyperbola. The figure shows also the line ζ' generating the hyperbola as image $f'(\zeta')$ by means of the other quadratic transformation f', which is defined by the same circle pencil and the other asymptotic direction e'. The hyperbola is this time generated by the points $V' = f'(U')$ for U' running on line ζ'. The relation of equation (7.1) is still valid, the participating there reflection being R_{LK} w.r.t. to the line of centers of the pencil, where K is the center of the circle $\kappa = (LZZ')$: $f' = f \circ R_{LK}$.

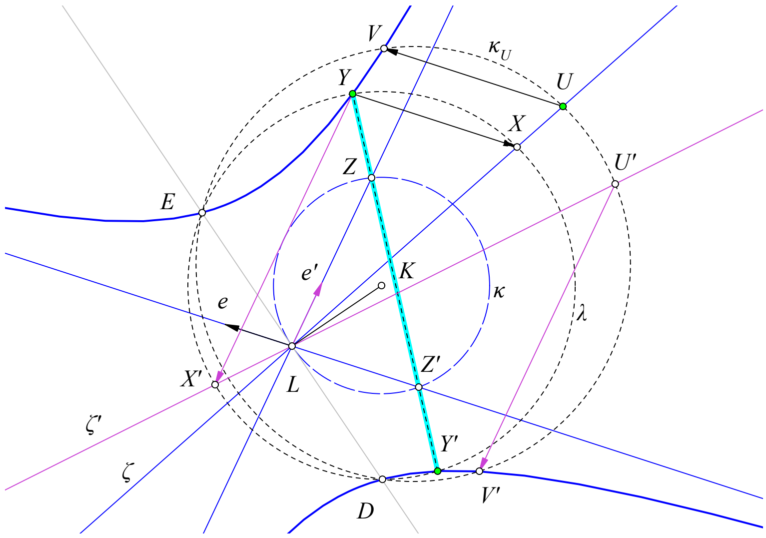

Figure 17. A kind of *"canonical"* generation from YY' (II)

8. Inverse construction for parabolas

Results analogous to those obtained so far for hyperbolas are valid also for parabolas. Here again, as noticed in section 6, for two intersection points $\{X, X'\}$ of the variable circle κ_X through $\{A, B\}$ and the line ζ, the corresponding points $\{X' = f(X), Y' = f(Y)\}$ define chords in a fixed direction ζ'. This direction is the reflection of the direction of ζ

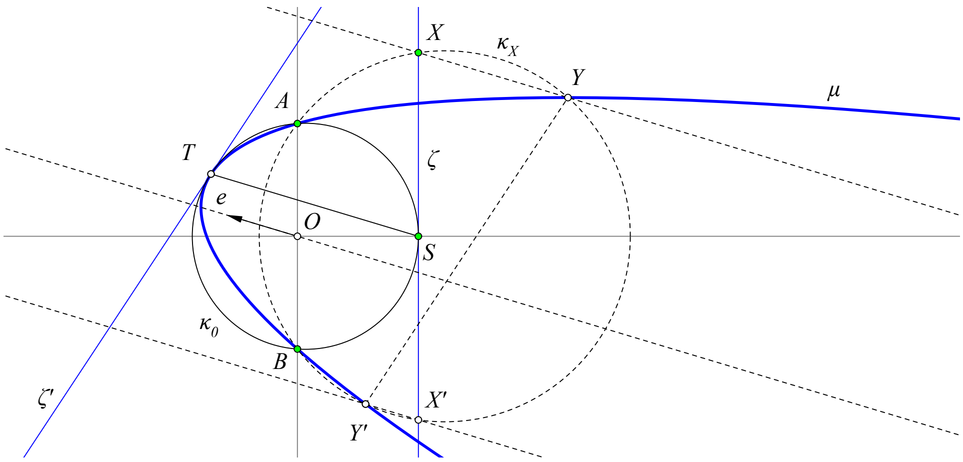

Figure 18. Determination of the parabola from the chord AB

w.r. to Je, vector e being parallel to its directrix (See Figure 18). Besides, the circle of the pencil passing through S is also tangent to the line ζ as well as to the parabola at a point T, at which the tangent is parallel to ζ'. Thus, starting with a chord AB of the parabola we can reconstruct it by means of the recipe for f w.r. to the pencil of circles through $\{A, B\}$ and the line ζ. The location of this line is determined by first drawing the circle κ_0 of the pencil which is tangent to the parabola, or, alternatively, find the point T on the parabola, at which the tangent has the known direction of line ζ'. Having T

and drawing from T the parallel to the axis, we locate then the point S and the line ζ, needed for the recipe of $\mu = f(\zeta)$. We formulate this as a theorem.

Theorem 8.1. *For every chord AB of a parabola μ, there is a line ζ defining it as image $\mu = f(\zeta)$ of a quadratic transformation w.r. to the pencil of intersecting type with base points $\{A, B\}$ and the direction e of its axis.*

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