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HAMILTONIANS DERIVATIONS OVER JACOBI BRACKET

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ABSTRACT. The Jacobi pair and the Jacobi vector field on a smooth manifold are introduced. We define the operators of cohomology for a Jacobi manifold and for a locally conformal symplectic manifold. These operators allow us to explore the hamiltonian vector fields. Finally, we caracterize the notion of globally hamiltonian vector fields and locally hamiltonian vector fields on a Jacobi manifold and on a locally conformal symplectic manifold in terms of derivations over the Jacobi bracket.

1. INTRODUCTION

All the objects that we consider are assumed to be C^∞-smooth and we follow the usual notation of differential geometric literature [3]. Let M be a smooth manifold, $C^\infty(M)$ the commutative algebra of smooth functions on M and E the $C^\infty(M)$-module. A first order differential operator on $C^\infty(M)$ with coeffcients in E is a $\mathbb{R}$-linear map $\varphi : C^\infty(M) \longrightarrow E$ such that

$$\varphi(f \cdot g) = \varphi(f) \cdot g + f \cdot \varphi(g) - f \cdot g \cdot \varphi\left(1_{C^\infty(M)}\right)$$

for any $f, g \in C^\infty(M)$. We denote by $Diff_{\mathbb{R}}[C^\infty(M), E]$ the $C^\infty(M)$-module of first order differential operators on $C^\infty(M)$ with coeffcients in E and

$$Der_{\mathbb{R}}[C^\infty(M), E] = \{\varphi \in Diff_{\mathbb{R}}[C^\infty(M), E] \,/\, \varphi\left(1_{C^\infty(M)}\right) = 0\}.$$

When $E = C^\infty(M)$, we denote by $Diff_{\mathbb{R}}[C^\infty(M)]$ the $C^\infty(M)$-module of first order differential operators of $C^\infty(M)$ and $Der_{\mathbb{R}}[C^\infty(M)]$ the $C^\infty(M)$-module of derivations on $C^\infty(M)$. Let $\mathfrak{X}(M)$ is a $C^\infty(M)$-module and vector fields acts as derivations on smooth functions, that is the map

$$D : \mathfrak{X}(M) \times C^\infty(M) \longrightarrow C^\infty(M), (X, f) \longmapsto D_X(f) := X(f)$$

satisfies the equation $D_X(f \cdot g) = D_X(f) \cdot g + f \cdot D_X(g)$, for any $f, g \in C^\infty(M)$.
Jacobi algebras were first introduced by Kirillov [2] under the name local Lie algebras and independently by Lichnerowicz [5] as the algebraic structure on the ring of C^∞ functions on a certain kind of smooth manifolds, called Jacobi manifolds. Jacobi algebras are a generalisation of Poisson algebras. The Jacobi bracket is a first order differential operator on the commutative algebra endowed with a Lie bracket.

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The purpose of this paper is to caracterize the notion of globally hamiltonian vector fields and locally hamiltonian vector fields on a Jacobi manifold and on a locally conformal symplectic manifold in terms of derivations over the Jacobi bracket. We generalize the classical notions of hamiltonian vector fields on a symplectic manifold [7].
The paper is organized as follows. In Section 2, we briefly recall the universal property of first order differential operators and we recover the Jacobi pair on a smooth manifold. In section 3, we introduce the cohomology associated with the adjoint representation and we explore the notion of hamiltonian vector fields on a Jacobi manifold. In section 4, we caracterize the Jacobi vector fields in terms of the derivations over the Jacobi bracket. Finally, in Section 5, we describe the Jacobi structure associated with a locally conformal symplectic manifold. We define the operator of cohomology which generalizes the de Rham cohomology and we caracterize the globally hamiltonian vector fields and the locally hamiltonian vector fields on a locally conformal symplectic manifold.

2. Preliminaries

Let M be a smooth manifold and denote by $\Omega_{\mathbb{R}}[C^\infty(M)]$ the module of Kähler differentials of commutative algebra $C^\infty(M)$, that is, the quotient space $\Omega_{\mathbb{R}}[C^\infty(M)] = I/I^2$, where I is the $C^\infty(M)$-submodule of $C^\infty(M) \bigotimes_{\mathbb{R}} C^\infty(M)$ generated by the elements of the form
$f \otimes 1_{C^\infty(M)} - 1_{C^\infty(M)} \otimes f$ with $f \in C^\infty(M)$ [1] and [8].
The linear map $\delta_M : C^\infty(M) \longrightarrow \Omega_{\mathbb{R}}[C^\infty(M)]$ defined by

$$\delta_M(f) = \overline{f \otimes 1_{C^\infty(M)} - 1_{C^\infty(M)} \otimes f}$$

is the canonical derivation which the image of δ_M generates the $C^\infty(M)$-module $\Omega_{\mathbb{R}}[C^\infty(M)]$, that is, for $\alpha \in \Omega_{\mathbb{R}}[C^\infty(M)]$, $\alpha = \sum\limits_{i \in I:finite} f_i \cdot \delta_M(g_i)$, with $f_i, g_i \in C^\infty(M)$.
The map $\Delta_M : C^\infty(M) \longrightarrow C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$ such that $\Delta_M(f) = f + \delta_M(f)$, for any f in $C^\infty(M)$, is a first order differential operator and the image of Δ_M generates the $C^\infty(M)$-module $C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$ i.e., for $x \in C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$, $x = f + \alpha$ where $\alpha = \sum\limits_{i \in I:finite} f_i \delta_M(g_i)$ with $f_i, g_i \in C^\infty(M)$.

Theorem 2.1. [8] *Universal property of the pair $(C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)], \Delta_M)$. For every $C^\infty(M)$-module E and for any first order differential operator $\varphi : C^\infty(M) \longrightarrow E$, there exists a unique $C^\infty(M)$-linear map $\widetilde{\varphi} : C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow E$ such that $\widetilde{\varphi} \circ \Delta_M = \varphi$. Moreover, the linear mapping*

$$Hom_{C^\infty(M)}(C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)], E) \longrightarrow Diff_{\mathbb{R}}(C^\infty(M), E), \psi \longmapsto \psi \circ \Delta_M$$

is an isomorphism of $C^\infty(M)$-modules.

Let $\Lambda[C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]] = \bigoplus\limits_{n \in \mathbb{N}} \Lambda^n [C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$ be the exterior algebra of the $C^\infty(M)$-module $C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$.

For any integer $p \geq 1$, we recall that a skew-symmetric $\mathbb{R}$-multilinear map

$$\varphi : [C^\infty(M)]^p = C^\infty(M) \times C^\infty(M) \times C^\infty(M) \times ... \times C^\infty(M) \longrightarrow E$$

is a skew-symmetric p-differential operator if the map

$$\varphi^i = \varphi\left(f_1, ..., \widehat{f_i}, ..., f_p\right) : C^\infty(M) \longrightarrow E, f_i \longmapsto \varphi\left(f_1, f_2, ..., f_{i-1}, f_i, f_{i+1}, ..., f_p\right)$$

is a first order differential operator for any $i = 1, 2, ..., p$, for any $f_1, f_2, ..., f_p \in C^\infty(M)$ [8].

Theorem 2.2. [8] *For any $C^\infty(M)$-module E and for any skew-symmetric p-differential operator $\varphi : [C^\infty(M)]^p \longrightarrow E$, there exists a unique skew-symmetric $C^\infty(M)$-multilinear map*

$$\widetilde{\varphi} : [C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]]^p \longrightarrow E$$

of degree p such that

$$\widetilde{\varphi}\left(\Delta_M(f_1), \Delta_M(f_2), ..., \Delta_M(f_p)\right) = \varphi\left(f_1, f_2, ..., f_p\right) \tag{2.1}$$

for any $f_1, f_2, ..., f_p \in C^\infty(M)$.

A Jacobi bracket on a manifold M is a Lie bracket $\{.,.\}$ on $C^\infty(M)$ satisfying the Leibniz identity

$$\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\} - gh\{f, 1_{C^\infty(M)}\} \tag{2.2}$$

for any $f, g, h \in C^\infty(M)$. A Jacobi manifold is a manifold equipped with a Jacobi bracket. The Leibniz identity means that, for a given function $f \in C^\infty(M)$ on a Jacobi manifold M, the inner derivation $ad(f) : C^\infty(M) \longrightarrow C^\infty(M), g \longmapsto \{f, g\}$ is a first order differential operator [2],[5]. If M be a Jacobi manifold, we denote $\xi = ad(1_{C^\infty(M)})$ the fundamental vector field of the Jacobi manifold. For any f and g in $C^\infty(M)$, we have

$$\xi\{f, g\} = \{\xi(f), g\} + \{f, \xi(g)\}, \xi(fg) = \xi(f)g + f \cdot \xi(g) \tag{2.3}$$

and the map $ad : C^\infty(M) \longrightarrow Diff_{\mathbb{R}}[C^\infty(M)], f \longmapsto ad(f)$ is a first order differential operator. Thus, by the Theorem 2.1, there exists a unique $C^\infty(M)$-linear map

$$\widetilde{ad} : C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow Diff_{\mathbb{R}}[C^\infty(M)]$$

such that

$$\widetilde{ad} \circ \Delta_M = ad. \tag{2.4}$$

Theorem 2.3. *The following statements are equivalent:*
(1) M is a Jacobi manifold.
(2) There exists a skew-symmetric 2-form

$$\omega_M : [C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]] \times [C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]] \longrightarrow C^\infty(M)$$

such that, for any f and g in $C^\infty(M)$,

$$\{f, g\} = \omega_M(\Delta_M(f), \Delta_M(g)) \tag{2.5}$$

defines a $\mathbb{R}$-Lie algebra structure on $C^\infty(M)$.
(3) There exists a skew symmetric 2-form

$$\pi : \Omega_{\mathbb{R}}[C^\infty(M)] \times \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow C^\infty(M)$$

and a vector field ξ on M such that, for any f and g in $C^\infty(M)$,

$$\{f, g\} = \pi(\delta_M(f), \delta_M(g)) + f\xi(g) - g\xi(f) \tag{2.6}$$

defines a $\mathbb{R}$-Lie algebra structure on $C^\infty(M)$.

Proof. (1)⇒(2) If M is a Jacobi manifold, the bracket $\{,\}$ is a skew-symmetric 2-differential operator. By the Theorem 2.2, there exists a unique $\omega_M \in \Lambda^2([C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)])$ such that

$$\{f, g\} = \omega_M(\Delta_M(f), \Delta_M(g))$$

for all $f, g \in C^\infty(M)$.
(2)⇒(3) If $\{f, g\} = \omega_M(\Delta_M(f), \Delta_M(g))$ and since $\Delta_M(f) = f + \delta_M(f)$, we get

$$\{f, g\} = f \cdot \omega_M(1_{C^\infty(M)}, \delta_M(g)) + g \cdot \omega_M(\delta_M(f), 1_{C^\infty(M)})) + \omega_M(\delta_M(f), \delta_M(g)).$$

Since

$$\omega_M(1_{C^\infty(M)}, f) + \omega_M(1_{C^\infty(M)}, \delta_M(f)) = \omega_M(1_{C^\infty(M)}, \Delta_M(f)) = \{1_{C^\infty(M)}, f\},$$

then there exists a vector field $\xi = ad(1_{C^\infty(M)})$ with

$$\xi(f) = ad(1_{C^\infty(M)})(f) = \{1_{C^\infty(M)}, f\} = \omega_M(1_{C^\infty(M)}, \delta_M(f)), \tag{2.7}$$

and there exists a skew-symmetric 2-form

$$\omega_{|\Omega_{\mathbb{R}}[C^\infty(M)]} = \pi : \Omega_{\mathbb{R}}[C^\infty(M)] \times \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow C^\infty(M),$$

such that

$$\begin{aligned}\{f, g\} &= \pi(\delta_M(f), \delta_M(g)) + f \cdot \omega_M(1_{C^\infty(M)}, \delta_M(g)) + g \cdot \omega_M(\delta_M(f), 1_{C^\infty(M)})) \\ &= \pi(\delta_M(f), \delta_M(g)) + f\xi(g) - g\xi(f).\end{aligned}$$

(3)⇒(1) If the bracket

$$\{f, g\} = \pi(\delta_M(f), \delta_M(g)) + f\xi(g) - g\xi(f)$$

defines a $\mathbb{R}$-Lie algebra structure on $C^\infty(M)$, then

$$\begin{aligned}ad(f)\,(g \cdot h) &= \{f, g \cdot h\} \\ &= \pi(\delta_M(f), \delta_M(g \cdot h)) + f\xi(g \cdot h) - g \cdot h\xi(f) \\ &= g \cdot ad(f)\,(h) + ad(f)\,(g) \cdot h - gh \cdot ad(f)\left(1_{C^\infty(M)}\right)\end{aligned}$$

i.e., $ad(f)$ is a differential operator. Therefore, M is a Jacobi manifold. □

The skew-symmetric 2-form ω_M on $C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$ is called Jacobi 2-form of the Jacobi manifold M and the pair (M, ω_M) is called Jacobi manifold.
In this case we say that the pair (π, ξ) defines a Jacobi structure on M and (M, π, ξ) is a Jacobi manifold. If $\xi = 0$, the pair (M, π) is called Poisson manifold. A first order differntial operator φ on a Jacobi manifold (M, ω_M) is said of Jacobi if $\mathfrak{L}_\varphi \omega_M = 0$.

3. Hamiltonian vector fields on Jacobi manifold

If (M, ω_M) is a Jacobi manifold, for $f \in C^\infty(M)$, the map $X_f : C^\infty(M) \longrightarrow C^\infty(M)$, such that

$$X_f\,(g) = \omega_M(\Delta_M\,(f)\,, \delta_M(g)) = \{f, g\} - g \cdot \{f, 1_{C^\infty(M)}\} \tag{3.1}$$

is a vector field on M called hamiltonian vector field associated with f. Moreover, the map

$$\Phi : C^\infty(M)] \longrightarrow Der_{\mathbb{R}}[C^\infty(M)], f \longmapsto X_f = ad(f) - \{f, 1_{C^\infty(M)}\}$$

is a morphism of $\mathbb{R}$-Lie algebras, that is, $[X_f, X_g] = X_{\{f,g\}}$, for all $f, g \in C^\infty(M)$. The set, $\mathfrak{X}_{Ham}(M)$, of the hamiltonians derivations is a Lie subalgebra of $Der_{\mathbb{R}}[C^\infty(M)]$. It is easy to see that Φ is a first order differential operator. It is follows that

$$X_{f\cdot g} = f \cdot X_g + g \cdot X_f - f \cdot g \cdot X_{1_{C^\infty(M)}}$$

for all $f, g \in C^\infty(M)$. When (M, ω_M) is a Jacobi manifold, then the map

$$ad : (C^\infty(M), \{,\}) \longrightarrow (Diff_{\mathbb{R}}\left[(C^\infty(M)\right], [,]), f \longmapsto ad(f)$$

is a adjoint representation of $C^\infty(M)$ into $C^\infty(M)$ i.e., $ad(\{f,g\}) = [ad(f), ad(g)]$ for any $f, g \in C^\infty(M)$.
For $p \in \mathbb{N}$, we denote by $\Lambda^p_J(M) = \mathfrak{L}^p_{sks}[C^\infty(M), C^\infty(M)]$ the $C^\infty(M)$-module of skew-symmetric multilinear forms of degree p from $C^\infty(M)$ into $C^\infty(M)$. We have $\Lambda^0_J(M) = C^\infty(M)$ and we denote by $\Lambda_J(M) = \bigoplus_{p=0}^{n} \Lambda^p_J(M)$, the algebra of skew-symmetric multilinear forms. Let $d_{ad} : \Lambda_J(M) \longrightarrow \Lambda_J(M)$ be the operator of cohomology associated with the adjoint representation ad and let

$$d_\Phi : \mathfrak{L}_{sks}[C^\infty(M), C^\infty(M)] \longrightarrow \mathfrak{L}_{sks}[C^\infty(M), C^\infty(M)]$$

be the operator of cohomology associated with the representation Φ. Thus, for any $\eta \in \Lambda^p_J(M)$ and for any $f_1, f_2, ..., f_{p+1} \in C^\infty(M)$,

$$\begin{aligned}(d_{ad}\eta)(f_1, f_2, ..., f_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i-1} ad(f_i)[\eta(f_1, f_2, ...\widehat{f_i}..., f_{p+1})] \\ &\quad + \sum_{1\le i<j\le p+1} (-1)^{i+j}\, \eta(\{f_i, f_j\}, f_1, ..., \widehat{f_i}, ...\widehat{f_j}, ..., f_{p+1})\end{aligned} \tag{3.2}$$

where $\widehat{f_i}$ means that the term f_i is omitted. For any $f, g \in C^\infty(M)$, $(d_{ad}f)(g) = -\{f, g\}$. In particular, $\left(d_{ad}1_{C^\infty(M)}\right)(f) = \{f, 1_{C^\infty(M)}\}$.
For any $\eta \in \mathfrak{L}_{sks}[C^\infty(M), C^\infty(M)]$, we have

$$d_{ad}\eta = d_\Phi\eta + [d_{ad}(1_{C^\infty(M)})]\Lambda\eta.$$

When M is a Jacobi manifold with bracket $\{,\}$, a vector field $X \in \mathfrak{X}(M)$ is locally hamiltonian if X is closed for the cohomology associated with the representation ad, that is $d_{ad}X = 0$.

Proposition 3.1. *When M is a Jacobi manifold with bracket $\{,\}$, then a locally hamiltonian vector field $X \in \mathfrak{X}(M)$ is the derivation of the Jacobi algebra $C^\infty(M)$.*

Proof. If $d_{ad}X = 0$, then for any f and $g \in C^\infty(M)$,

$$\begin{aligned}0 &= (d_{ad}X)(f, g) \\ &= ad(f)[X(g)] - ad(g)[X(f)] - X(\{f, g\}) \\ &= \{f, X(g)\} - \{g, X(f)\} - X(\{f, g\}),\end{aligned}$$

that is,

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\},$$

for any f and $g \in C^\infty(M)$, □

When M is a Jacobi manifold with bracket $\{,\}$, a vector field $X \in \mathfrak{X}(M)$ is globally hamiltonian if X is exact for the cohomology associated with the adjoint representation ad that is, there exists $f \in C^{\infty}(M)$ such that $X = d_{ad}(f)$.

Proposition 3.2. *When M is a Jacobi manifold with bracket $\{,\}$, then a globally hamiltonian vector field $X \in \mathfrak{X}(M)$ is the derivation interior of the Jacobi algebra $C^{\infty}(M)$.*

Proof. Let $X \in \mathfrak{X}(M)$ the vector field be a globally hamiltonian, there exists $f \in C^{\infty}(M)$ such that $X = d_{ad}(f)$. For any $g \in C^{\infty}(M)$, we have

$$X(g) = (d_{ad}f)(g) = -ad(f)(g)$$

i.e., $X = -ad(f)$. Thus, X is globally hamiltonian if $X = -ad(f)$ with $f \in C^{\infty}(M)$, that is, X is the derivation interior of the Jacobi algebra $C^{\infty}(M)$. □

4. Jacobi vector fields

For any derivation $D : C^{\infty}(M) \longrightarrow C^{\infty}(M)$, the Lie derivative with respect to D is the map

$$\mathfrak{L}_D : \Lambda^p(\Omega_{\mathbb{R}}[C^{\infty}(M)]) \longrightarrow \Lambda^p(\Omega_{\mathbb{R}}[C^{\infty}(M)])$$

such that for $\eta \in \Lambda^p(\Omega_{\mathbb{R}}[C^{\infty}(M)])$ and $x_1, ..., x_p \in \Omega_{\mathbb{R}}[C^{\infty}(M)]$,

$$(\mathfrak{L}_D\eta)(x_1, ..., x_p) = D\left[\eta(x_1, ..., x_p)\right] - \sum_{i=1}^{p} \eta(x_1, ..., \mathfrak{L}_D x_i, x_{i+1}, ..., x_p). \tag{4.1}$$

For any $D \in Der_{\mathbb{R}}[C^{\infty}(M)]$ and $f \in C^{\infty}(M)$, we have

$$\mathfrak{L}_D\delta_M(f) = \delta_M\left[D(f)\right]. \tag{4.2}$$

A vector field X on a Jacobi manifold (M, π, ξ) is said to be a Jacobi vector field if $\mathfrak{L}_X\pi = 0$ and $\mathfrak{L}_X\xi = 0$. When $X \in \mathfrak{X}(M)$ is a Jacobi vector field, then $X[\xi(f)] = \xi[X(f)]$ and

$$X\left[\pi\left(\delta_M(f), \delta_M(g)\right)\right] = \pi\left(\delta_M[X(f)], \delta_M(g)\right) + \pi\left(\delta_M(f), \delta_M[X(g)]\right)$$

for any $f, g \in C^{\infty}(M)$.

Theorem 4.1. *Let $X \in \mathfrak{X}(M)$ be a vector field on a Jacobi manifold, then the following statements are equivalent:*
(i) X is a Jacobi vector field;
(ii) X is a derivation over the Jacobi bracket;
(iii) $[X, X_f] = X_{X(f)}$.

Proof. $(i) \Rightarrow (ii)$ If $X \in \mathfrak{X}(M)$ is a Jacobi vector field, then $\mathfrak{L}_X\pi = 0$ and $\mathfrak{L}_X\xi = 0$. For all $f, g \in C^{\infty}(M)$, we have, from (2.6),

$$\begin{aligned} X(\{f, g\}) &= X\left[\pi\left(\delta_M(f), \delta_M(g)\right) + f \cdot \xi(g) - g \cdot \xi(f)\right] \\ &= X\left[\pi\left(\delta_M(f), \delta_M(g)\right)\right] + X(f) \cdot \xi(g) + f \cdot X\left[\xi(g)\right] \\ &\quad - X(g) \cdot \xi(f) - g \cdot X\left[\xi(f)\right] \end{aligned}$$

i.e.,

$$\begin{aligned} X(\{f, g\}) &= \pi\left(\delta_M[X(f)], \delta_M(g)\right) + X(f) \cdot \xi(g) - g \cdot \xi[X(f)] \\ &\quad + \pi\left(\delta_M(f), \delta_M[X(g)]\right) + f \cdot \xi\left[X(g)\right] - X(g) \cdot \xi(f). \end{aligned}$$

Therefore,
$$X\{f,g\} = \{X(f),g\} + \{f,X(g)\}.$$
(ii)⇒(iii) When X is the derivation over the Jacobi bracket $(C^\infty(M), \{,\})$, we have for all $g \in C^\infty(M)$,
$$[X, X_f](g) = X[X_f(g)] - X_f[X(g)].$$
From (3.1)
$$\begin{aligned}[X, X_f](g) &= \{X(f),g\} + g \cdot \xi[X(f)] \\ &= X_{X(f)}(g).\end{aligned}$$
(iii)⟹(i) When $[X, X_f] = X_{X(f)}$, we have for all $f \in C^\infty(M)$,

$$\begin{aligned}\mathfrak{L}_X\xi(f) &= \{X(1_{C^\infty(M)}),f\} + \{1_{C^\infty(M)}, X(f)\} - \{1_{C^\infty(M)}, X(f)\} \\ &= 0,\end{aligned}$$
and for all $f,g \in C^\infty(M)$ and $\pi \in \Lambda^2(\Omega_{\mathbb{R}}[C^\infty(M)])$, from (4.1),
$$\begin{aligned}&\mathfrak{L}_X\pi(\delta_M(f), \delta_M(g)) \\ =\ & X[\pi(\delta_M(f), \delta_M(g))] - \pi\left(\mathfrak{L}_X\delta_M(f), \delta_M(g)\right) - \pi\left(\delta_M(f), \mathfrak{L}_X\delta_M(g)\right)\end{aligned}$$
By (2.6), we get
$$\begin{aligned}\mathfrak{L}_X\pi(\delta_M(f), \delta_M(g)) &= -f \cdot \xi[X(g)] + g \cdot \xi[X(f)] - g \cdot \xi[X(f)] + f \cdot \xi[X(g)] \\ &= 0.\end{aligned}$$

□

Proposition 4.1. *Let (M, π, ξ) be a Jacobi manifold. Then, all locally hamiltonian vector fields are Jacobi vector fields.*

Proof. Let X be a locally hamiltonian vector field on a Jacobi manifold M with bracket $\{,\}$. X is the derivation of the Jacobi algebra $C^\infty(M)$ and from the Theorem 4.1, X is a Jacobi vector field. □

Proposition 4.2. *Let (M, π, ξ) be a Jacobi manifold. Then, all globally hamiltonian vector fields are Jacobi vector fields.*

Proof. Let X be a globally hamiltonian vector field on a Jacobi manifold M, there exists $f \in C^\infty(M)$ such that $X = d_{ad}(f)$. Then
$$d_{ad}X = d_{ad}\left(d_{ad}(f)\right) = 0,$$
that is X is a locally hamiltonian vector field. By the Theorem 4.1, X is a Jacobi vector field. □

The first group of cohomology $H^1_{Jac}(M)$ is the quotient of the space of Jacobi vector fields by the hamiltonians vectors fields.
Let μ be a volume form on an orientable manifold M and let X be a vector fields on M [3]. The divergence operator of X with respect to μ is the map $div_\mu X$ such that

$$(div_\mu X)\mu = \mathfrak{L}_X\mu.$$
For any X and $Y \in \mathfrak{X}(M)$, for any f in $C^\infty(M)$, we have

$$div_\mu([X,Y]) = X(div_\mu Y) - Y(div_\mu X), \tag{4.3}$$

$$div_\mu(f \cdot X) = X(f) + f \cdot div_\mu X \tag{4.4}$$

and for $f > 0$,

$$div_{f\cdot\mu}(X) = X(log f) + div_\mu X. \tag{4.5}$$

Proposition 4.3. *Let (M, π, ξ, μ) be a Jacobi manifold, equipped with a volume form μ, then, the map*

$$X_\mu : C^\infty(M) \to C^\infty(M), f \mapsto div_\mu X_f$$

is a Jacobi vector field. Moreover, this Jacobi vector field X_μ is a 1-cocycle for the cohomology associated with the adjoint representation.

Proof. For any f and g in $C^\infty(M)$, from $X_{f\cdot g} = f \cdot X_g + g \cdot X_f - f \cdot g \cdot X_{1_{C^\infty(M)}}$, we have

$$X_\mu(f \cdot g) = div_\mu(f \cdot X_g + g \cdot X_f - f \cdot g \cdot X_{1_{C^\infty(M)}})$$

By direct computations, using (3.1) and (4.4), we obtain

$$X_\mu(f \cdot g) = f \cdot X_\mu(g) + g \cdot X_\mu(f). \tag{4.6}$$

We also get,

$$X_\mu(\{f, g\}) = \{X_\mu(f), g\} + \{f, X_\mu(g)\}. \tag{4.7}$$

By (4.1),

$$\begin{aligned} \mathfrak{L}_{X_\mu}\xi(f) &= X_\mu(\xi(f)) - \xi(\mathfrak{L}_{X_\mu}(f)) \\ &= X_\mu(\{1_{C^\infty(M)}, f\}) - \xi(X_\mu(f)) \end{aligned}$$

From (4.7), we get $\mathfrak{L}_{X_\mu}\xi(f) = 0$, for all $f \in C^\infty(M)$.
Using again (4.1), for any f and g in $C^\infty(M)$, we find

$$\begin{aligned} \left(\mathfrak{L}_{X_\mu}\pi\right)(\delta_M(f), \delta_M(g)) &= X_\mu[\pi(\delta_M(f), \delta_M(g))] - \pi(\mathfrak{L}_{X_\mu}\delta_M(f), \delta_M(g)) \\ &\quad - \pi(\delta_M(f), \mathfrak{L}_{X_\mu}\delta_M(g)) \end{aligned}$$

From (4.2), (4.6), (4.7) and by using the formula (2.6), we get,

$$\mathfrak{L}_{X_\mu}\pi(\delta_M(f), \delta_M(g)) = 0,$$

for any f and g in $C^\infty(M)$. Thus, X_μ is a Jacobi vector field.
From (3.2), for any $f, g \in C^\infty(M)$,

$$\begin{aligned} (d^1_{ad}X_\mu)(f, g) &= ad(f)[X_\mu(g)] - ad(g)[X_\mu(f)] - X_\mu(\{f, g\}) \\ &= \{f, X_\mu(g)\} - \{X_\mu(f), g\} - X_\mu(\{f, g\}). \end{aligned}$$

From (4.7), $(d^1_{ad}X_\mu)(f, g) = 0$, for any $f, g \in C^\infty(M)$. Hence, X_μ is a 1-cocycle for the cohomology associated with *ad*. □

The vector field X_μ is called modular Jacobi vector field.

Proposition 4.4. *The Jacobi cohomology class of the modular vector field X_μ is independent of the choice of volume form μ.*

Proof. For any $f > 0$ and $g \in C^{\infty}(M)$, we obtain

$$X_{f\cdot\mu}(g) - X_{\mu}(g) = -X_{logf}(g).$$

This means that, the vector fields $X_{f\cdot\mu}$ and X_{μ} differ by hamiltonian vector field, that is, there is $h \in C^{\infty}(M)$ such that $-X_{logf} = d^0(h)$. Thus, $X_{f\cdot\mu} - X_{\mu} = d^0(h)$ is a Jacobi 1-coboundary, it follows that, $[X_{f\cdot\mu}] = [X_{\mu}]$. Therefore, the Jacobi cohomology class of the modular form is therefore independent of the chosen volume form. □

5. Hamiltonian vector fields on a locally conformal symplectic Manifold

Locally conformal symplectic structures were introduced by Lee in [4] and then studied extensively by Libermann [6] and Vaisman [9].
A locally conformal symplectic structure on M is a pair (α, ω) of a differential closed 1-form α and a nondegenerate differential 2-form ω on M such that

$$d\omega = -\alpha \wedge \omega. \tag{5.1}$$

The 1-form α is known as the Lee form. When $\alpha = 0$, then M is a symplectic manifold.
Since the 2-form ω is nondegenerate, then the map $\omega^{\flat} : \mathfrak{X}(M) \longrightarrow \Lambda^1(M), X \longmapsto i_X\omega$ such that $\left(\omega^{\flat}(X)\right)(Y) = \omega(X, Y)$ is an isomorphism of $C^{\infty}(M)$-modules, for all $Y \in \mathfrak{X}(M)$. Thus, for all $f \in C^{\infty}(M)$, since $df + f\alpha \in \Lambda^1(M)$, there exists a unique vector field $X_f \in \mathfrak{X}(M)$ such that

$$\omega^{\flat}(X_f) = i_{X_f}\omega = df + f\alpha$$

thas is

$$\begin{aligned}(i_{X_f}\omega)(Y) = \omega(X_f, Y) &= (df)(Y) + f\cdot\alpha(Y) \\ &= Y(f) + f\cdot\alpha(Y)\end{aligned} \tag{5.2}$$

for all $Y \in \mathfrak{X}(M)$.

$$\omega(X, X_f) = -X(f) - f\cdot\alpha(X) \tag{5.3}$$

We consider the bracket $\{,\} : C^{\infty}(M) \times C^{\infty}(M) \longrightarrow C^{\infty}(M)$ such that

$$\{f, g\} = -X_g(f) - f\cdot\alpha(X_g) \tag{5.4}$$

for all $f, g \in C^{\infty}(M)$. For any X, X_f and X_g in $\in \mathfrak{X}(M)$ with f and g in $C^{\infty}(M)$, the relation

$$d\omega(X, X_f, X_g) = -\alpha \wedge \omega(X, X_f, X_g)$$

becomes

$$\{f\cdot\alpha(X), g\} + \{f, g\cdot\alpha(X)\} = 2[X(\{f,g\}) + \{f,g\}\cdot\alpha(X) - \{X(f), g\} - \{f, X(g)\}]. \tag{5.5}$$

Lemma 5.1. *If (M, α, ω) is a locally conformal symplectic manifold. Then for any $f \in C^{\infty}(M)$, the map $\varphi_f : C^{\infty}(M) \longrightarrow C^{\infty}(M)$ such that*

$$\varphi_f(g) = \{f, g\} = -\omega(X_f, X_g) = -X_g(f) - f\cdot\alpha(X_g) \tag{5.6}$$

is a first order differential operator for all $g \in C^{\infty}(M)$.

Proof. For any $f, g, h \in C^\infty(M)$, we have
$$\{1_{C^\infty(M)}, f\} = -\alpha(X_f). \tag{5.7}$$
and
$$\{f, g \cdot h\} = -\{g \cdot h, f\} = X_f(g \cdot h) + g \cdot h \cdot \alpha(X_f).$$
Since X_f is a derivation, by using (5.6) and (5.7), we get
$$\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\} - gh \cdot \left\{f, 1_{C^\infty(M)}\right\}. \tag{5.8}$$
Hence, φ_f is a first order differential operator.

□

For any X_f, X_g and X_h in $\mathfrak{X}(M)$, with f, g and $h \in C^\infty(M)$, we have
$$\alpha\left[X_f, X_g\right] = X_f\alpha(X_g) - X_g\alpha(X_f).$$
Since α is closed est, using the Cartan formula, we have
$$\mathfrak{L}_X\alpha(X_f) = i_X d\alpha(X_f) + d i_X\alpha(X_f) = d\alpha(X, X_f) = 0.$$

Lemma 5.2. *If (M, α, ω) is a locally conformal symplectic manifold. Then for all $f, g, h \in C^\infty(M)$, we have*
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0. \tag{5.9}$$

Proof. A direct calculation gives
$$\begin{aligned} d\omega\left(X_f, X_g, X_h\right) = &(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}) + \{g, h\}\,\alpha(X_f) \\ &- \{f, g\}\,\alpha(X_h) + \{f, g\}\,\alpha(X_h), \end{aligned}$$
and
$$-(\alpha \wedge \omega)\left(X_f, X_g, X_h\right) = \alpha(X_f)\{g, h\} - \alpha(X_g)\{f, h\} + \alpha(X_h)\{f, g\}.$$
By (5.1), we get
$$\begin{aligned} &\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} + \{g, h\} \cdot \alpha(X_f) - \{f, h\} \cdot \alpha(X_g) \\ &\quad + \{f, g\} \cdot \alpha(X_h) = \alpha(X_f) \cdot \{g, h\} - \alpha(X_g) \cdot \{f, h\} + \alpha(X_h) \cdot \{f, g\}, \end{aligned}$$
that is
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

□

Theorem 5.1. *All locally conformal symplectic manifold is a Jacobi manifold.*

Proof. From the Lemma 5.2, we deduce that the pair $(C^\infty(M), \{,\})$ is a Lie algebra and from, the Lemma 5.1, the map $ad(f)$ is a first order differential operator for any $f \in C^\infty(M)$. Therefore, M is a Jacobi manifold. □

Proposition 5.1. *For all $f, g \in C^\infty(M)$, we have*
$$\alpha(X_{\{f,g\}}) = \{f, \alpha(X_g)\} - \{g, \alpha(X_f)\}. \tag{5.10}$$

Proof. From (5.7) and (5.9), we have
$$\begin{aligned} \alpha(X_{\{f,g\}}) &= -\{1_{C^\infty(M)}, \{f, g\}\} \\ &= \{f, \alpha(X_g)\} - \{g, \alpha(X_f)\}. \end{aligned}$$

□

Proposition 5.2. *The map $C^{\infty}(M) \longrightarrow Der_{\mathbb{R}}\left[(C^{\infty}(M)\right], f \longmapsto X_f$ is a first order differential operator. Moreover, for all $f, g \in C^{\infty}(M)$, we have*

$$[X_f, X_g] = X_{\{f,g\}}.$$

Proof. For all $f, g, h \in C^{\infty}(M)$, we have

$$X_{f\cdot g}(h) = -\{h, f\cdot g\} - h\cdot\alpha(X_{f\cdot g}).$$

Since $\alpha(X_{f\cdot g}) = f\cdot\alpha(X_g) + g\cdot\alpha(X_f)$ and by using (5.8), we get

$$X_{f\cdot g}(h) = (f\cdot X_g + g\cdot X_f - fg\cdot X_{1_{C^{\infty}(M)}})(h)$$

for all $h \in C^{\infty}(M)$. Thus, we have

$$X_{f\cdot g} = f\cdot X_g + g\cdot X_f - f\cdot g\cdot X_{1_{C^{\infty}(M)}}. \tag{5.11}$$

For all $f, g, h \in C^{\infty}(M)$, we have

$$\begin{aligned}([X_f, X_g] - X_{\{f,g\}})(h) &= [X_f, X_g](h) - X_{\{f,g\}}(h)\\ &= (X_f\circ X_g - X_g\circ X_f)(h) + \{h, \{f,g\}\} + h\cdot\alpha(X_{\{f,g\}})\end{aligned}$$

From (5.10),

$$\begin{aligned}([X_f, X_g] - X_{\{f,g\}})(h) &= X_f[X_g(h)] - X_g[X_f(h)] + \{h, \{f,g\}\}\\ &\quad + h\cdot[\{f, \alpha(X_g)\} - \{g, \alpha(X_f)\}]\end{aligned}$$

i.e.,

$$\begin{aligned}([X_f, X_g] - X_{\{f,g\}})(h) &= X_f[-\{h,g\} - h\cdot\alpha(X_g)] - X_g[-\{h,f\} - h\cdot\alpha(X_f)]\\ &\quad + \{h, \{f,g\}\} + h\cdot\{f, \alpha(X_g)\} - h\cdot\{g, \alpha(X_f)\}\end{aligned}$$

By a direct computations and using (5.4), we find $([X_f, X_g] - X_{\{f,g\}})(h) = 0$, for all $f, g, h \in C^{\infty}(M)$. Thus, $[X_f, X_g] = X_{\{f,g\}}$. □

When M is a smooth manifold, α a diffential 1-form on M, for any $X \in \mathfrak{X}(M)$, the map

$$\rho_\alpha(X) : C^{\infty}(M) \longrightarrow C^{\infty}(M)$$

such that

$$[\rho_\alpha(X)](f) = X(f) + f\cdot\alpha(X) \tag{5.12}$$

is a first order differential operator for all $f \in C^{\infty}(M)$ and the map

$$\rho_\alpha : \mathfrak{X}(M) \longrightarrow Diff_{\mathbb{R}}(C^{\infty}(M)), X \longmapsto \rho_\alpha(X)$$

is a morphism of $C^{\infty}(M)$-modules. Moreover,

$$[\rho_\alpha(X)](f) - f[\rho_\alpha(X)](1_{C^{\infty}(M)}) = X(f).$$

Proposition 5.3. *If (M, α, ω) is a locally conformal symplectic manifold, then the map*

$$\rho_\alpha : \mathfrak{X}(M) \longrightarrow Diff_{\mathbb{R}}[C^{\infty}(M)], X \longmapsto \rho_\alpha(X)$$

is a Lie algebras homomorphism.

Proof. By a direct computations and using (5.12), we find,

$$\begin{aligned}(\rho_\alpha([X,Y]) - [\rho_\alpha(X),\rho_\alpha(Y)])(f) &= f\cdot[X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])]\\ &= f\cdot(d\alpha)(X,Y),\end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$ and for any $f \in C^\infty(M)$. Since α is d-closed i.e., $d\alpha = 0$, then

$$\rho_\alpha([X,Y]) = [\rho_\alpha(X),\rho_\alpha(Y)].$$

□

Let $d_{\rho_\alpha} : \mathfrak{L}^p_{sks}(\mathfrak{X}(M), C^\infty(M)) \longrightarrow \mathfrak{L}^{p+1}_{sks}(\mathfrak{X}(M), C^\infty(M))$ be the cohomology operator associated with the representation ρ_α defined for $\eta \in \mathfrak{L}^p_{sks}(\mathfrak{X}(M), C^\infty(M))$ and $X_1, ..., X_{p+1} \in \mathfrak{X}(M)$ by

$$\begin{aligned}(d_{\rho_\alpha}\eta)(X_1,...,X_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i-1}\rho_\alpha(X_i)[\eta(X_1,...,\hat{X}_i,...,X_{p+1})]\\ &\quad + \sum_{1\le i<j\le p+1}(-1)^{i+j}\eta([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...,X_{p+1}).\end{aligned}$$

For all differential form η on M, we have

$$d_{\rho_\alpha}\eta = d\eta + \alpha\wedge\eta \text{ and } d_{\rho_\alpha}(1_{C^\infty(M)}) = \alpha.$$

Definition 5.1. *Let (M, α, ω) be a locally conformal symplectic manifold. We said that a vector field X on M is locally hamiltonian if the form $i_X\omega$ is closed for the cohomology associated with the representation ρ_α i.e., $d_{\rho_\alpha}(i_X\omega) = 0$.*

Proposition 5.4. *Il $X \in \mathfrak{X}(M)$ is locally hamiltonian, then $\rho_\alpha(X)$ is a derivation of Jacobi algebra $(C^\infty(M), \{,\})$ i.e., for all $f, g \in C^\infty(M)$,*

$$\rho_\alpha(X)(\{f,g\}) = \{\rho_\alpha(X)(f), g\} + \{f, \rho_\alpha(X)(g)\}. \tag{5.13}$$

Proof. If $d_{\rho_\alpha}(i_X\omega) = 0$, then for any $Y, Z \in \mathfrak{X}(M)$, $d_{\rho_\alpha}i_X\omega\,(Y,Z) = 0$. In particular for any $X_f, X_g \in \mathfrak{X}(M)$ with $f, g \in C^\infty(M)$, we have

$$\begin{aligned}0 &= d_{\rho_\alpha}i_X\omega\left(X_f,X_g\right)\\ &= \rho_\alpha(X_f)[i_X\omega(X_g)] - \rho_\alpha(X_g)[i_X\omega(X_f)] - i_X\omega\left([X_f,X_g]\right)\end{aligned}$$

From (5.3) and (5.12), we get

$$\begin{aligned}0 &= -\{X(f),g\} - \{f\cdot\alpha(X),g\} - \{f,X(g)\}\\ &\quad -\{f,g\cdot\alpha(X)\} + X(\{f,g\}) + \{f,g\}\cdot\alpha(X)\end{aligned}$$

that is

$$X(\{f,g\}) + \{f,g\}\cdot\alpha(X) = \{X(f) + f\cdot\alpha(X), g\} + \{f, X(g) + g\cdot\alpha(X)\}.$$

Therefore,

$$\rho_\alpha(X)(\{f,g\}) = \{\rho_\alpha(X)(f), g\} + \{f, \rho_\alpha(X)(g)\}.$$

□

Proposition 5.5. *If $X \in \mathfrak{X}(M)$ is locally hamiltonian vector field on a locally conformal symplectic manifold (M, α, ω), then*

$$X(\{f,g\}) = \{X(f), g\} + \{f, X(g)\} - \{f,g\}\cdot\alpha(X). \tag{5.14}$$

Proof. From (5.13), we have

$$X(\{f,g\}) + \{f,g\} \cdot \alpha(X) = \{X(f) + f \cdot \alpha(X), g\} + \{f, X(g) + g \cdot \alpha(X)\}.$$

This means that

$$\{f \cdot \alpha(X), g\} + \{f, g \cdot \alpha(X)\} = X(\{f,g\}) - \{X(f), g\} - \{f, X(g)\} + \{f,g\} \cdot \alpha(X)$$

By (5.5), we obtain

$$X(\{f,g\}) = \{X(f), g\} + \{f, X(g)\} - \{f,g\} \cdot \alpha(X).$$

□

When $\alpha = 0$, then $\rho_\alpha = id_{\mathfrak{X}(M)}$ and (M, ω) is a symplectic manifold. In this case, a vector field X on M is locally hamiltonian if the form $i_X\omega$ is closed for the de Rham cohomology i.e., X is a derivation of the Lie algebra $C^\infty(M)$ induced by the structure of Poisson defined by the symplectic manifold.

Proposition 5.6. *For any $f \in C^\infty(M), X \in \mathfrak{X}(M)$, we have*
i)

$$\{\alpha(X), f\} = 0. \tag{5.15}$$

ii)

$$\alpha(X) \cdot \alpha(X_f) = 0. \tag{5.16}$$

Proof. i) Since α is closed, we have

$$0 = d\alpha(X, X_f) = X[\alpha(X_f)] - X_f[\alpha(X)] - \alpha([X, X_f]).$$

Using (5.6) and (5.7), we get

$$X(-\{1_{C^\infty(M)}, f\}) + \{\alpha(X), f\} + \alpha(X) \cdot \alpha(X_f) - \alpha(X_{X(f)}) = 0$$

i.e., by (5.14), we have

$$\begin{aligned} &-\{X(1_{C^\infty(M)}), f\} - \{1_{C^\infty(M)}, X(f)\} + \{1_{C^\infty(M)}), f\} \cdot \alpha(X) \\ &+\{\alpha(X), f\} - \{1_{C^\infty(M)}, f\} \cdot \alpha(X) + \{1_{C^\infty(M)}, X(f)\} = 0. \end{aligned}$$

Hence, $\{\alpha(X), f\} = 0$.
ii)

$$0 = \mathfrak{L}_X\alpha(X_f) = X[\alpha(X_f)] - \alpha(\mathfrak{L}_X(X_f)).$$

By (5.6) and by a direct computations, we obtain $\alpha(X) \cdot \alpha(X_f) = 0$. □

Proposition 5.7. *For any $f \in C^\infty(M)$, we have $[\rho_\alpha(X), X_f] = X_{\rho_\alpha(X)(f)}$.*

Proof. For any $g \in C^\infty(M)$, we have

$$\begin{aligned} [\rho_\alpha(X), X_f](g) &= \rho_\alpha(X)[X_f(g)] - X_f[\rho_\alpha(X)(g)] \\ &= \rho_\alpha(X)[-\{g,f\} - g \cdot \alpha(X_f)] + \{\rho_\alpha(X)(g), f\} + \rho_\alpha(X)(g) \cdot \alpha(X_f) \\ &= \rho_\alpha(X)(\{f,g\}) - \rho_\alpha(X)[g \cdot \alpha(X_f)] \\ &\quad -\{f, \rho_\alpha(X)(g)\} + \rho_\alpha(X)(g) \cdot \alpha(X_f) \end{aligned}$$

Using (5.6), (5.7) and (5.13), we get

$$\begin{aligned} [\rho_\alpha(X), X_f](g) &= \{\rho_\alpha(X)(f), g\} + g \cdot \{\rho_\alpha(X)(1_{C^\infty(M)}), f\}) \\ &\quad +g \cdot \{1_{C^\infty(M)}, \rho_\alpha(X)(f)\} + g \cdot \alpha(X_f)\alpha(X), \end{aligned}$$

that is

$$
\begin{aligned}
[\rho_\alpha(X), X_f](g) &= -\{g, \rho_\alpha(X)(f)\} - g\cdot\alpha(X_{\rho_\alpha(X)(f)}) \\
&\quad + g\cdot\{\alpha(X), f\} + g\cdot\alpha(X_f)\alpha(X).
\end{aligned}
$$

From (5.15) and (5.16), we get

$$
\begin{aligned}
[\rho_\alpha(X), X_f](g) &= -\{g, \rho_\alpha(X)(f)\} - g\cdot\alpha(X_{\rho_\alpha(X)(f)}) \\
&= X_{\rho_\alpha(X)(f)}(g)
\end{aligned}
$$

for any $g \in C^\infty(M)$. □

Proposition 5.8. *$\mathfrak{L}_{\rho_\alpha(X)}\omega = 0$ if and only if $\rho_\alpha(X)$ is a derivation of Jacobi algebra $(C^\infty(M), \{,\})$.*

Proof. $\mathfrak{L}_{\rho_\alpha(X)}\omega = 0$ if and only if, for $f, g \in C^\infty(M)$,

$$
\begin{aligned}
0 &= \mathfrak{L}_{\rho_\alpha(X)}\omega(X_f, X_g) \\
&= \rho_\alpha(X)[\omega(X_f, X_g)] - \omega(\mathfrak{L}_{\rho_\alpha(X)}X_f, X_g) - \omega(X_f, \mathfrak{L}_{\rho_\alpha(X)}X_g) \\
&= \rho_\alpha(X)(\{f,g\}) - \omega([\rho_\alpha(X), X_f], X_g) - \omega(X_f, [\rho_\alpha(X), X_g]).
\end{aligned}
$$

Since $[\rho_\alpha(X), X_f] = X_{\rho_\alpha(X)(f)}$, we have

$$
\begin{aligned}
\rho_\alpha(X)(\{f,g\}) &= \omega(X_{\rho_\alpha(X)(f)}, X_g) + \omega(X_f, X_{\rho_\alpha(X)(g)}) \\
&= \{\rho_\alpha(X)(f), g\} + \{f, \rho_\alpha(X)(g)\}.
\end{aligned}
$$

□

Definition 5.2. *A vector field X on a locally conformal symplectic manifold (M, α, ω) is globally hamiltonian if the 1-form $i_X\omega$ is d_{ρ_α}-exact i.e., if there exists $f \in C^\infty(M)$ such that $i_X\omega = d_{\rho_\alpha} f$.*

Proposition 5.9. *If X is a globally hamiltonian vector field on (M, α, ω), then $\rho_\alpha(X)$ is a derivation interior of Jacobi algebra $(C^\infty(M)), \{,\})$.*

Proof. If $i_X\omega = d_{\rho_\alpha} f$, then for any $Y \in \mathfrak{X}(M)$, $i_X\omega(Y) = d_{\rho_\alpha} f(Y)$. In particular for any $X_g \in \mathfrak{X}(M)$ with $g \in C^\infty(M)$, $i_X\omega(X_g) = d_{\rho_\alpha} f(X_g)$ i.e.,

$$
\omega(X, X_g) = [\rho_\alpha(X_g)](f) = X_g(f) + f\cdot\alpha(X_g)
$$

i.e.,

$$
\begin{aligned}
-X(g) - g\cdot\alpha(X) &= X_g(f) + f\cdot\alpha(X_g) \\
&= -\{f, g\}
\end{aligned}
$$

$$
X(g) + g\cdot\alpha(X) = \{f, g\},
$$

that is,

$$
\rho_\alpha(X)(g) = \{f, g\} = ad(f)(g)
$$

for any $g \in C^\infty(M)$. Thus, $\rho_\alpha(X) = ad(f)$. □

If $\alpha = 0$, then $\rho_\alpha(X) = X$ and we recover the notion of a globally hamiltonian vector field on a symplectic manifold (M, ω).

Proposition 5.10. *If X is a globally hamiltonian vector field on a locally conformal symplectic manifold, then $\mathfrak{L}_{\rho_\alpha(X)}\omega_M = 0$.*

Proof. Let X be a globally hamiltonian vector field on M, then there exists $f \in C^\infty(M)$ such that $\rho_\alpha(X) = ad(f)$. For any $g, h \in C^\infty(M)$,

$$\begin{aligned}\left(\mathfrak{L}_{\rho_\alpha(X)}\omega_M\right)(\Delta_M(g), \Delta_M(h)) &= (\mathfrak{L}_{ad(f)}\omega_M)(\Delta_M(g), \Delta_M(h)) \\ &= ad(f)(\omega_M(\Delta_M(g), \Delta_M(h))) \\ &\quad -(\omega_M\left(\mathfrak{L}_{ad(f)}\Delta_M(g), \Delta_M(h)\right)) \\ &\quad -(\omega_M\left(\Delta_M(g), \mathfrak{L}_{ad(f)}\Delta_M(h)\right)).\end{aligned}$$

By (2.5), we have

$$\begin{aligned}\left(\mathfrak{L}_{\rho_\alpha(X)}\omega_M\right)(\Delta_M(g), \Delta_M(h)) &= ad(f)(\{g,h\}) - \omega_M(\Delta_M(\{f,g\}), \Delta_M(h)) \\ &\quad -\omega_M(\Delta_M(g), \Delta_M(\{f,h\})) \\ &= \{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} \\ &= 0.\end{aligned}$$

Therefore, $\rho_\alpha(X)$ is a first order differntial operator of Jacobi. □

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