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THE GEOMETRY OF $\langle\xi\rangle$-ONE VARIABLE COMPLEX FUNCTIONS IN THE SPACE $FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C})$

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1. Preliminaries and results. Essential elements and notions of the theory of one variable complex $\langle\xi\rangle$-functions in the space $FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C})$, definitions (Cojan), definitions and remarks. $\langle\xi\rangle$-Cauchy-Riemann conditions, a property of a $\langle\xi\rangle$-function which is $\langle\xi\rangle$-holomorphic on a rectangle, representation $\langle\xi\rangle$-formulas of $\langle\xi\rangle$-functions which are $\langle\xi\rangle$-polygenic in the sense of Călugăreanu, a result of $\prec \langle\xi\rangle$-subordination Miller-Mocanu-Robertson and a corollary for the Călugăreanu-Cojan $\langle\xi\rangle$-operators.

2. Definitions 1 (Cojan). Consider the class of semi-discontinuous functions

$$F_d(\mathbb{R}^2, td) = \{\Omega : \Omega \in C^0(D_1 \cup D_2, \mathbb{R}^2)\},$$

for every

$$(x, y) \in D_1 \cup D_2 \subset Cl(D_1) \cup Cl(D_2) \subset \mathbb{R}^2, \Omega(x, y) \in \mathbb{R}^2,$$

where the domains D_1, D_2 satisfy $D_1 \cap D_2 = \emptyset, Fr(D_1) \cap Fr(D_2) \neq \emptyset$, and we use the canonical identifications

$$(x, y) = x + iy = z.$$

Let $F^{-1}(\mathbb{R}^2, td)$ be the class of invertible functions. Assume that

$$\Psi : \mathbb{R}^2 \to \mathbb{R}, \Psi \in C^1(D, \mathbb{R}^2), \text{ for every } (x, y) \in D \subset Cl(D) \subset \mathbb{R}^2, \Psi(x, y) \in \mathbb{R}.$$

Consider the following conditions :
1) If $z_0 := x_0 + iy_0 = (x_0, y_0) \in U_0 \times V_0 \subset U \times V \subset D$,
$\psi(x_0, y_0) = 0$, where $U_0, V_0, U, V, D, U_0 \times V_0, U \times V$ are domains, then

$$\frac{\partial \psi(x_0, y_0)}{\partial y} \neq 0;$$

2) There exists an open neighborhood $(x_0, y_0) \in U_0 \times V_0$ and a unique function

$$y = f : \mathbb{R} \to \mathbb{R}, f \in C^1(U, \mathbb{R}), \text{ for every } x \in U_0 \subset U \subset \mathbb{R}, f(x) \in \mathbb{R}$$

for which

$$y_0 = f(x_0), \ \forall\ x \in U_0,\ \psi(x, f(x)) = 0.$$

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That is there exists

$$\frac{df}{dx} = -\frac{\partial\psi}{\partial x} : \frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{df}{dx} \in C^0(D, \mathbb{R}).$$

The partial derivatives of ψ and the derivative of f are continuous on D ([3], page 16). If the function $y = f(x)$ satisfies the above conditions, then it is called invertible. In this case

$$\emptyset \neq F_d(\mathbb{R}^2, td) \cap F^{-1}(\mathbb{R}^2, td) =: F\langle\xi\rangle(\mathbb{C}, td\mathbb{C})$$

is the class of one complex variable $\langle\xi\rangle$-functions which are $\langle\xi\rangle$-discontinuous, $td\mathbb{C}$ is the topology on $\mathbb{C}$ and td is the topology on $\mathbb{R}^2$, different from the trivial topology and the topology consisting in all subsets of $\mathbb{C} = (\mathbb{R}^2, +, \cdot)$.

3. Definitions and remarks 2. We will use the abreviations Cl for the closure of a set and Fr for the boundary of a set.
Let $f\langle\xi\rangle : \mathbb{C} \to \mathbb{C}, f\langle\xi\rangle \in F\langle\xi\rangle(\mathbb{C}, td\mathbb{C})$. If for every $z_1, z_2 \in \mathbb{C}$ i.e. $z_1 \neq z_2 \Rightarrow f\langle\xi\rangle(z_1) \neq f\langle\xi\rangle(z_2)$, then $< \xi >$-function $f\langle\xi\rangle$ is $\langle\xi\rangle$-injective. If for every $y\langle\xi\rangle \in \mathbb{C}$ there exists at least one $z,\ z \in \mathbb{C}$, that is $y\langle\xi\rangle = f\langle\xi\rangle(z)$, then the $< \xi >$-function $f\langle\xi\rangle$ is $\langle\xi\rangle$-surjective. If the $< \xi >$-function is $\langle\xi\rangle$-injective and $\langle\xi\rangle$-surjective, then it is called $\langle\xi\rangle$-bijective.
A complex $\langle\xi\rangle$-discontinuous $\langle\xi\rangle$-function $f\langle\xi\rangle(z) \in F\langle\xi\rangle(\mathbb{C}, td\mathbb{C})$ is $\langle\xi\rangle$-holomorphic at point $z_0 \in U$, where U is a domain, $f\langle\xi\rangle : \mathbb{C} \to \mathbb{C}, f\langle\xi\rangle \in F\langle\xi\rangle(z_0, td\mathbb{C})$. for every $z \in D_1 \cup D_2 \subset U \subset \mathbb{C}$, if there exists the limit

$$\lim_{\substack{z \to z_0, \\ z \neq z_0}} \frac{f\langle\xi\rangle(z) - f\langle\xi\rangle(z_0)}{z - z_0} = \frac{d\langle\xi\rangle f\langle\xi\rangle(z_0)}{d\langle\xi\rangle z}$$

$$= \frac{d\langle\xi\rangle f\langle\xi\rangle}{d\langle\xi\rangle z} = \frac{dg}{dz} + (1+z)^{-1} S(f\langle\xi\rangle, \xi),\ z \neq -1.$$

The *oscillation* of $f\langle\xi\rangle$ at ξ is defined as

$$S(f\langle\xi\rangle, \xi) = \lim_{\substack{z \to \xi, \\ z \in Cl(D_1)}} f\langle\xi\rangle(z) - \lim_{\substack{z \to \xi, \\ z \in Cl(D_2)}} f\langle\xi\rangle(z) \neq 0.$$

The domains D_1 and D_2 satisfy $D_1 \cap D_2 = \emptyset$, that is we have

$$Fr(D_1) \cap Fr(D_2) = \{\xi\} \neq \emptyset,\ z \in V(\xi) \cap (D_1 \cup D_2) \neq \emptyset,$$

where $V(\xi)$ is a neighborhood of the $\langle\xi\rangle$-discontinuity point $\xi \in U$ and $g \in \lambda(U, td\mathbb{C})$, the set of functions which are holomorphic on U. Consider $H\langle\xi\rangle(U, td\mathbb{C})$ the set of $\langle\xi\rangle$-functions which are $\langle\xi\rangle$-holomorphic on U and denote

$$F\langle\xi\rangle(\mathbb{C}, td\mathbb{C}) \cup F\langle\xi\rangle(z_0, td\mathbb{C}) \cup H\langle\xi\rangle(U, td\mathbb{C}) = FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C})$$

the space of $\langle\xi\rangle$ - one variable complex functions or the set of $\langle\xi\rangle$-functions which are $\langle\xi\rangle$-holomorphic on $\mathbb{C}$, $FH\langle\xi\rangle(\mathbb{R}^2, td)$ the set of $\langle\xi\rangle$-functions which are $\langle\xi\rangle$-holomorphic on $\mathbb{R}^2$, and $F\langle\xi\rangle(z_0, td\mathbb{C})$ the set of $\langle\xi\rangle$-functions $\langle\xi\rangle$-holomorphic at the point $z_0 \in U$.
If $f\langle\xi\rangle, f\langle\xi\rangle \in FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C})$, then $f\langle\xi\rangle \pm g\langle\xi\rangle$, $f\langle\xi\rangle \cdot g\langle\xi\rangle$, $\dfrac{f\langle\xi\rangle}{g\langle\xi\rangle}$, the $\langle\xi\rangle$-composition defined by

$$g\langle\xi\rangle(f\langle\xi\rangle(z)) = (g\langle\xi\rangle \circ f\langle\xi\rangle)(z),$$

and $f^{-1}\langle\xi\rangle$ are $\langle\xi\rangle$-holomorphic $\langle\xi\rangle$-functions, and we have

$$\frac{d\langle\xi\rangle(f\langle\xi\rangle \pm g\langle\xi\rangle)}{d\langle\xi\rangle z} = \frac{d\langle\xi\rangle f\langle\xi\rangle}{d\langle\xi\rangle z} \pm \frac{d\langle\xi\rangle g\langle\xi\rangle}{d\langle\xi\rangle z},$$

$$\frac{d\langle\xi\rangle(\langle\xi\rangle \cdot g\langle\xi\rangle)}{d\langle\xi\rangle z} = \frac{d\langle\xi\rangle f\langle\xi\rangle}{d\langle\xi\rangle z} \cdot g\langle\xi\rangle + f\langle\xi\rangle \cdot \frac{d\langle\xi\rangle g\langle\xi\rangle}{d\langle\xi\rangle z},$$

$$\frac{d\langle\xi\rangle(g\langle\xi\rangle \circ f\langle\xi\rangle)}{d\langle\xi\rangle z} = \frac{d\langle\xi\rangle g\langle\xi\rangle}{d\langle\xi\rangle f\langle\xi\rangle} \cdot \frac{d\langle\xi\rangle f\langle\xi\rangle}{d\langle\xi\rangle z}.$$

If $f\langle\xi\rangle$ is a $\langle\xi\rangle$-bijection and $f\langle\xi\rangle : U \to V$ is $\langle\xi\rangle$-holomorphic at the point $z_0 \in U$ with

$$\frac{d\langle\xi\rangle f\langle\xi\rangle(z_0)}{d\langle\xi\rangle z} \neq 0,$$

then $f^{-1}\langle\xi\rangle$ is $\langle\xi\rangle$-holomorphic at the point $w_0 = f\langle\xi\rangle(z_0)$, that is

$$\frac{d\langle\xi\rangle f^{-1}\langle\xi\rangle(w_0)}{d\langle\xi\rangle z} = \left(\frac{d\langle\xi\rangle f\langle\xi\rangle(z_0)}{d\langle\xi\rangle z}\right)^{-1}.$$

The $\langle\xi\rangle$-function

$$FH\langle\xi\rangle(\mathbb{R}^2, td) \ni \exists\, u\langle\xi\rangle : \mathbb{R}^2 \supset N \to \mathbb{R}^2$$

is $\langle\xi\rangle$-differentiable at the point $(x_0, y_0) \in N$ if for every

$$V(x_0, y_0) \cap N \neq \emptyset,$$

we have

$$\Delta u\langle\xi\rangle = \frac{\partial\langle\xi\rangle u\langle\xi\rangle}{\partial\langle\xi\rangle x}\Delta x + \frac{\partial\langle\xi\rangle u\langle\xi\rangle}{\partial\langle\xi\rangle y}\Delta y + \beta.$$

Let

$$\beta = \beta_1\Delta x + \beta_2\Delta y,\ \beta_1 \to 0,\ \beta_2 \to 0$$

for $\Delta x \to 0$, $\Delta y \to 0$ and

$$d\langle\xi\rangle u\langle\xi\rangle = \frac{\partial\langle\xi\rangle u\langle\xi\rangle}{\partial\langle\xi\rangle x}d\langle\xi\rangle x + \frac{\partial\langle\xi\rangle u\langle\xi\rangle}{\partial\langle\xi\rangle y}d\langle\xi\rangle y$$

is called the $\langle\xi\rangle$-differential of the $\langle\xi\rangle$-function at (x_0, y_0).

4. Theorem 1. The $\langle\xi\rangle$-function $f\langle\xi\rangle(z) = u\langle\xi\rangle(x,y) + iv\langle\xi\rangle(x,y) \in FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C})$ is $\langle\xi\rangle$-holomorphic at the point $z_0 = (x_0, y_0)$ if and only if the $\langle\xi\rangle$-functions $u\langle\xi\rangle(x,y), v\langle\xi\rangle(x,y) \in FH\langle\xi\rangle(\mathbb{R}^2, td\mathbb{C})$ are $\langle\xi\rangle$-differentiable at the fixed point (x_0, y_0) and satisfy the $\langle\xi\rangle$- Cauchy-Riemann conditions

$$\frac{\partial\langle\xi\rangle u\langle\xi\rangle}{\partial\langle\xi\rangle x} = \frac{\partial\langle\xi\rangle v\langle\xi\rangle}{\partial\langle\xi\rangle y},\ \frac{\partial\langle\xi\rangle u\langle\xi\rangle}{\partial\langle\xi\rangle y} = -\frac{\partial\langle\xi\rangle v\langle\xi\rangle}{\partial\langle\xi\rangle x}.$$

Proof. The limit

$$\frac{d\langle\xi\rangle f\langle\xi\rangle(z_0)}{d\langle\xi\rangle z} = \lim_{\substack{z\to z_0,\\ z\neq z_0}} \frac{f\langle\xi\rangle(z) - f\langle\xi\rangle(z_0)}{z - z_0}$$

$$= \lim_{\Delta z\to 0} \frac{\Delta f\langle\xi\rangle(z)}{\Delta z} = A\langle\xi\rangle + ib\langle\xi\rangle$$

is finite if and only if

$$(a) \qquad \frac{\Delta f\langle\xi\rangle(z)}{\Delta z} = A\langle\xi\rangle + iB\langle\xi\rangle + \varepsilon,\ \Delta z \neq 0,$$

where $\varepsilon \to 0$ for $\Delta z \to 0$. If $\varepsilon = \varepsilon_1 + i\varepsilon_2$, then $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ when $\Delta x \to 0$, $\Delta y \to 0$. If we identify the real and the imaginary part in (a), then

$$\Delta u\langle\xi\rangle = A\langle\xi\rangle\Delta x - B\langle\xi\rangle\Delta y + \varepsilon_1\Delta x - \varepsilon_2\Delta y,$$

$$\Delta v\langle\xi\rangle = B\langle\xi\rangle\Delta x + A\langle\xi\rangle\Delta y + \varepsilon_2\Delta x + \varepsilon_1\Delta y,$$

$$\Delta f\langle\xi\rangle = \Delta u\langle\xi\rangle + i\Delta v$$

$$\Leftrightarrow A\langle\xi\rangle = \frac{\partial\langle\xi\rangle u\langle\xi\rangle}{\partial\langle\xi\rangle x} = \frac{\partial\langle\xi\rangle v\langle\xi\rangle}{\partial\langle\xi\rangle y},$$

$$B\langle\xi\rangle = -\frac{\partial\langle\xi\rangle u\langle\xi\rangle}{\partial\langle\xi\rangle u} = \frac{\partial\langle\xi\rangle v\langle\xi\rangle}{\partial\langle\xi\rangle x}.$$

$$FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C}) \ni \exists\, f\langle\xi\rangle :\ \mathbb{C} \supset U \supset D_1 \cup D_2 \to \mathbb{C},$$

$$\frac{d\langle\xi\rangle f\langle\xi\rangle}{d\langle\xi\rangle z} = \frac{dg}{dz} + (1+z)^{-1}S(f\langle\xi\rangle, \xi),\ z \neq -1$$

with

$$S(f\langle\xi\rangle, \xi) := \lim_{\substack{z\to\xi,\\ z\in Cl(D_1)}} f\langle\xi\rangle(z) - \lim_{\substack{z\to\xi,\\ z\in Cl(D_2)}} f\langle\xi\rangle(z) \neq 0.$$

$$g(z) \in \lambda(U, td\mathbb{C}),$$

where U is a domain (Definitions and Remarks 2). We have the domains $U \supset D_1 \cap D_2 = \emptyset$, with the property

$$Fr(D_1) \cap Fr(D_2) = \{\xi\} \neq \emptyset,$$
$$z \in V(\xi) \cap (D_1 \cup D_2) \neq \emptyset. \qquad \square$$

5. Theorem 2. ($I\langle\xi\rangle(R)$-Cauchy). Let U be a domain and,

$$FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C}) \ni \exists\, f\langle\xi\rangle : \mathbb{C}\ \supset U \supset R \to \mathbb{C}.$$

If the $\langle\xi\rangle$-integral of every constant $\langle\xi\rangle$-function and of every $\langle\xi\rangle$-function of the considered space is zero on R_n, $n \geq 1$, then

$$I\langle\xi\rangle(R) = \int_{\partial R} f\langle\xi\rangle(z)d\langle\xi\rangle z = 0,$$

where ∂R is the boundary of the rectangle R.

Proof. Divide the rectangle into four rectangles $R(1)$, $R(2)$, $R(3)$, $R(4)$, such that the $\langle\xi\rangle$-integral acts twice on each interior side with opposite orientation, hence the sum of two such integrals is zero. Then

$$I\langle\xi\rangle(R) = \sum_{j=1}^{4} I\langle\xi\rangle(R(j))$$

and

$$|I\langle\xi\rangle(R(j))| < 4^{-1}|I\langle\xi\rangle(R)|,\ 1 \leq j \leq 4,$$

$$|I\langle\xi\rangle(R)| \leq \sum_{j=1}^{4} |I\langle\xi\rangle(R(j))| < |I\langle\xi\rangle(R)|,$$

which is not possible. Therefore, for at least one rectangle $R(j)$ we have

$$|I\langle\xi\rangle(R(j))| \geq 4^{-1}|I\langle\xi\rangle(R)|,$$

and we denote it by R_1. Consider $R_2 \subset R_1$ with

$$|I\langle\xi\rangle(R_2)| \geq 4^{-1}|I\langle\xi\rangle(R_1)| \geq 4^{-2}|I\langle\xi\rangle(R)|$$

and inductively for $n \geq 1$,

$$R_1 \supset \ldots \supset R_n \supset \ldots$$

with

$$|I\langle\xi\rangle(R_n)| \geq 4^{-n}|I\langle\xi\rangle(R)|.$$

Because the sequence of diameters of R_n is convergent to 0, the intersection $\bigcap_{n=1}^{\infty} R_n$ consists in a point $w \in R$. Because the $\langle\xi\rangle$-function is $\langle\xi\rangle$-holomorphic at w, there exists $\dfrac{d\langle\xi\rangle f\langle\xi\rangle}{d\langle\xi\rangle w}$, that is for every $\varepsilon > 0$ there exists β such that

$$|z - w| < \beta \Rightarrow \left|(f\langle\xi\rangle(z) - f\langle\xi\rangle(w)) - (z - w)\frac{d\langle\xi\rangle f\langle\xi\rangle}{d\langle\xi\rangle w}\right| < |z - w|\varepsilon.$$

We obtain

$$|I\langle\xi\rangle(R_n)| = \left|\int_{\partial R_n} \left[f\langle\xi\rangle(z) - f\langle\xi\rangle(w) - (z - w)\frac{d\langle\xi\rangle f\langle\xi\rangle}{d\langle\xi\rangle w}\right] d\langle\xi\rangle z\right|$$

$$< \varepsilon \int_{\partial R_n} |z - w|\, |d\langle\xi\rangle z|.$$

Considering d_n and P_n the length of the diagonal and the perimeter of R_n, and similarly d and P for R, then it follows that for every $\varepsilon > 0$,

$$|I\langle\xi\rangle(R_n)| \leq 4^{-n}\varepsilon P_n d_n$$
$$\Rightarrow |I\langle\xi\rangle(R)| \leq P\varepsilon d$$
$$\Rightarrow I\langle\xi\rangle(R) = 0.$$

We have

$$\frac{d\langle\xi\rangle f\langle\xi\rangle}{d\langle\xi\rangle z} = \frac{dg}{dz} + (1+z)^{-1}S(f\langle\xi\rangle, \xi),\ z \neq -1,$$

where $g(z) \in \lambda(U, td\mathbb{C})$ and the domains $U \supset U_1 \cup U_2 = \emptyset$, that is

$$Fr(U_1) \cap Fr(U_2) = \{\xi\} \neq \emptyset,$$
$$z \in V(\xi) \cap R \cap (U_1 \cup U_2) \neq \emptyset,$$

with

$$S(f\langle\xi\rangle, \xi) = \lim_{\substack{z\to\xi,\\ z\in Cl(U_1)}} f\langle\xi\rangle(z) - \lim_{\substack{z\to\xi,\\ z\in Cl(U_2)}} f\langle\xi\rangle(z) \neq 0. \qquad \square$$

6. Definition 3. If

(1): $f_1\langle\xi\rangle(0) = f_3\langle\xi\rangle(f_2\langle\xi\rangle(0))$,
(2): $\dfrac{d\langle\xi\rangle f_1\langle\xi\rangle(0)}{d\langle\xi\rangle z} < f_3\langle\xi\rangle(0)$,
the domains $D_k \cap U_k \neq \emptyset$, $E_k \cap U_k \neq \emptyset$, co-domains

$$f_k\langle\xi\rangle(D_k \cap U_k) \neq \emptyset,$$
$$f_k\langle\xi\rangle(E_k \cap U_k) \neq \emptyset,$$
$$f_3\langle\xi\rangle(f_2\langle\xi\rangle(D_2 \cap U_2)) \neq \emptyset,$$
$$f_3\langle\xi\rangle(f_2\langle\xi\rangle(E_2 \cap U_2)) \neq \emptyset$$

and if the co-domain of $f_2\langle\xi\rangle$ is equal to the definition domain of $f_3\langle\xi\rangle$,

$$FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C}) \ni \exists\, f_k\langle\xi\rangle : \mathbb{C} \supset D_k \cap U_k \to \mathbb{C},$$
$$FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C}) \ni \exists\, f_k\langle\xi\rangle : \mathbb{C} \supset E_k \cap U_k \to \mathbb{C},\ 1 \leq k \leq 2,$$
$$FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C}) \ni \exists\, f_3\langle\xi\rangle : \mathbb{C} \supset f_2\langle\xi\rangle(D_2 \cap U_2) \ni f_2\langle\xi\rangle(z) \to \mathbb{C},$$
$$FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C}) \ni \exists\, f_3\langle\xi\rangle : \mathbb{C} \supset f_2\langle\xi\rangle(E_2 \cap U_2) \ni f_2\langle\xi\rangle(z) \to \mathbb{C},$$

and if

$$f_1\langle\xi\rangle(z) = f_3\langle\xi\rangle(f_2\langle\xi\rangle(z)),$$

we say that $f_1\langle\xi\rangle$ is Miller-Mocanu-Robertson $\langle\xi\rangle$-subordonated to the $\langle\xi\rangle$-function $f_3\langle\xi\rangle$, and we denote by $f_1\langle\xi\rangle \prec \langle\xi\rangle f_3\langle\xi\rangle$.

7. Proposition 1. Suppose that the $\langle\xi\rangle$-functions

$$f_1\langle\xi\rangle(z) = f_3\langle\xi\rangle(f_2\langle\xi\rangle(z))$$

satisfy the $\langle\xi\rangle$-conditions in Definition 3 and

$$f_1\langle\xi\rangle(z) = 18z - 12$$

if

$$z \in D_1 \cap U_1,\ f_1\langle\xi\rangle(z) = 2z - 1,$$
$$z \in E_1 \cap U_1,\ f_2\langle\xi\rangle(z) = 3z - 2,$$
$$z \in D_2 \cap U_2,\ f_2\langle\xi\rangle(z) = z - 1,$$
$$z \in E_2 \cap U_2,\ f_3\langle\xi\rangle(z) = 6z,$$
$$f_2\langle\xi\rangle(z) \in f_2(D_2 \cap U_2),$$
$$f_3\langle\xi\rangle(z) = 2z + 1,$$
$$f_2\langle\xi\rangle(z) \in f_2\langle\xi\rangle(E_2 \cap U_2).$$

If

$$\frac{dg_1(0)}{dz} = \frac{dg_2(0)}{dz} = 0,$$
$$g_1(z) \in \lambda(D_1 \cap U_1, td\mathbb{C}),$$
$$g_2(z) \in \lambda(E_1 \cap U_1, td\mathbb{C})$$

and

$$S_1(f_1\langle\xi\rangle, \xi_1) = -2,$$
$$S_2(f_1\langle\xi\rangle, f_1\langle\xi\rangle(\xi_1)) = \frac{1}{2},\ z \neq -1,$$

then

$$f_1\langle\xi\rangle(z) \prec \langle\xi\rangle f_3\langle\xi\rangle(z).$$

Proof. $f_1\langle\xi\rangle(0) = -12$,

$$f_3\langle\xi\rangle(f_2\langle\xi\rangle(0)) = -12,$$
$$f_1\langle\xi\rangle(0) = f_3\langle\xi\rangle(f_2\langle\xi\rangle(0)),$$
$$f_1\langle\xi\rangle(0) = -1,$$
$$f_3\langle\xi\rangle(f_2\langle\xi\rangle(0)) = -1,$$
$$f_1\langle\xi\rangle(0) = f_3\langle\xi\rangle(f_2\langle\xi\rangle(0)).$$

Therefore, the relation (1) in Definition 3 is verified.

$$\frac{d\langle\xi\rangle f_1\langle\xi\rangle}{d\langle\xi\rangle z} = \frac{dg_1}{dz} + (1+z)^{-1}S_1(f_1\langle\xi\rangle, \xi_1),$$
$$\frac{d\langle\xi\rangle f_1\langle\xi\rangle(0)}{d\langle\xi\rangle z} = -2,$$
$$f_3\langle\xi\rangle(0) = 0,$$
$$\frac{d\langle\xi\rangle f_1\langle\xi\rangle(0)}{d\langle\xi\rangle z} < f_3\langle\xi\rangle(0),$$
$$S_1(f_1\langle\xi\rangle, \xi_1) = \lim_{\substack{z\to\xi_1,\\ z\in Cl(D_1\cap U_1)}} f_1\langle\xi\rangle(z) - \lim_{\substack{z\to\xi_1,\\ z\in Cl(E_1\cap U_1)}} f_1\langle\xi\rangle(z) = -2.$$

Consider the domains $(D_1 \cap U_1) \cap (E_1 \cap U_1) = \emptyset$, that is

$$Fr(D_1 \cap U_1) \cap Fr(E_1 \cap U_1) = \{\xi_1\} \neq \emptyset,$$
$$z \in V(\xi_1) \cap (D_1 \cap U_1) \cup (E_1 \cap U_1) \neq \emptyset,$$

$$\frac{d\langle\xi\rangle f_1\langle\xi\rangle}{d\langle\xi\rangle z} = \frac{dg_2}{dz} + (1+z)^{-1}S_2(f_1\langle\xi\rangle, f_1\langle\xi\rangle(\xi_1)),$$
$$\frac{d\langle\xi\rangle f_1\langle\xi\rangle(0)}{d\langle\xi\rangle z} = \frac{1}{2},$$
$$f_3\langle\xi\rangle(0) = 1,$$
$$\frac{d\langle\xi\rangle f_1\langle\xi\rangle(0)}{d\langle\xi\rangle z} < f_3\langle\xi\rangle(0),\ z \in E_1 \cap U_1,$$

and we denote

$$f_1\langle\xi\rangle(D_1 \cap U_1) = U_3,$$
$$f_1\langle\xi\rangle(E_1 \cap U_1) = U_4.$$

Consider the domains $U_3 \cap U_4 = \emptyset$, that is

$$Fr(U_3) \cap Fr(U_4) = \{f_1\langle\xi\rangle(\xi_1)\} \neq \emptyset,$$
$$f_1\langle\xi\rangle(z) \in f_1\langle\xi\rangle(V(\xi_1)) \cap (U_3 \cup U_4) \neq \emptyset,$$

$f_1\langle\xi\rangle(V(\xi_1))$ is a neighborhood of the $\langle\xi\rangle$-discontinuity point

$$f_1\langle\xi\rangle(\xi_1) \in f_1\langle\xi\rangle(U_1) \subset \mathbb{C}.$$

This shows that the relation (2) in Definition 3 is also verified, hence

$$f_1\langle\xi\rangle(z) \prec \langle\xi\rangle f_3\langle\xi\rangle(z).$$

□

8. Definitions and Remark 4. We define the $\langle\xi\rangle$-integral on a $\langle\xi\rangle$-differentiable path. If

$$FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C}) \ni \exists\, \omega\langle\xi\rangle : \mathbb{R} \supset [a, b] \ni t$$

$$\to \omega\langle\xi\rangle(t) = x\langle\xi\rangle(t) + iy\langle\xi\rangle(t) \in U \subset \mathbb{C},$$

a $\langle\xi\rangle$-path in the domain U, having the origin z_0 and the extremity z_1, is every $\langle\xi\rangle$-function with

$$\omega\langle\xi\rangle(a) = z_0,\ \omega\langle\xi\rangle(b) = z_1.$$

The $\langle\xi\rangle$-path is closed if $z_0 = z_1$ and we say that the $\langle\xi\rangle$-path is piecewise $\langle\xi\rangle$-differentiable if there is

$$a = t_0 < \ldots < t_n = b,$$

where on (t_j, t_{j+1}), $\dfrac{d\langle\xi\rangle\omega\langle\xi\rangle}{d\langle\xi\rangle t}$ has the right limit at t_j and the left limit at t_{j+1}, $0 \le j \le n-1$.

Define the $\langle\xi\rangle$-integral on the $\langle\xi\rangle$-differentiable path $\omega\langle\xi\rangle$ by,

$$\int\limits_{\omega\langle\xi\rangle} f\langle\xi\rangle(z)d\langle\xi\rangle z = \sum_{j=0}^{n-1}\int\limits_{t_j}^{t_{j+1}} f\langle\xi\rangle(\omega\langle\xi\rangle(t))d\langle\xi\rangle\omega\langle\xi\rangle(t)$$

$$= \sum_{j=0}^{n-1}\int\limits_{t_j}^{t_{j+1}} (f\langle\xi\rangle \circ \omega\langle\xi\rangle)(t)\left(\frac{d\langle\xi\rangle x\langle\xi\rangle}{d\langle\xi\rangle t} + i\,\frac{d\langle\xi\rangle y\langle\xi\rangle}{d\langle\xi\rangle t}\right)d\langle\xi\rangle t,$$

$$FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C}) \ni \exists\, f\langle\xi\rangle :\ \mathbb{C} \supset U \to \mathbb{C}.$$

If the co-domain U of $\omega\langle\xi\rangle$ is equal to the definition domain of $f\langle\xi\rangle$, then we have the composition of the $\langle\xi\rangle$-functions. The $\langle\xi\rangle$-index $\mathrm{Ind}\langle\xi\rangle(\omega\langle\xi\rangle, z_0)$ of the closed $\langle\xi\rangle$-differentiable path, which does not contain the point z_0 of U, is the number

$$\mathrm{Ind}\langle\xi\rangle(\omega\langle\xi\rangle, z_0) = \frac{1}{2\pi i}\int\limits_{\omega\langle\xi\rangle}\frac{d\langle\xi\rangle z}{z - z_0},\ z \neq z_0.$$

Clearly, the $\langle\xi\rangle$-integral is linear, that is for every $f\langle\xi\rangle, g\langle\xi\rangle \in FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C})$ and for every $p, q \in \mathbb{C}$, we have

$$\int\limits_{\omega\langle\xi\rangle}(pf\langle\xi\rangle(z) + qg\langle\xi\rangle(z))d\langle\xi\rangle z = p\int\limits_{\omega\langle\xi\rangle} f\langle\xi\rangle(z)d\langle\xi\rangle z + q\int\limits_{\omega\langle\xi\rangle} g\langle\xi\rangle(z)d\langle\xi\rangle z.$$

9. Definitions and Remarks 5. Consider the closed piecewise differentiable $\langle\xi\rangle$-paths

$$FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C}) \ni \exists\, \omega_k\langle\xi\rangle :\ \mathbb{R} \ni t \to \omega_k\langle\xi\rangle(t)$$

$$= x_k\langle\xi\rangle(t) + iy_k\langle\xi\rangle(t) \in U \subset \mathbb{C},$$

$$\omega_k^0\langle\xi\rangle(t) = x_k^0\langle\xi\rangle(t) + iy_k^0\langle\xi\rangle(t),$$

where we denote $(x_k^0\langle\xi\rangle(t), y_k^0\langle\xi\rangle(t)) = u_k^0\langle\xi\rangle$ is fixed and the domains $E_k \subset \mathbb{C}$, $1 \le k \le 2$. If

$$g\langle\xi\rangle(z) = P\langle\xi\rangle(x,y) + iQ\langle\xi\rangle(x,y) \in FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C}),$$

then the oscillation at ξ is

$$S(g\langle\xi\rangle, \xi) = \lim_{\substack{z\to\xi,\\ z\in Cl(E_1)}} g\langle\xi\rangle(z) - \lim_{\substack{z\to\xi,\\ z\in Cl(E_2)}} g\langle\xi\rangle(z) \neq 0,$$

$$E_1 \cap E_2 = \emptyset$$

that is

$$Fr(E_1) \cap Fr(E_2) = \{\xi\} \neq \emptyset,$$
$$z \in V(\xi) \cap (E_1 \cup E_2) \neq \emptyset.$$

We define the $\langle\xi\rangle$-derivative by

$$\frac{d\langle\xi\rangle g\langle\xi\rangle}{d\langle\xi\rangle z} = \frac{df}{dz} + (1+z)^{-1}S(g\langle\xi\rangle, \xi) + \delta\langle\xi\rangle[g\langle\xi\rangle] + \exp(-2i\varphi)\rho\langle\xi\rangle[g\langle\xi\rangle]$$
$$= \frac{df}{dz} + (1+z)^{-1}S(g\langle\xi\rangle, \xi)$$
$$+\frac{1}{2}\left(\frac{\partial\langle\xi\rangle P\langle\xi\rangle}{\partial\langle\xi\rangle x} + \frac{\partial\langle\xi\rangle Q\langle\xi\rangle}{\partial\langle\xi\rangle y} + i\left(\frac{\partial\langle\xi\rangle Q\langle\xi\rangle}{\partial\langle\xi\rangle x} - \frac{\partial\langle\xi\rangle P\langle\xi\rangle}{\partial\langle\xi\rangle y}\right)\right)$$
$$+\frac{1}{2}\exp(-2i\varphi)\left(\frac{\partial\langle\xi\rangle P\langle\xi\rangle}{\partial\langle\xi\rangle x} - \frac{\partial\langle\xi\rangle Q\langle\xi\rangle}{\partial\langle\xi\rangle y} + i\left(\frac{\partial\langle\xi\rangle Q\langle\xi\rangle}{\partial\langle\xi\rangle x} + \frac{\partial\langle\xi\rangle P\langle\xi\rangle}{\partial\langle\xi\rangle y}\right)\right),$$
$$z \neq -1,\ f(z) \in \lambda(E_1 \cup E_2, td\mathbb{C}),$$

and the angle is the limit between the segment Δz and and the axis Ox.

10. Teorema 3. Consider the complex Călugăreanu $\langle\xi\rangle$-polygenic $\langle\xi\rangle$-functions which are $\langle\xi\rangle$-discontinuous. If

$$\rho\langle\xi\rangle[g\langle\xi\rangle(z)] < A,\ z \in E_1 \cup E_2 \subset \mathbb{C}$$

applying the definitions and Remarks 4 and 5, for the closed piecewise differentiable $\langle\xi\rangle$-paths,

$$\omega_k\langle\xi\rangle \in FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C}),$$

which does not contain $u_k^0\langle\xi\rangle$, $1 \leq k \leq 2$, $z \neq u_k^0\langle\xi\rangle$ that is

$$2g\langle\xi\rangle(z) = i\rho\langle\xi\rangle[g\langle\xi\rangle(z)] + \pi \in FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C}),$$

then we obtain the $\langle\xi\rangle$-formulas $(p_k\langle\xi\rangle)$ of representation of the considered $\langle\xi\rangle$-functions.

Proof. It is obvious that we have

$$2\int\limits_{\omega_k\langle\xi\rangle} \frac{g\langle\xi\rangle(z)}{z - u_k^0\langle\xi\rangle} d\langle\xi\rangle z = i\int\limits_{\omega_k\langle\xi\rangle} \frac{\rho\langle\xi\rangle[g\langle\xi\rangle(z)]}{z - u_k^0\langle\xi\rangle} d\langle\xi\rangle z + \pi\int\limits_{\omega_k\langle\xi\rangle} \frac{1}{z - u_k^0\langle\xi\rangle} d\langle\xi\rangle z$$
$$\Rightarrow 2\int\limits_{\omega_k\langle\xi\rangle} \frac{g\langle\xi\rangle(z)}{z - u_k^0\langle\xi\rangle} d\langle\xi\rangle z = i\int\limits_{\omega_k\langle\xi\rangle} \frac{\rho\langle\xi\rangle[g\langle\xi\rangle(z)]}{z - u_k^0\langle\xi\rangle} d\langle\xi\rangle z$$
$$+2\pi^2 i\, Ind\langle\xi\rangle(\omega_k\langle\xi\rangle, u_k^0\langle\xi\rangle)$$
$$\Rightarrow (p_k\langle\xi\rangle):\ \frac{1}{\pi i}\int\limits_{\omega_k\langle\xi\rangle} \frac{g\langle\xi\rangle(z)}{z - u_k^0\langle\xi\rangle} d\langle\xi\rangle z$$
$$= \frac{1}{2\pi}\int\limits_{\omega_k\langle\xi\rangle} \frac{\rho\langle\xi\rangle[g\langle\xi\rangle(z)]}{z - u_k^0\langle\xi\rangle} d\langle\xi\rangle z + \pi Ind\langle\xi\rangle(\omega_k\langle\xi\rangle, u_k^0\langle\xi\rangle),\ 1 \leq k \leq 2. \qquad \square$$

11. Corollary 1. (Călugăreanu-Cojan $\langle\xi\rangle$-operators). Let

$$FH\langle\xi\rangle(\mathbb{C}, td\mathbb{C}) \ni \exists\, f\langle\xi\rangle, g\langle\xi\rangle :\ \mathbb{C} \supset D \to \mathbb{C},$$

be the $\langle\xi\rangle$-index of the closed piecewise $\langle\xi\rangle$-differentiable $\langle\xi\rangle$-path in the domain D relative to the point $u_0\langle\xi\rangle$, $z \neq u_0\langle\xi\rangle$,

$$Ind\langle\xi\rangle(\omega\langle\xi\rangle, u_0\langle\xi\rangle) = \frac{1}{2\pi i}\int\limits_{\omega\langle\xi\rangle}\frac{1}{z-u_0\langle\xi\rangle}d\langle\xi\rangle z$$

that is $\omega\langle\xi\rangle$ does not contain $u_0\langle\xi\rangle$ and

$$\omega\langle\xi\rangle \in FH\langle\xi\rangle(\mathbb{N}, td\mathbb{C}).$$

Considering the Călugăreanu $\langle\xi\rangle$-integral $\langle\xi\rangle$-operator $(p\langle\xi\rangle)$

$$(p\langle\xi\rangle) = \frac{1}{2\pi}\int\limits_{\omega\langle\xi\rangle}\frac{\rho\langle\xi\rangle[g\langle\xi\rangle(z)]}{z-u_0\langle\xi\rangle}d\langle\xi\rangle z + \pi\mathrm{Ind}\langle\xi\rangle(\omega\langle\xi\rangle, u_0\langle\xi\rangle)$$

from Teorema 4 for $\mathrm{Ind}\langle\xi\rangle(\omega\langle\xi\rangle, u_0\langle\xi\rangle)$ and the Cojan $\langle\xi\rangle$-operators

$$T_1\langle\xi\rangle(f\langle\xi\rangle, g\langle\xi\rangle)(\omega\langle\xi\rangle) = \int\limits_{\omega\langle\xi\rangle} f\langle\xi\rangle(z)\frac{d\langle\xi\rangle g\langle\xi\rangle}{d\langle\xi\rangle z}d\langle\xi\rangle z$$

$$+\int\limits_{\omega\langle\xi\rangle}\frac{1}{z-u_0\langle\xi\rangle}d\langle\xi\rangle z,$$

$$T_2\langle\xi\rangle(f\langle\xi\rangle, h)(\omega\langle\xi\rangle) = \int\limits_{\omega\langle\xi\rangle} f\langle\xi\rangle(z)\frac{dh}{dz}d\langle\xi\rangle z,$$

$$h(z) \in \lambda(D, td\mathbb{C},$$

$$T_3\langle\xi\rangle(f\langle\xi\rangle)(\omega\langle\xi\rangle) = \int\limits_{\omega\langle\xi\rangle}(1+z)^{-1}f\langle\xi\rangle(z)d\langle\xi\rangle z,\ z \neq -1,$$

then the $\langle\xi\rangle$-relation $(p\langle\xi\rangle, T\langle\xi\rangle)$ holds.

Proof.

$$T_1\langle\xi\rangle(f\langle\xi\rangle, g\langle\xi\rangle)(\omega\langle\xi\rangle)$$

$$= \int\limits_{\omega\langle\xi\rangle} f\langle\xi\rangle(z)\left(\frac{dh}{dz} + (1+z)^{-1}S(g\langle\xi\rangle, \xi)\right)d\langle\xi\rangle z + 2\pi i\, Ind\langle\xi\rangle(\omega\langle\xi\rangle, u_0\langle\xi\rangle)$$

$$\Rightarrow \frac{1}{2i}T_1\langle\xi\rangle(f\langle\xi\rangle, g\langle\xi\rangle)(\omega\langle\xi\rangle) = \frac{1}{2i}T_2\langle\xi\rangle(f\langle\xi\rangle, h)(\omega\langle\xi\rangle)$$

$$+\frac{1}{2i}S(g\langle\xi\rangle, \xi)T_3\langle\xi\rangle(f\langle\xi\rangle)(\omega\langle\xi\rangle) + \pi Ind\langle\xi\rangle(\omega\langle\xi\rangle, u_0\langle\xi\rangle).$$

Replacing $\pi Ind\langle\xi\rangle(\omega\langle\xi\rangle, u_0\langle\xi\rangle)$ in $(p\langle\xi\rangle)$ we obtain the $\langle\xi\rangle$-relation

$$T_2\langle\xi\rangle(f\langle\xi\rangle, h)(\omega\langle\xi\rangle) + T_3\langle\xi\rangle(f\langle\xi\rangle(\omega\langle\xi\rangle))S(g\langle\xi\rangle, \xi) - T_1\langle\xi\rangle(f\langle\xi\rangle, g\langle\xi\rangle)(\omega\langle\xi\rangle)$$

$$= \frac{1}{2\pi}\int\limits_{\omega\langle\xi\rangle}\frac{\rho\langle\xi\rangle[g\langle\xi\rangle(z)]}{z-u_0\langle\xi\rangle}d\langle\xi\rangle - \frac{1}{\pi}\int\limits_{\omega\langle\xi\rangle}\frac{g\langle\xi\rangle(z)}{z-u_0\langle\xi\rangle}d\langle\xi\rangle z,$$

that is a representation for the Călugăreanu-Cojan $\langle\xi\rangle$-operators on a closed $\langle\xi\rangle$-piecewise differentiable $\langle\xi\rangle$-path.

We have the domains $D \supset D_1 \cap D_2 = \emptyset$, that is

$$Fr(D_1) \cap Fr(D_2) = \{\xi\} \neq \emptyset,$$
$$z \in V(\xi) \cap (D_1 \cup D_2) \neq \emptyset,$$

with the oscillation

$$S(g\langle\xi\rangle, \xi) = \lim_{\substack{z\to\xi,\\ z\in Cl(D_1)}} g\langle\xi\rangle(z) - \lim_{\substack{z\to\xi,\\ z\in Cl(D_2)}} g\langle\xi\rangle(z) \neq 0. \qquad \square$$

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