# SOME GENERALIZATIONS OF THE FEUERBACH'S THEOREM 

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#### Abstract

The celebrated theorem of Feuerbach states that the nine-point circle of a nonequilateral triangle is tangent to both its incircle and its three excircles. In this note, we give three generalizations of the Feuerbach's Theorem and synthetic proofs for them.


## 1. Introduction and motivations

Feuerbach's Theorem is a well known old theorem [1], [2], [7]. In [1], it is formulated in the following form.

Theorem 1. (Feuerbach, 1822) In a nonequilateral triangle, the nine-point circle of a triangle is tangent internally to the incircle, and externally to each of the excircles.
For three points $P, Q$ and $R$ in the plane, the circle with center $P$, the circle with diameter $P Q$, and the circumcircle of triangle $P Q R$ are denoted by $(P),(P Q)$, and $(P Q R)$ respectively.
For historical details of this Theorem 1, please see [1]. In this article, we study some generalizations of theorem 1 as following.

Theorem 2. Let $A B C$ is a triangle and $H$ is its orthocenter. Let $D$ is an arbitrary point, that is different from $H$ and not on $B C, C A, A B$ and $(A B C)$. Let $\ell_{a}$ is the line passes through the orthogonal projections of $A$ onto $D B$ and $D C$. We define $\ell_{b}$ and $\ell_{c}$ similarly. Then, the circumcircle of the triangle determined by the lines $\ell_{a}, \ell_{b}$ and $\ell_{c}$ is tangent to the pedal circle of $D$ with repect to triangle $A B C$.

Theorem 3. Let $A B C$ is a triangle and $H$ is its orthocenter. Let $D$ is an arbitrary point, that is different from $H$ and not on $B C, C A, A B$ and $(B C),(C A),(A B)$. Let $H_{a}, H_{b}$ and $H_{c}$ be the orthocenters of the triangles $D B C, D C A$, and $D A B$ respectively. Then, the circumcircle of the triangle determined by the three radical axes of each of pair of circles $\left(\left(D H_{a}\right),(B C)\right),\left(\left(D H_{b}\right),(C A)\right)$ and $\left(\left(D H_{c}\right),(A B)\right)$ is tangent to the nine-point circle of triangle $A B C$.

Theorem 4. Let $A B C$ is a triangle and $O$ is its circumcenter. Let $D$ is an arbitrary point, that is different from $H$ and not on $B C, C A, A B$ and $(A B C)$. Let $D_{a}, D_{b}$ and $D_{c}$ are the orthogonal projections of $D$ onto $B C, C A$, and $A B$ respectively. Then, the nine-point circle of triangle determined by the lines passing through the midpoints of $B C, C A$ and $A B$ respectively and parallel to $D_{b} D_{c}, D_{c} D_{a}$ and $D_{a} D_{b}$ is tangent to $\left(D_{a} D_{b} D_{c}\right)$.

[^0]

Figure 1.


Figure 2.


Figure 3.


Figure 4.
It is directly seen that for $D$ is incenter or excenter of triangle $A B C$, the Theorem 2, Theorem 3 and Theorem 4 return Feuerbach's Theorem; hence they are extensions of the Theorem 1, indeed.
In this article, we present the purely synthetic proofs of them. In the proofs, we shall make use of the notion of directional angles. As in [3, chapter 2, $\S \& 8$-14], the directed angle from the line $x$ to the line $y$ denote by $(x, y)$, the directed angle from the non-zero
vector $\vec{x}$ to the non-zero vector $\vec{y}$ denote by $(\vec{x}, \vec{y})$. To facilitate the readers, we would like to present a few results related to these two concepts that will be used in this article.
(i) For three lines $x, y, z$, we have $(x, y) \equiv(x, z)+(z, y)(\bmod \pi)$.
(see: [3], Theorem 311 and Attention 312),
(ii) For three vectors $\vec{x}, \vec{y}, \vec{z}$, we have $(\vec{x}, \vec{y}) \equiv(\vec{x}, \vec{z})+(\vec{z}, \vec{y})(\bmod 2 \pi)$.
(see: [3], Theorem 255 and Attention 256),
(iii) For two lines $x, y$, we have $(x, y) \equiv-(y, x)(\bmod \pi)$.
(see: [3], Theorem 324),
(iv) For two vectors $\vec{x}, \vec{y}$, we have $(\vec{x}, \vec{y}) \equiv \pi(\bmod 2 \pi)$ if and only if $\vec{x}, \vec{y}$ have opposite directions.
(see: [3], Theorem 315),
(v) For two lines $x, y$, we have $(x, y) \equiv 0(\bmod \pi)$ if and only if $x$ and $y$ are either parallel or coincident.
(see: [3], Theorem 322),
(vi) If the lines $x, y, x^{\prime}$ such that $(x, y) \equiv\left(x^{\prime}, y\right)(\bmod \pi)$, then $x$ and $x^{\prime}$ are either parallel or coincident.
(see: [3], Theorem 327),
(vii) For two lines $x, y$, we have $(x, y) \equiv \frac{\pi}{2}(\bmod \pi)$ if and only if $x \perp y$.
(see: [3], Theorem 32),
(viii) If the lines $x, y, x^{\prime}, y^{\prime}$ such that $x \perp x^{\prime}$ and $y \perp y^{\prime}$, then $(x, y) \equiv\left(x^{\prime}, y^{\prime}\right)(\bmod \pi)$.
(see: [3], Theorem 326),
(ix) If $A B C$ is a triangle, then the following conditions are equivalent:

1) $A B=A C$.
2) $(B A, B C) \equiv(C B, C A)(\bmod \pi)$.
3) $2(B A, B C) \equiv(\overrightarrow{B A}, \overrightarrow{A C})(\bmod 2 \pi)$.
(see: [4], Theorem 36 and Theorem 37),
(x) If points $A, B, C, D$ are not linear, such that $A B=A D$ and $C B=C D$, then
a) $(A B, A C) \equiv(A C, A D)(\bmod \pi)$.
b) $(B A, B C) \equiv(D C, D A)(\bmod \pi)$.
(see: [4], Theorem 38 and Theorem 41),
(xi) If $A, B$ are two distinct points belong to circle $(O)$ and $M$ is an arbitrary point, then $M$ belongs to $(O)$ if and only if $(M A, M B) \equiv \frac{1}{2}(\overrightarrow{O A}, \overrightarrow{O B})(\bmod \pi)$.
(see: [4], Theorem 58),
(xii) If $A B C$ is a triangle and $M$ is an arbitrary point, then $M$ belong to circle $(A B C)$ if and only if $(M A, M B) \equiv(C A, C B)(\bmod \pi)$.
(see: [4], Theorem 59),
(xiii) If $A B C$ is a triangle and $t$ is an arbitrary line, then $t$ is tangent to circle $(A B C)$ if and only if $(t, A B) \equiv(C A, C B)(\bmod \pi)$.
(see: [4], Theorem 61).

## 2. PRELIMINARY RESULTS

To prove the theorems, we need some lemmas.
Lemma 5. (Miquel point of the compelte quadrilateral). The circumcircles of all four triangles of a complete quadrangle meet a point.
(See: [6], 3.1. Steiner's Theorem 1 and the Miquel Point, p. 38, or [4], problem 85).
Lemma 6. Let $A B C$ is a triangle, and $d$ is an arbitrary line through $A$ and not perpendicular to $B C$. Let $H$ is the orthogonal projection of $A$ onto $B C$; let $K, L$ are the orthogonal projections of $B, C$ onto $d$ respectively. Then
(a) The center of circle (HKL) belongs to nine-point circle of triangle ABC.
(b) $(H K, H L) \equiv(A C, A B)(\bmod \pi)$.

Proof. Let us denote $M, N, P$ are the midpoints of $B C, C A, A B$ respectively; $I$ is the center of circle $(H K L)$. Then $M N \| A B$ and $M P \| A C$.
(1)


Figure 5.
From $A H \perp B C, B K \perp d, C L \perp d$, we get $P H=P K$ and $N H=N L$.
Combining with $I H=I K=I L$, we obtain $I P, I N$ are the perpendicular bisectors of $H K, H L$ respectively.
(2)

Again, from $A H \perp B C, B K \perp d$ and $C L \perp d$, we deduce that groups of four points $(A, B, K, H)$ and ( $A, C, H, L$ ) are concyclic.
(3)
(a) We have,

$$
\begin{align*}
(I P, I N) & \equiv(H K, H L)(\bmod \pi)  \tag{2}\\
& \equiv(H K, A K)+(A L, H L)(\bmod \pi) \\
& \equiv(H B, A B)+(A C, H C)(\bmod \pi) \\
& \equiv(A C, A B)(\bmod \pi) \\
& \equiv(M P, M N)(\bmod \pi)
\end{align*}
$$

It follows that $I$ belongs to circle ( $M N P$ ), the nine-point circle of triangle $A B C$.
(b) We have,

$$
\begin{align*}
(H K, H L) & \equiv(H K, H A)+(H A, H L)(\bmod \pi) \\
& \equiv(B K, B A)+(C A, C L)(\bmod \pi)  \tag{3}\\
& \equiv(C A, B A)(\bmod \pi)
\end{align*}
$$

(by $B K \| C L$ ).

Lemma 5 is proved.
Lemma 7. If any of the three points in $A, B, C$, and $D$ are not collinear, then the nine- point circles of triangles $A B C, B C D, C D A, D A B$ meet at one point.
(See: [4], Lemma 2 in problem 88, or [5] 396. Theorem, p. 242, or [8], Theorem 1, 1).
This point is variably known as the Euler, Euler-Poncelet, and Poncelet point of four points $A, B, C, D$. Note that this is case when the Poncelet point of four points not define: If $A, B, C$, and $D$ be four points in the plane that form an orthocentric system, which means that each of the four points $A, B, C, D$ is the orthocenter of the triangle formed by the other three points, then the triangles $A B C, B C D, C D A, D A B$ all share the same nine-point circle, and hence the Poncelet point of the four points $A, B, C, D$ cannot be uniquely defined.
Lemma 8. Let four points $A, B, C$ and $D$ be non-orthocentric system, and let $H$ is the orthocenter of triangle $A B C$. Then the Poncelet point of the points $A, B, C, D$ is simultaneously the Poncelet point of the points $H, B, C, D$, as well as the Poncelet point of the points $H, C, D, A$, as well as the Poncelet point of the points $H, D, A, B$, and belongs to the nine-point circles of triangles $H A D, H B D, H C D$.
(See: [8], Theorem 3, 1).
Lemma 9. Let four points $A, B, C$ and $D$ be non-orthocentric system, and $P$ is the Poncelet of the points $A, B, C, D$. Denote $X, Y, Z$ the orthogonal projections of $D$ onto $B C, C A, A B$, respectively. Let $H, K$ be the orthogonal projections of $B, C$ on to $A D$ respectively. Let I be the center circle (XHK). Then KY, HZ, PI have a common point on the circle (XYZ).
Proof. Let us denote $M$ is the midpoint of $C D$. According to Lemma 7, $P$ belongs to the nine-point circles of triangles $D B C$ and $D C A$. From this, by Lemma 6(a), we get $P$ is the second intersection of (MIX) and (MKY).
Denote by $T$ as the second intersection of (PXM) and (XHK). We show that PI, XT, KY are concurrent.
We have,

$$
\begin{array}{rlr}
(M T, M K) & \equiv(M T, M I)+(M I, M K)(\bmod \pi) & \\
& \equiv(X T, X I)+(M X, M I)(\bmod \pi) & (\text { by } X \in(M T I), I X=I K, M X=M K) \\
& \equiv(T I, T X)+(T X, T I))(\bmod \pi) & (b y I X=I T ; T \in(I X M)) \\
& \equiv(T I, T I)(\bmod \pi) & \\
& \equiv 0(\bmod \pi) . &
\end{array}
$$

Therefore $M T$ coincides $M K$, i.e. $M$ belongs to $K T$.
Let us denote $E$ is the orthogonal projection of $B$ onto $D Y$; and $F$ is the intersection of $E H$ and $K Y$.


Figure 6.

From $B, D, E, H, X$ all belong to circle $(B D)$, and $C, D, K, Y, X$ all belong to circle $(C D)$, we get $X$ is the Miquel point of the complete quadrilateral DKYFEH (by Lemma 5). Thus, $F$ belongs to circle (XHK).
Therefore,

$$
\begin{aligned}
(P K, P I) & \equiv(P K, P M)+(P M, P I)(\bmod \pi) \\
& \equiv(Y K, Y M)+(X M, X I)(\bmod \pi) \\
& \equiv(K M, K Y)+(X M, X I)(\bmod \pi) \\
& \equiv(K M, K F)+(K I, K M)(\bmod \pi) \\
& \equiv(K I, K F)(\bmod \pi) \\
& \equiv(F K, F I)(\bmod \pi)
\end{aligned}
$$

(by $Y \in(P K M) ; X \in(P M I))$
(by $M K=M Y$ )
(by $F \in K Y ; M X=M K, I X=I K$ )
(by $I K=I F$ ).
It follows that $P$ belongs to circle (FIK).
Therefore PI, XT, KY are concurrent at the radical center of three circles (IMX), (FIK) and (XKH).
Similarly, PI, XT, HZ are also concurrent.
Hence $P I, H Z, K Y$ are concurrent.
On the other hand,

$$
\begin{aligned}
(H Z, Y K) & \equiv(H Z, H D)+(D K, Y K)(\bmod \pi) \\
& \equiv(X Z, X D)+(D X, Y X)(\bmod \pi) \\
& \equiv(X Z, X Y)(\bmod \pi)
\end{aligned}
$$

(by $D \in H K$ )
(by $X \in(B D) \equiv(H Z D) ; X \in(D K Y)$ )

It follows that the intersection point of $H Z$ and $K Y$ is on circle ( $X Y Z$ ). The Lemma 9 is proved.

Lemma 10. The Poncelet point of four points $A, B, C$ and $D$ lies on the pedal circle of the point $A$ with respect to triangle $B C D$, on the pedal circle of the point $B$ with respect to triangle $C D A$, on the pedal circle of the point $C$ with respect to triangle $D A B$, on the pedal circle of the point $D$ with respect to triangle $A B C$.
(See: [8], Theorem 4, 1 or [5], 396. Theorem and 397. Theorem, p. 242).

## 3. MAIN RESULTS

## PROOF OF THEOREM 2

Let us denote $A_{b}, A_{c}$ are the orthogonal projections of $A$ onto $D B, D C$ respectively; we difene four points $B_{c}, B_{a}$ and $C_{a}, C_{b}$ similarly; $A_{a}=B_{a} B_{c} \cap C_{a} C_{b}, B_{b}=C_{a} C_{b} \cap A_{b} A_{c}$ and $C_{c}=A_{b} A_{c} \cap B_{c} B_{a} ; D a, D_{b}, D_{c}$ are the orthogonal projections of $D$ onto $B C, C A, A B$ respectively. We show that circles $\left(A_{a} B_{b} C_{c}\right)$ and $\left(D_{a} D_{b} D_{c}\right)$ are tangent to each other. Denote by $A_{0}=B_{a} D_{c} \cap C_{a} D_{b}, B_{0}=C_{b} D_{a} \cap A_{b} D_{c}$ and $C_{0}=A_{c} D_{b} \cap B_{c} D_{a}$.


Figure 7.
According to Lemma 9, the point $A_{0}=B_{a} D_{c} \cap C_{a} D_{b}$ belongs to ( $D_{a} D_{b} D_{c}$ ), the pedal circle of $D$ with respect to triangle $A B C$.
Similarly, $B_{0}$ and $C_{0}$ also belong to $\left(D_{a} D_{b} D_{c}\right)$.
Therefore $\left(A_{0} B_{0} C_{0}\right)$ and $\left(D_{a} D_{b} D_{c}\right)$ are coincident.

We have,

$$
\begin{array}{rlr}
\left(B_{a} A_{a}, B_{a} C_{a}\right) & \equiv\left(B_{a} B_{c}, B_{a} D\right)(\bmod \pi) & \left(\text { by } B_{c} \in B_{a} A_{a}, D \in B_{a} C_{a}\right) \\
& \equiv\left(B B_{c}, B D\right)(\bmod \pi) & \left(\text { by } B \in(B D) \equiv\left(B_{a} B_{c} D\right)\right) \\
& \equiv\left(B B_{c}, B C_{b}\right)(\bmod \pi) & \left(\text { by } C_{b} \in B D\right) \\
& \equiv\left(C B_{c}, C C_{b}\right)(\bmod \pi) & \left(\text { by } C \in(B C) \equiv\left(B B_{c} C_{b}\right)\right) \\
& \equiv\left(C D, C C_{b}\right)(\bmod \pi) & \left(\text { by } D \in C B_{c}\right) \\
& \equiv\left(C_{a} D, C_{a} C_{b}\right)(\bmod \pi) & \left(\text { by } D \in C C_{b}\right) \\
& \equiv\left(C_{a} B_{a}, C_{a} A_{a}\right)(\bmod \pi) & \left(\text { by } B_{a} \in C_{a} D, A_{a} \in C_{a} C_{b}\right) .
\end{array}
$$

It follows that $A_{a} B_{a}=A_{a} C_{a}$.
Since,

$$
\begin{aligned}
\left(A_{0} B_{0}, A_{0} D_{b}\right) & \equiv\left(D_{a} B_{0}, D_{a} D_{b}\right)(\bmod \pi) \\
& \equiv\left(D_{a} C_{b}, D_{a} D_{b}\right)(\bmod \pi) \\
& \equiv\left(C_{a} C_{b}, C_{a} D_{b}\right)(\bmod \pi) \\
& \equiv\left(A_{a} B_{b}, A_{0} D_{b}\right)(\bmod \pi)
\end{aligned}
$$

(by $C_{b} \in D_{a} B_{0}$ )
$\left(\right.$ by $\left.C_{a} \in(C D) \equiv\left(D_{a} C_{b} D_{b}\right)\right)$
(by $A_{a} B_{b}=C_{a} C_{b}, A_{0} \in C_{a} D_{b}$ ).

We decude that $A_{0} B_{0}$ and $A_{a} B_{b}$ are either parallel or coincident.
Similarly, $B_{0} C_{0}$ and $B_{b} C_{c}$ are either parallel or coincident; $C_{0} A_{0}$ and $C_{c} A_{a}$ are either parallel or coincident.
Thus, triangles $A_{0} B_{0} C_{0}$ and $A_{a} B_{b} C_{c}$ have correspondingly parallel sides or coincident sides.
This means that the triangles $A_{0} B_{0} C_{0}$ and $A_{a} B_{b} C_{c}$ are images of each other through a homothety or a translation.
Let us denote $P$ is the Poncelet point of four points $A, B, C$ and $D$.
According to Lemma 10, we have $P$ belongs to circle ( $D_{a} D_{b} D_{c}$ ).
We have,

$$
\begin{align*}
\left(\overrightarrow{A_{a} B_{a}}, \overrightarrow{A_{a} C_{a}}\right) & \equiv\left(\overrightarrow{A_{a} B_{a}}, \overrightarrow{B_{a} A_{a}}\right)+\left(\overrightarrow{B_{a} A_{a}}, \overrightarrow{A_{a} C_{a}}\right)(\bmod 2 \pi)  \tag{7}\\
& \equiv \pi+2\left(B_{a} A_{a}, B_{a} C_{a}\right)(\bmod 2 \pi)  \tag{5}\\
& \equiv \pi+2\left(B_{a} B_{c}, B_{a} D\right)(\bmod 2 \pi) \\
& \equiv 2 \frac{\pi}{2}+2\left(B B_{c}, B D\right)(\bmod 2 \pi) \\
& \equiv 2\left(C C_{b}, B D\right)+2\left(C D, C C_{b}\right)(\bmod 2 \pi) \\
& \equiv 2(C D, B D)(\bmod 2 \pi) \\
& \equiv 2\left(D_{a} B_{a}, D_{a} C_{a}\right)(\bmod 2 \pi)
\end{align*}
$$

(by $B_{c} \in B_{a} A_{a}, D \in B_{a} C_{a}$ )
(by $B \in(B D) \equiv\left(B_{a} B_{c} D\right)$ )
(by $C C_{b} \perp B D, B B_{c} \perp C D$ )
(by Lemma 6(b)).
From this, combining with (5) we obtain $A_{a}$ is the center of circle ( $D_{a} B_{a} C_{a}$ ). According to Lemma 9 for the points $A, B, C$ and $D$, the lines $B_{a} D_{c}, C_{a} D_{b}$ and $P A_{a}$ are concurrent. Hence $P$ belongs to the line $A_{a} A_{0}$.
Similarly, $P$ also belongs to the lines $B_{b} B_{0}$ and $C_{c} C_{0}$. This means that $A_{a} A_{0}, B_{b} B_{0}$ and $C_{c} C_{0}$ are concurent at the point $P$. From (6), we get $P$ is the center of the homothety that transforming $A_{a} B_{b} C_{c}$ to $A_{0} B_{0} C_{0}$. But $P$ belongs to $\left(A_{0} B_{0} C_{0}\right)$ by (7), this implies that the circles $\left(A_{a} B_{b} C_{a}\right)$ and $\left(A_{0} B_{0} C_{0}\right)$ are tangent at $P$. And by (4), we decude that $\left(A_{a} B_{b} C_{c}\right)$ is tangent to $\left(D_{a} D_{b} D_{c}\right)$.

Our proof is completed.

## PROOF OF THEOREM 3

Suppose that $(E)$ is the nine-point circle of triangle $A B C$; let us denote $A A^{b}, B B^{b}, C C^{c}$ are the altitudes of triangle $A B C ; B B^{c}, C C^{a}$ are the altitudes of triangle $H_{a} B C$. We define $C C^{b}, A A^{c}$ and $A A^{b}, B B^{a}$ similarly; $M_{a}, M_{b}, M_{c}$ are the midpoints of $B C, C A, A B$ respectively.
It is easy to see that $A^{a}, B^{b}, C^{c}, M_{a}, M_{b}, M_{c}$ all belong to circle $(E)$, and $B^{a} C^{a}, C^{b} A^{b}, A^{c} B^{c}$ respectively are the radical axes of groups of two circles $\left.\left(\left(D H_{a}\right),(B C)\right),\left(D H_{b}\right),(C A)\right)$, $\left(\left(D H_{C}\right),(A B)\right)$.
Denote by $A^{0}=A^{c} B^{c} \cap A^{b} C^{b}, B^{0}=B^{a} C^{a} \cap B^{c} A^{c}, C^{0}=C^{b} A^{b} \cap C^{a} B^{a}$. We show that ( $A^{0} B^{0} C^{0}$ ) is tangent to $(E)$.
Indeed, let us denote $O_{a}, O_{b}, O_{c}$ are the centers of circles $\left(A^{a} B^{a} C^{a}\right),\left(B^{b} C^{b} A^{b}\right),\left(C^{c} A^{c} B^{c}\right)$ respectively. By Lemma $6(a)$, the points $O_{a}, O_{b}, O_{c}$ all belong to $(E)$. It follows that $\left(O_{a} O_{b} O_{c}\right)$ and ( $E$ ) are coincident.
From $A B^{c}$ and $C B^{a}$ are all perpendicular to $B^{a} B^{c}$, we get the perpendicular bisector of $B^{a} B^{c}$ passes through the midpoint of $A C$. Similarly, the perpendicular bisector of $B^{a} B^{c}$ also passes through midpoint of $A H_{a}$. Denote by $N_{a}$ is the midpoint of $A H_{a}$, then the points $O_{b}, M_{b}, N_{a}$ all belong to the perpendicular bisector of $B^{a} B^{c}$. Thus $N_{a}, O_{b}, M_{b}$ are collinear.
Similarly, $N_{a}, O_{c}, M_{c}$ are also collinear.
Therefore,

$$
\begin{aligned}
\left(O_{b} N_{a}, O_{b} O_{c}\right) & \equiv\left(O_{b} M_{b}, O_{b} O_{c}\right)(\bmod \pi) \\
& \equiv\left(M_{c} M_{b}, M_{c} O_{c}\right)(\bmod \pi) \\
& \equiv\left(B C, B H_{a}\right)(\bmod \pi) \\
& \equiv\left(B C, B C^{a}\right)(\bmod \pi) \\
& \equiv\left(B^{a} C, B^{a} C^{a}\right)(\bmod \pi) \\
& \equiv\left(O_{b} N_{a}, B^{0} C^{0}\right)(\bmod \pi)
\end{aligned}
$$

$$
\begin{array}{r}
\left(\text { by } M_{b} \in N_{a} O_{b}\right) \\
\left(\text { by } O_{b}, O_{c}, M_{b}, M_{c} \in(E)\right) \\
\text { (by } \left.M_{c} M_{b}\left\|B C, M_{c} O_{c}\right\| B H_{a}\right) \\
\left(\text { by } C_{a} \in B H_{a}\right) \\
\left(\text { by } B^{a} \in(B C) \equiv\left(B C C^{a}\right)\right) \\
\text { (by } \left.B^{a} C \| O_{b} N_{a}, B^{0} C^{0} \equiv B^{a} C^{a}\right) .
\end{array}
$$

Hence, $O_{b} O_{c}$ and $B^{0} C^{0}$ are either parallel or coincident.
Similarly, $O_{a} O_{b}$ and $A^{0} B^{0}$ are either parallel or coincident; $O_{c} O_{a}$ and $C^{0} A^{0}$ are either parallel or coincident.
Thus, triangles $O_{a} O_{b} O_{c}$ and $A^{0} B^{0} C^{0}$ have correspondingly parallel sides or coincident sides.
This means that the triangle $O_{a} O_{b} O_{c}$ and $A^{0} B^{0} C^{0}$ are images of each other through a homothety or a translation.
Denote by $P$ is the Poncelet point of the points $A, B, C$ and $D$.
Cause $D$ is the orthocenter of triangle $H_{c} A B$ and from Lemma 8 , we get $P$ is the Poncelet point of the points $H_{c}, A, B, D$.
On the other hand, $A$ is orthocenter of triangle $B D H_{b}$. According to Lemma 8, the Poncelet point of four points $A, D, H_{b}, H_{c}$ simultaneously the Poncelet point of four points $A, B, D, H_{c}$. Hence, $P$ is the Poncelet point of four points $A, D, H_{b}, H_{c}$.


Figure 8.

According to Lemma 9 for the points $A, D, H_{b}, H_{c}$, the lines $A^{b} C^{b}, A^{c} B^{c}$ and $O_{a} P$ are concurrent. Hence, $P$ belongs to $O_{a} A^{0}$.
Similarly, the point $P$ also belongs to circles $O_{b} B^{0}$ and $O_{c} C^{0}$.
Therefore, $P$ is common point of the lines $O_{a} A^{0}, O_{b} B^{0}, O_{c} C^{0}$, combining with (9), we obtain $P$ is the center of the homothety that transforming $O_{a} O_{b} O_{c}$ to $A^{0} B^{0} C^{0}$. This means that the circles $\left(O_{a} O_{b} O_{c}\right)$ and $\left(A^{0} B^{0} C^{0}\right)$ are tangent with $P$ is tangentcy point. From (8), we get $\left(A^{0} B^{0} C^{0}\right)$ is tangent to $(E)$.
Our proof is completed.
Remark 11. The pedal circle of $D$ with respect to triangle $H_{a} H_{b} H_{c}$ is tangent to nine-point circle of triangle $A B C$ at point $P$.
Proof. Let us denote $D^{a}, D^{b}, D^{c}$ are the orthogonal projections of $D$ onto $H_{b} H_{c}, H_{c} H_{a}, H_{a} H_{b}$ respectively; We show that $\left(D^{a} D^{b} D^{c}\right)$ and $(E)$ are tangent to each other.
Indeed, from $P$ is the Poncelet point of four points $A, D, H_{b}, H_{c}$, we get $P$ belongs to the nine-point circle of triangle $D H_{b} H_{c}$.
Similarly, $P$ also belongs to the nine-point circles of the triangles $D H_{c} H_{a}$ and $D H_{a} H_{b}$.
Thus, from Lemma 7 , we get $P$ is the Poncelet point of the points $D, H_{a}, H_{b}, H_{c}$. And from Lemma 10, we get $P$ belongs to the pedal circle of $D$ with respect to triangle $H_{a} H_{b} H_{c}$.
Hence $P$ belongs to circle $\left(D^{a} D^{b} D^{c}\right)$.
(10)

According to Lemma 10, $P$ belongs to the pedal circle of $A$ with respect to triangle $B C D$. From $P$ is the Poncelet point of four points $A, D, H_{b}, H_{c}$ and by Lemma 10, we get $P$


Figure 9.
belongs to the pedal circle of $D$ with respect to triangle $A H_{b} H_{c}$. According to Lemma 9, the point $A^{0}=A^{c} B^{c} \cap A^{b} C^{b}$ belongs to the pedal circle of $D$ with respect to triangle $A H_{b} H_{c}$.
Hence, $P, A^{0}, A^{a}, D^{a}, B^{c}, C^{b}$ are concyclic.
Similarly, $P, B^{0}, B^{b}, D^{b}, C^{a}, A^{c}$ are also concyclic.
Drawing the tangent line $P t$ at $P$ of circle ( $E$ ), then by Theorem 3, $P t$ is tangent line at $P$ of circle ( $\left.A^{0} B^{0} C^{0}\right)$.
Therefore,

$$
\begin{aligned}
\left(P t, P D^{a}\right) & \equiv\left(P t, P A^{0}\right)+\left(P A^{0}, P D^{a}\right)(\bmod \pi) \\
& \equiv\left(B^{0} P, B^{0} A^{0}\right)+\left(C^{b} A^{0}, C^{b} D^{a}\right)(\bmod \pi) \\
& \equiv\left(B^{0} P, B^{0} A^{c}\right)+\left(C^{b} A^{b}, C^{b} D^{a}\right)(\bmod \pi) \\
& \equiv\left(D^{b} P, D^{b} A^{c}\right)+\left(D A^{b}, D D^{a}\right)(\bmod \pi) \\
& \equiv\left(D^{b} P, D^{b} A^{c}\right)+\left(H_{b} A^{b}, H_{b} D^{a}\right)(\bmod \pi) \\
& \equiv\left(D^{b} P, D^{b} A^{c}\right)+\left(H_{c} A^{c}, H_{c} D^{a}\right)(\bmod \pi) \\
& \equiv\left(D^{b} P, D^{b} A^{c}\right)+\left(D^{b} A^{c}, D^{b} D^{a}\right)(\bmod \pi) \\
& \equiv\left(D^{b} P, D^{b} D^{a}\right)(\bmod \pi) .
\end{aligned}
$$

(by (13); by (11))
(by $A^{c} \in A^{0} B^{0} ; A^{b} \in C^{b} A^{0}$ )
(by (12); $\left.D \in\left(D H_{b}\right) \equiv\left(C^{b} A^{b} D^{a}\right)\right)$
(by $\left.H_{b} \in\left(D H_{b}\right) \equiv\left(C^{b} A^{b} D^{a}\right)\right)$
(by $H_{c} A^{c} \| H_{b} A^{b}, H_{b} D^{a} \equiv H_{c} D^{a}$ )
(by $D^{b} \in\left(D H_{c}\right) \equiv\left(H_{c} A^{c} D^{a}\right)$ )

Therefore, $P t$ is tangent to $\left(P D^{a} D^{b}\right)$. From this, combining with (10), we obtain $\left(D^{a} D^{b} D^{c}\right)$ and $(E)$ are tangent to each other at $P$.

The Remark is proved.

## PROOF OF THEOREM 4

Let us denote $M_{a}, M_{b}, M_{c}$ are the midpoints of $B C, C A, A B$ respectively; the line passes through $M_{b}$ and parallel to $D_{c} D_{a}$, and the line passes through $M_{c}$ and parallel to $D_{a} D_{b}$ meet in $A_{A}$; we define $B_{B}$ and $C_{C}$ similarly. We show that the nine-point circle of triangle $A_{A} B_{B} C_{C}$ and $\left(D_{a} D_{b} D_{c}\right)$ are tangent to each other.
Denote by $A^{\prime}, B^{\prime}, C^{\prime}$ are the reflections of $A, B, C$ respectively in $A_{A}, B_{B}, C_{C}$.


Figure 10.

From $A_{A}, M_{b}, M_{c}$ respectively are the midpoints of $A A^{\prime}, C A, A B$, we get $A^{\prime} B \| A_{A} M_{c}$ and $A^{\prime} C \| A_{A} M_{b}$.
Since $D$ do not belongs to $B C, C A, A B$ and ( $A B C$ ), it follows that there exist isogonal conjugate of $D$ with respect to triangle $A B C$. Let $F$ is the isogonal conjugate of $D$ with respect to triangle $A B C$, then $F B$ and $F C$ are respectively perpendicular to $D_{c} D_{a}, D_{a} D_{b}$ (well-known).

From (14) and (15), we decude that $A^{\prime}$ is the orthocenter of triangle $E B C$.
Similarly, $B^{\prime}, C^{\prime}$ respectively are the orthocenter of triangles $F C A, F A B$.
Denote by $B_{A}, C_{A}$ are the intersections of $A E$ respectively with $B C^{\prime}, C B^{\prime}$; we define $C_{B}, A_{B}$ and $A_{C}, B_{C}$ similarly; and let $A_{O}=B_{C} B_{A} \cap C_{A} C_{B}, B_{O}=C_{A} C_{B} \cap A_{B} A_{C}, C_{O}=A_{B} A_{C} \cap$ $B_{C} B_{A}$.

Let us denote $F_{a}, F_{b}, F_{c}$ are the orthogonal projections of $F$ onto $B C, C A, A B$ respectively. By Theorem 2, the pedal circle of $F$ with respect to triangle $A B C$ is tangen to ( $A_{O} B_{O} C_{O}$ ). (16)

On the other hand, from $D$ and $F$ are isogonal conjugate in triangle $A B C$, we get the circle $\left(D_{a} D_{b} D_{c}\right)$ is also the pedal circle of $F$ with respect to triangle $A B C$.
From (16) and (17), we get $\left(A_{O} B_{O} C_{O}\right)$ and $\left(D_{a} D_{b} D_{c}\right)$ are tangent each other.
In the same way as the proof of Theorem 2, we have $A_{0}$ is the center of circle ( $F_{B} B_{A} C_{A}$ ) and $B_{B} C_{C}$ is the perpendicular bisector of $B_{A} C_{A}$. Therefore $B_{B} C_{C}$ is either the internal or external bisector of angle $\angle C_{O} A_{O} B_{O}$.
Similarly, $C_{C} A_{A}, A_{A} B_{B}$ respectively be either the internal or external bisectors of angles $\angle A_{O} B_{O} C_{O}, \angle B_{O} C_{O} A_{O}$.
This means that each of three points $A_{A}, B_{B}, C_{C}$ be either the incenter or excenter of triangle $A_{O} B_{O} C_{O}$. Therefore, $\left(A_{O} B_{O} C_{O}\right)$ is the nine-point circle of triangle $A_{A} B_{B} C_{C}$.
Combining (18) and (19), we obtain the nine-point circle of triangle $A_{A} B_{B} C_{C}$ is tangent to circle $\left(D_{a} D_{b} D_{c}\right)$.
Our proof is completed.
Remark 12. From the proof of Theorem 4, we get the following result:
Let $A B C$ is a triangle and $H$ is its orthocenter. Let $D$ is an arbitrary point, that is different from $H$ and not on $B C, C A, A B$ and $(A B C)$. Then, the nine-point circle of the triangle determined by the lines through the midpoints of $B C, C A, A B$ and respectively be perpendicular to $D A, D B, D C$ is tangent to pedal circle of $D$ with respect to triangle $A B C$.

Remark 13. From Remark 12, we obtain another generalization Feuerbach's theorem:
Let $A B C$ be a triangle and $O$ is its circumcenter. Let $D$ is an arbitrary point, that is different from $O$ and not on the $B C, C A, A B$ and $(A B C)$. Let $D_{a}, D_{b}, D_{c}$ are the orthogonal projections of $D$ onto $B C, C A, A B$ respectively. Let $P$ is the anticomplement of $D$ with respect to triangle $D_{a} D_{b} D_{c}$. Let $k_{a}$ is the line passes through $D_{a}$ and parallel to $D_{b} D_{c}$. We define $k_{b}$ and $k_{c}$ similarly. Then, the pedal circle of $P$ with respect to triangle determined by the lines $k_{a}, k_{b}$ and $k_{c}$ is tangent to nine-point circle of triangle $A B C$.
When $D$ coincides incenter or excenter of triangle $A B C$, the Remark return Feurebach's Theorem.

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