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SOME GENERALIZATIONS OF THE FEUERBACH'S THEOREM

NGUYEN NGOC GIANG AND LE VIET AN

ABSTRACT. The celebrated theorem of Feuerbach states that the nine-point circle of a nonequilateral triangle is tangent to both its incircle and its three excircles. In this note, we give three generalizations of the Feuerbach's Theorem and synthetic proofs for them.

1. INTRODUCTION AND MOTIVATIONS

Feuerbach's Theorem is a well known old theorem [1], [2], [7]. In [1], it is formulated in the following form.

Theorem 1. (Feuerbach, 1822) *In a nonequilateral triangle, the nine-point circle of a triangle is tangent internally to the incircle, and externally to each of the excircles.*

For three points *P*, *Q* and *R* in the plane, the circle with center *P*, the circle with diameter *PQ*, and the circumcircle of triangle *PQR* are denoted by (P), (PQ), and (PQR) respectively.

For historical details of this Theorem 1, please see [1]. In this article, we study some generalizations of theorem 1 as following.

Theorem 2. Let ABC is a triangle and H is its orthocenter. Let D is an arbitrary point, that is different from H and not on BC, CA, AB and (ABC). Let ℓ_a is the line passes through the orthogonal projections of A onto DB and DC. We define ℓ_b and ℓ_c similarly. Then, the circumcircle of the triangle determined by the lines ℓ_a , ℓ_b and ℓ_c is tangent to the pedal circle of D with repect to triangle ABC.

Theorem 3. Let ABC is a triangle and H is its orthocenter. Let D is an arbitrary point, that is different from H and not on BC, CA, AB and (BC), (CA), (AB). Let H_a , H_b and H_c be the orthocenters of the triangles DBC, DCA, and DAB respectively. Then, the circumcircle of the triangle determined by the three radical axes of each of pair of circles ((DH_a), (BC)), ((DH_b), (CA)) and ((DH_c), (AB)) is tangent to the nine-point circle of triangle ABC.

Theorem 4. Let ABC is a triangle and O is its circumcenter. Let D is an arbitrary point, that is different from H and not on BC, CA, AB and (ABC). Let D_a , D_b and D_c are the orthogonal projections of D onto BC, CA, and AB respectively. Then, the nine-point circle of triangle determined by the lines passing through the midpoints of BC, CA and AB respectively and parallel to D_bD_c , D_cD_a and D_aD_b is tangent to $(D_aD_bD_c)$.

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Figure 1.



Figure 2.



Figure 4.

It is directly seen that for *D* is incenter or excenter of triangle *ABC*, the Theorem 2, Theorem 3 and Theorem 4 return Feuerbach's Theorem; hence they are extensions of the Theorem 1, indeed.

In this article, we present the purely synthetic proofs of them. In the proofs, we shall make use of the notion of *directional* angles. As in [3, chapter 2, §§8-14], the directed angle from the line x to the line y denote by (x, y), the directed angle from the non-zero

vector \overrightarrow{x} to the non-zero vector \overrightarrow{y} denote by $(\overrightarrow{x}, \overrightarrow{y})$. To facilitate the readers, we would like to present a few results related to these two concepts that will be used in this article.

(i) For three lines x, y, z, we have $(x, y) \equiv (x, z) + (z, y) \pmod{\pi}$.

(see: [3], Theorem 311 and Attention 312),

(ii) For three vectors \vec{x} , \vec{y} , \vec{z} , we have $(\vec{x}, \vec{y}) \equiv (\vec{x}, \vec{z}) + (\vec{z}, \vec{y}) \pmod{2\pi}$.

(see: [3], Theorem 255 and Attention 256),

(iii) For two lines x, y, we have $(x, y) \equiv -(y, x) \pmod{\pi}$.

(see: [3], Theorem 324), (iv) For two vectors \vec{x} , \vec{y} , we have $(\vec{x}, \vec{y}) \equiv \pi \pmod{2\pi}$ if and only if \vec{x} , \vec{y} have opposite directions.

(see: [3], Theorem 315),

(v) For two lines x, y, we have $(x, y) \equiv 0 \pmod{\pi}$ if and only if x and y are either parallel or coincident.

(see: [3], Theorem 322),

(vi) If the lines x, y, x' such that $(x, y) \equiv (x', y) \pmod{\pi}$, then x and x' are either parallel or coincident.

(see: [3], Theorem 327),

(vii) For two lines x, y, we have $(x, y) \equiv \frac{\pi}{2} \pmod{\pi}$ if and only if $x \perp y$.

(see: [3], Theorem 32),

(viii) If the lines x, y, x', y' such that $x \perp x'$ and $y \perp y'$, then $(x, y) \equiv (x', y') \pmod{\pi}$.

(see: [3], Theorem 326),

(ix) If ABC is a triangle, then the following conditions are equivalent:

1) AB = AC.

2) $(BA, BC) \equiv (CB, CA) \pmod{\pi}$.

3) $2(BA, BC) \equiv (\overrightarrow{BA}, \overrightarrow{AC}) \pmod{2\pi}$.

(see: [4], Theorem 36 and Theorem 37),

(x) If points A, B, C, D are not linear, such that AB = AD and CB = CD, then

a) $(AB, AC) \equiv (AC, AD) \pmod{\pi}$.

b) $(BA, BC) \equiv (DC, DA) \pmod{\pi}$.

(see: [4], Theorem 38 and Theorem 41),

(xi) If A, B are two distinct points belong to circle (O) and M is an arbitrary point, then M belongs to (O) if and only if (MA, MB) $\equiv \frac{1}{2}(\overrightarrow{OA}, \overrightarrow{OB}) \pmod{\pi}$.

(see: [4], Theorem 58),

(xii) If ABC is a triangle and M is an arbitrary point, then M belong to circle (ABC) if and only if $(MA, MB) \equiv (CA, CB) \pmod{\pi}$.

(see: [4], Theorem 59),

(xiii) If ABC is a triangle and t is an arbitrary line, then t is tangent to circle (ABC) if and only if $(t, AB) \equiv (CA, CB) \pmod{\pi}$.

(see: [4], Theorem 61).

2. PRELIMINARY RESULTS

To prove the theorems, we need some lemmas.

Lemma 5. (Miguel point of the compelte quadrilateral). The circumcircles of all four triangles of a complete quadrangle meet a point.

(See: [6], 3.1. Steiner's Theorem 1 and the Miquel Point, p. 38, or [4], problem 85).

Lemma 6. Let ABC is a triangle, and d is an arbitrary line through A and not perpendicular to BC. Let H is the orthogonal projection of A onto BC; let K, L are the orthogonal projections of B, C onto d respectively. Then

(a) The center of circle (HKL) belongs to nine-point circle of triangle ABC.

(b) $(HK, HL) \equiv (AC, AB) \pmod{\pi}$.

Proof. Let us denote M, N, P are the midpoints of BC, CA, AB respectively; I is the center of circle (*HKL*). Then $MN \parallel AB$ and $MP \parallel AC$. (1)



From $AH \perp BC$, $BK \perp d$, $CL \perp d$, we get PH = PK and NH = NL. Combining with IH = IK = IL, we obtain IP, IN are the perpendicular bisectors of HK, HL respectively. (2) Again, from $AH \perp BC$, $BK \perp d$ and $CL \perp d$, we deduce that groups of four points (A, B, K, H) and (A, C, H, L) are concyclic. (3) (a) We have,

$$(IP, IN) \equiv (HK, HL) \pmod{\pi}$$
(by (2))

$$\equiv (HK, AK) + (AL, HL) \pmod{\pi}$$
(by $A \in KL$)

$$\equiv (HB, AB) + (AC, HC) \pmod{\pi}$$
(by (3))

$$\equiv (AC, AB) \pmod{\pi}$$
(by $H \in BC$)

$$\equiv (MP, MN) \pmod{\pi}$$
(by (1)).

It follows that *I* belongs to circle (*MNP*), the nine-point circle of triangle *ABC*. (b) We have,

$$(HK, HL) \equiv (HK, HA) + (HA, HL) \pmod{\pi}$$
$$\equiv (BK, BA) + (CA, CL) \pmod{\pi}$$
$$(by (3))$$
$$\equiv (CA, BA) \pmod{\pi}$$
$$(by BK \parallel CL)$$

Lemma 5 is proved.

Lemma 7. *If any of the three points in A, B, C, and D are not collinear, then the nine- point circles of triangles ABC, BCD, CDA, DAB meet at one point.*

(See: [4], Lemma 2 in problem 88, or [5] 396. Theorem, p. 242, or [8], Theorem 1, 1). This point is variably known as the **Euler**, **Euler-Poncelet**, and **Poncelet** point of four points *A*, *B*, *C*, *D*. Note that this is case when the Poncelet point of four points not define: If *A*, *B*, *C*, and *D* be four points in the plane that form an orthocentric system, which means that each of the four points *A*, *B*, *C*, *D* is the orthocenter of the triangle formed by the other three points, then the triangles *ABC*, *BCD*, *CDA*, *DAB* all share the same nine-point circle, and hence the Poncelet point of the four points *A*, *B*, *C*, *D* cannot be uniquely defined.

Lemma 8. Let four points A, B, C and D be non-orthocentric system, and let H is the orthocenter of triangle ABC. Then the Poncelet point of the points A, B, C, D is simultaneously the Poncelet point of the points H, B, C, D, as well as the Poncelet point of the points H, C, D, A, as well as the Poncelet point of the point of the points H, D, A, B, and belongs to the nine-point circles of triangles HAD, HBD, HCD.

(See: [8], Theorem 3, 1).

Lemma 9. Let four points A, B, C and D be non-orthocentric system, and P is the Poncelet of the points A, B, C, D. Denote X, Y, Z the orthogonal projections of D onto BC, CA, AB, respectively. Let H, K be the orthogonal projections of B, C on to AD respectively. Let I be the center circle (XHK). Then KY, HZ, PI have a common point on the circle (XYZ).

Proof. Let us denote *M* is the midpoint of *CD*. According to Lemma 7, *P* belongs to the nine-point circles of triangles *DBC* and *DCA*. From this, by Lemma 6(a), we get *P* is the second intersection of (*MIX*) and (*MKY*).

Denote by *T* as the second intersection of (PXM) and (XHK). We show that *PI*, *XT*, *KY* are concurrent.

We have,

$$(MT, MK) \equiv (MT, MI) + (MI, MK) \pmod{\pi}$$

$$\equiv (XT, XI) + (MX, MI) \pmod{\pi}$$
 (by $X \in (MTI), IX = IK, MX = MK$)

$$\equiv (TI, TX) + (TX, TI)) \pmod{\pi}$$
 (by $IX = IT; T \in (IXM)$)

$$\equiv (TI, TI) \pmod{\pi}$$

$$\equiv 0 \pmod{\pi}.$$

Therefore *MT* coincides *MK*, i.e. *M* belongs to *KT*.

Let us denote *E* is the orthogonal projection of *B* onto *DY*; and *F* is the intersection of *EH* and *KY*.



Figure 6.

From *B*, *D*, *E*, *H*, *X* all belong to circle (*BD*), and *C*, *D*, *K*, *Y*, *X* all belong to circle (*CD*), we get *X* is the Miquel point of the complete quadrilateral *DKYFEH* (by Lemma 5). Thus, *F* belongs to circle (*XHK*).

Therefore,

$$(PK, PI) \equiv (PK, PM) + (PM, PI) \pmod{\pi}$$

$$\equiv (YK, YM) + (XM, XI) \pmod{\pi}$$
 (by $Y \in (PKM); X \in (PMI)$)

$$\equiv (KM, KY) + (XM, XI) \pmod{\pi}$$
 (by $MK = MY$)

$$\equiv (KM, KF) + (KI, KM) \pmod{\pi}$$
 (by $F \in KY; MX = MK, IX = IK$)

$$\equiv (KI, KF) \pmod{\pi}$$

$$\equiv (FK, FI) \pmod{\pi}$$
 (by $IK = IF$).

It follows that *P* belongs to circle (*FIK*). Therefore *PI*, *XT*, *KY* are concurrent at the radical center of three circles (*IMX*), (*FIK*) and (*XKH*). Similarly, *PI*, *XT*, *HZ* are also concurrent. Hence *PI*, *HZ*, *KY* are concurrent. On the other hand, (HZ, YK) = (HZ, HD) + (DK, YK)(mod =)

$$(HZ, YK) \equiv (HZ, HD) + (DK, YK) \pmod{\pi}$$

$$\equiv (XZ, XD) + (DX, YX) \pmod{\pi}$$

$$\equiv (XZ, XY) \pmod{\pi}.$$

$$(by \ X \in (BD) \equiv (HZD); X \in (DKY))$$

$$\equiv (XZ, XY) \pmod{\pi}.$$

It follows that the intersection point of *HZ* and *KY* is on circle (*XYZ*). The Lemma 9 is proved. \Box

Lemma 10. The Poncelet point of four points A, B, C and D lies on the pedal circle of the point A with respect to triangle BCD, on the pedal circle of the point B with respect to triangle CDA, on the pedal circle of the point C with respect to triangle DAB, on the pedal circle of the point D with respect to triangle ABC.

(See: [8], Theorem 4, 1 or [5], 396. Theorem and 397. Theorem, p. 242).

3. MAIN RESULTS

PROOF OF THEOREM 2

Let us denote A_b , A_c are the orthogonal projections of A onto DB, DC respectively; we difene four points B_c , B_a and C_a , C_b similarly; $A_a = B_a B_c \cap C_a C_b$, $B_b = C_a C_b \cap A_b A_c$ and $C_c = A_b A_c \cap B_c B_a$; Da, D_b , D_c are the orthogonal projections of D onto BC, CA, AB respectively. We show that circles $(A_a B_b C_c)$ and $(D_a D_b D_c)$ are tangent to each other. Denote by $A_0 = B_a D_c \cap C_a D_b$, $B_0 = C_b D_a \cap A_b D_c$ and $C_0 = A_c D_b \cap B_c D_a$.



Figure 7.

According to Lemma 9, the point $A_0 = B_a D_c \cap C_a D_b$ belongs to $(D_a D_b D_c)$, the pedal circle of D with respect to triangle ABC. Similarly, B_0 and C_0 also belong to $(D_a D_b D_c)$. Therefore $(A_0 B_0 C_0)$ and $(D_a D_b D_c)$ are coincident. (4) We have,

$$(B_{a}A_{a}, B_{a}C_{a}) \equiv (B_{a}B_{c}, B_{a}D) \pmod{\pi} \qquad (by \ B_{c} \in B_{a}A_{a}, D \in B_{a}C_{a})$$

$$\equiv (BB_{c}, BD) \pmod{\pi} \qquad (by \ B \in (BD) \equiv (B_{a}B_{c}D))$$

$$\equiv (BB_{c}, BC_{b}) \pmod{\pi} \qquad (by \ C_{b} \in BD)$$

$$\equiv (CB_{c}, CC_{b}) \pmod{\pi} \qquad (by \ C \in (BC) \equiv (BB_{c}C_{b}))$$

$$\equiv (CD, CC_{b}) \pmod{\pi} \qquad (by \ D \in CB_{c})$$

$$\equiv (C_{a}D, C_{a}C_{b}) \pmod{\pi} \qquad (by \ B_{a} \in C_{a}D, A_{a} \in C_{a}C_{b}).$$

It follows that $A_a B_a = A_a C_a$. (5) Since,

$$(A_0B_0, A_0D_b) \equiv (D_aB_0, D_aD_b) \pmod{\pi}$$

$$\equiv (D_aC_b, D_aD_b) \pmod{\pi}$$

$$\equiv (C_aC_b, C_aD_b) \pmod{\pi}$$

$$\equiv (A_aB_b, A_0D_b) \pmod{\pi}$$

$$(by C_a \in (CD) \equiv (D_aC_bD_b) \pmod{\pi}$$

$$(by A_aB_b = C_aC_b, A_0 \in C_aD_b)$$

We decude that A_0B_0 and A_aB_b are either parallel or coincident.

Similarly, B_0C_0 and B_bC_c are either parallel or coincident; C_0A_0 and C_cA_a are either parallel or coincident.

Thus, triangles $A_0B_0C_0$ and $A_aB_bC_c$ have correspondingly parallel sides or coincident sides.

This means that the triangles $A_0B_0C_0$ and $A_aB_bC_c$ are images of each other through a homothety or a translation. (6)

Let us denote *P* is the Poncelet point of four points *A*, *B*, *C* and *D*. According to Lemma 10, we have *P* belongs to circle $(D_a D_b D_c)$. (7) We have,

$$\begin{aligned} (A_a B'_a, A_a C'_a) &\equiv (A_a B'_a, B_a A'_a) + (B_a A'_a, A_a C'_a) (\text{mod } 2\pi) \\ &\equiv \pi + 2(B_a A_a, B_a C_a) (\text{mod } 2\pi) \\ &\equiv \pi + 2(B_a B_c, B_a D) (\text{mod } 2\pi) \\ &\equiv 2\frac{\pi}{2} + 2(BB_c, BD) (\text{mod } 2\pi) \\ &\equiv 2(CC_b, BD) + 2(CD, CC_b) (\text{mod } 2\pi) \\ &\equiv 2(CD, BD) (\text{mod } 2\pi) \\ &\equiv 2(D_a B_a, D_a C_a) (\text{mod } 2\pi) \end{aligned}$$

$$\begin{aligned} &(\text{by } B \in (BD) \equiv (B_a B_c D)) \\ &(\text{by } CC_b \perp BD, BB_c \perp CD) \\ &(\text{by } Lemma 6(b)). \end{aligned}$$

From this, combining with (5) we obtain A_a is the center of circle $(D_a B_a C_a)$. According to Lemma 9 for the points A, B, C and D, the lines $B_a D_c, C_a D_b$ and PA_a are concurrent. Hence P belongs to the line $A_a A_0$.

Similarly, *P* also belongs to the lines B_bB_0 and C_cC_0 . This means that A_aA_0, B_bB_0 and C_cC_0 are concurrent at the point *P*. From (6), we get *P* is the center of the homothety that transforming $A_aB_bC_c$ to $A_0B_0C_0$. But *P* belongs to $(A_0B_0C_0)$ by (7), this implies that the circles $(A_aB_bC_a)$ and $(A_0B_0C_0)$ are tangent at *P*. And by (4), we decude that $(A_aB_bC_c)$ is tangent to $(D_aD_bD_c)$.

Our proof is completed.

PROOF OF THEOREM 3

Suppose that (*E*) is the nine-point circle of triangle *ABC*; let us denote AA^b, BB^b, CC^c are the altitudes of triangle *ABC*; *BB^c*, *CC^a* are the altitudes of triangle *H_aBC*. We define *CC^b*, *AA^c* and *AA^b*, *BB^a* similarly; *M_a*, *M_b*, *M_c* are the midpoints of *BC*, *CA*, *AB* respectively.

It is easy to see that A^a , B^b , C^c , M_a , M_b , M_c all belong to circle (*E*), and B^aC^a , C^bA^b , A^cB^c respectively are the radical axes of groups of two circles ((*DH*_a), (*BC*)), (*DH*_b), (*CA*)), ((*DH*_C), (*AB*)).

Denote by $A^{0} = A^{c}B^{c} \cap A^{b}C^{b}$, $B^{0} = B^{a}C^{a} \cap B^{c}A^{c}$, $C^{0} = C^{b}A^{b} \cap C^{a}B^{a}$. We show that $(A^{0}B^{0}C^{0})$ is tangent to (E).

Indeed, let us denote O_a, O_b, O_c are the centers of circles $(A^a B^a C^a), (B^b C^b A^b), (C^c A^c B^c)$ respectively. By Lemma 6(a), the points O_a, O_b, O_c all belong to (E). It follows that $(O_a O_b O_c)$ and (E) are coincident. (8)

From AB^c and CB^a are all perpendicular to B^aB^c , we get the perpendicular bisector of B^aB^c passes through the midpoint of AC. Similarly, the perpendicular bisector of B^aB^c also passes through midpoint of AH_a . Denote by N_a is the midpoint of AH_a , then the points O_b , M_b , N_a all belong to the perpendicular bisector of B^aB^c . Thus N_a , O_b , M_b are collinear.

Similarly, *N*_{*a*}, *O*_{*c*}, *M*_{*c*} are also collinear. Therefore,

$$\begin{array}{ll} (O_b N_a, O_b O_c) \equiv (O_b M_b, O_b O_c) (\text{mod } \pi) & (\text{by } M_b \in N_a O_b) \\ \equiv (M_c M_b, M_c O_c) (\text{mod } \pi) & (\text{by } O_b, O_c, M_b, M_c \in (E)) \\ \equiv (BC, BH_a) (\text{mod } \pi) & (\text{by } M_c M_b \parallel BC, M_c O_c \parallel BH_a) \\ \equiv (BC, BC^a) (\text{mod } \pi) & (\text{by } C_a \in BH_a) \\ \equiv (O_b N_a, B^0 C^0) (\text{mod } \pi) & (\text{by } B^a C \parallel O_b N_a, B^0 C^0 \equiv B^a C^a). \end{array}$$

Hence, $O_b O_c$ and $B^0 C^0$ are either parallel or coincident.

Similarly, $O_a O_b$ and $A^0 B^0$ are either parallel or coincident; $O_c O_a$ and $C^0 A^0$ are either parallel or coincident.

Thus, triangles $O_a O_b O_c$ and $A^0 B^0 C^0$ have correspondingly parallel sides or coincident sides.

This means that the triangle $O_a O_b O_c$ and $A^0 B^0 C^0$ are images of each other through a homothety or a translation. (9)

Denote by *P* is the Poncelet point of the points *A*, *B*, *C* and *D*.

Cause *D* is the orthocenter of triangle H_cAB and from Lemma 8, we get *P* is the Poncelet point of the points H_c , *A*, *B*, *D*.

On the other hand, *A* is orthocenter of triangle BDH_b . According to Lemma 8, the Poncelet point of four points *A*, *D*, *H*_b, *H*_c simultaneously the Poncelet point of four points *A*, *B*, *D*, *H*_c. Hence, *P* is the Poncelet point of four points *A*, *D*, *H*_b, *H*_c.



Figure 8.

According to Lemma 9 for the points A, D, H_b, H_c , the lines $A^b C^b, A^c B^c$ and $O_a P$ are concurrent. Hence, *P* belongs to $O_a A^0$.

Similarly, the point *P* also belongs to circles $O_h B^0$ and $O_c C^0$.

Therefore, P is common point of the lines $O_a A^0, O_b B^0, O_c C^0$, combining with (9), we obtain P is the center of the homothety that transforming $O_a O_b O_c$ to $A^0 B^0 C^0$. This means that the circles $(O_a O_b O_c)$ and $(A^0 B^0 C^0)$ are tangent with P is tangent ypoint. From (8), we get $(A^0B^0C^0)$ is tangent to (E). Our proof is completed.

Remark 11. The pedal circle of D with respect to triangle $H_aH_bH_c$ is tangent to nine-point circle of triangle ABC at point P.

Proof. Let us denote D^a , D^b , D^c are the orthogonal projections of D onto H_bH_c , H_cH_a , H_aH_b respectively; We show that $(D^a D^b D^c)$ and (E) are tangent to each other.

Indeed, from P is the Poncelet point of four points A, D, H_b, H_c , we get P belongs to the nine-point circle of triangle DH_bH_c .

Similarly, P also belongs to the nine-point circles of the triangles DH_cH_a and DH_aH_b . Thus, from Lemma 7, we get P is the Poncelet point of the points D, H_a, H_b, H_c . And from Lemma 10, we get P belongs to the pedal circle of D with respect to triangle $H_a H_b H_c$. Hence *P* belongs to circle $(D^a D^b D^c)$. (10)

According to Lemma 10, *P* belongs to the pedal circle of *A* with respect to triangle *BCD*. From P is the Poncelet point of four points A, D, H_b, H_c and by Lemma 10, we get P



Figure 9.

belongs to the pedal circle of *D* with respect to triangle AH_bH_c . According to Lemma 9, the point $A^0 = A^cB^c \cap A^bC^b$ belongs to the pedal circle of *D* with respect to triangle AH_bH_c .

Hence, P, A^0 , A^a , D^a , B^c , C^b are concyclic. (11) Similarly, P, B^0 , B^b , D^b , C^a , A^c are also concyclic. (12) Drawing the tangent line Pt at P of circle (E), then by Theorem 3, Pt is tangent line at P of circle ($A^0B^0C^0$). (13) Therefore,

$$(Pt, PD^{a}) \equiv (Pt, PA^{0}) + (PA^{0}, PD^{a}) \pmod{\pi}$$

$$\equiv (B^{0}P, B^{0}A^{0}) + (C^{b}A^{0}, C^{b}D^{a}) \pmod{\pi}$$

$$\equiv (B^{0}P, B^{0}A^{c}) + (C^{b}A^{b}, C^{b}D^{a}) \pmod{\pi}$$

$$\equiv (D^{b}P, D^{b}A^{c}) + (DA^{b}, DD^{a}) \pmod{\pi}$$

$$\equiv (D^{b}P, D^{b}A^{c}) + (H_{b}A^{b}, H_{b}D^{a}) \pmod{\pi}$$

$$\equiv (D^{b}P, D^{b}A^{c}) + (H_{c}A^{c}, H_{c}D^{a}) \pmod{\pi}$$

$$\equiv (D^{b}P, D^{b}A^{c}) + (D^{b}A^{c}, D^{b}D^{a}) \pmod{\pi}$$

$$\equiv (D^{b}P, D^{b}D^{a}) \pmod{\pi}.$$

Therefore, *Pt* is tangent to (PD^aD^b) . From this, combining with (10), we obtain $(D^aD^bD^c)$ and (*E*) are tangent to each other at *P*.

The Remark is proved.

PROOF OF THEOREM 4

Let us denote M_a , M_b , M_c are the midpoints of BC, CA, AB respectively; the line passes through M_b and parallel to D_cD_a , and the line passes through M_c and parallel to D_aD_b meet in A_A ; we define B_B and C_C similarly. We show that the nine-point circle of triangle $A_AB_BC_C$ and $(D_aD_bD_c)$ are tangent to each other.

Denote by A', B', C' are the reflections of A, B, C respectively in A_A, B_B, C_C .



Figure 10.

From A_A , M_b , M_c respectively are the midpoints of AA', CA, AB, we get $A'B \parallel A_AM_c$ and $A'C \parallel A_AM_b$. (14)

Since *D* do not belongs to *BC*, *CA*, *AB* and (*ABC*), it follows that there exist isogonal conjugate of *D* with respect to triangle *ABC*. Let *F* is the isogonal conjugate of *D* with respect to triangle *ABC*, then *FB* and *FC* are respectively perpendicular to $D_c D_a$, $D_a D_b$ (well-known). (15)

From (14) and (15), we decude that A' is the orthocenter of triangle *EBC*. Similarly, B', C' respectively are the orthocenter of triangles *FCA*, *FAB*. Denote by B_A , C_A are the intersections of *AE* respectively with *BC'*, *CB'*; we define C_B , A_B and A_C , B_C similarly; and let $A_O = B_C B_A \cap C_A C_B$, $B_O = C_A C_B \cap A_B A_C$, $C_O = A_B A_C \cap B_C B_A$. Let us denote F_a , F_b , F_c are the orthogonal projections of F onto BC, CA, AB respectively. By Theorem 2, the pedal circle of F with respect to triangle ABC is tangen to $(A_OB_OC_O)$. (16)

On the other hand, from *D* and *F* are isogonal conjugate in triangle *ABC*, we get the circle $(D_a D_b D_c)$ is also the pedal circle of *F* with respect to triangle *ABC*. (17)

From (16) and (17), we get $(A_OB_OC_O)$ and $(D_aD_bD_c)$ are tangent each other. (18) In the same way as the proof of Theorem 2, we have A_O is the center of circle $(F_aB_AC_A)$, and B_BC_C is the perpendicular bisector of B_AC_A . Therefore B_BC_C is either the internal or external bisector of angle $\angle C_OA_OB_O$.

Similarly, $C_C A_A$, $A_A B_B$ respectively be either the internal or external bisectors of angles $\angle A_O B_O C_O$, $\angle B_O C_O A_O$.

This means that each of three points A_A , B_B , C_C be either the incenter or excenter of triangle $A_OB_OC_O$. Therefore, $(A_OB_OC_O)$ is the nine-point circle of triangle $A_AB_BC_C$. (19) Combining (18) and (19), we obtain the nine-point circle of triangle $A_AB_BC_C$ is tangent to circle $(D_aD_bD_c)$.

Our proof is completed.

Remark 12. From the proof of Theorem 4, we get the following result:

Let ABC is a triangle and H is its orthocenter. Let D is an arbitrary point, that is different from H and not on BC, CA, AB and (ABC). Then, the nine-point circle of the triangle determined by the lines through the midpoints of BC, CA, AB and respectively be perpendicular to DA, DB, DC is tangent to pedal circle of D with respect to triangle ABC.

Remark 13. From Remark 12, we obtain another generalization Feuerbach's theorem:

Let ABC be a triangle and O is its circumcenter. Let D is an arbitrary point, that is different from O and not on the BC, CA, AB and (ABC). Let D_a , D_b , D_c are the orthogonal projections of D onto BC, CA, AB respectively. Let P is the anticomplement of D with respect to triangle $D_aD_bD_c$. Let k_a is the line passes through D_a and parallel to D_bD_c . We define k_b and k_c similarly. Then, the pedal circle of P with respect to triangle determined by the lines k_a , k_b and k_c is tangent to nine-point circle of triangle ABC.

When *D* coincides incenter or excenter of triangle *ABC*, the Remark return Feurebach's Theorem.

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Banking University of Ho Chi Minh City, 36 Ton That Dam street, district 1, Ho Chi Minh City, Vietnam

Email address: nguyenngocgiang.net@gmail.com

VNU-HCM High School for the Gifted, 153 Nguyen Chi Thanh, district 7, Ho Chi Minh City, Vietnam.

Email address: levietan.spt@gmail.com