Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.12, (2023), Issue 1, pp.71-99

THE GENERALIZED NONCOMMUTATIVE RESIDUE AND THE KASTLER-KALAU-WALZE TYPE THEOREM

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ABSTRACT. In this paper, we define the generalized noncommutative residue of the Dirac operator. And we give the proof of the Kastler-Kalau-Walze type theorem for the generalized noncommutative residue on 4-dimensional and 6-dimensional compact manifolds with (resp.without) boundary.

1. INTRODUCTION

Until now, many geometers have studied noncommutative residues. In [6, 18], authors found noncommutative residues are of great importance to the study of noncommutative geometry. In [2], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Connes showed us that the noncommutative residue on a compact manifold M coincided with the Dixmier's trace on pseudodifferential operators of order $-\dim M$ in [3]. And Connes claimed the noncommutative residue of the square of the inverse of the Dirac operator was proportioned to the Einstein-Hilbert action. Kastler [9] gave a brute-force proof of this theorem. Kalau and Walze proved this theorem in the normal coordinates system simultaneously in [8] . Ackermann proved that the Wodzicki residue of the square of the inverse of the Dirac operator $\mathrm{Wres}(D^{-2})$ in turn is essentially the second coefficient of the heat kernel expansion of D^2 in [1].

On the other hand, Wang generalized the Connes' results to the case of manifolds with boundary in [14, 15], and proved the Kastler-Kalau-Walze type theorem for the Dirac operator and the signature operator on lower-dimensional manifolds with boundary [16]. In [16, 17], Wang computed $\widetilde{\mathrm{Wres}}[\pi^+D^{-1}\circ\pi^+D^{-1}]$ and $\widetilde{\mathrm{Wres}}[\pi^+D^{-2}\circ\pi^+D^{-2}]$, where the two operators are symmetric, in these cases the boundary term vanished. But for $\widetilde{\mathrm{Wres}}[\pi^+D^{-1}\circ\pi^+D^{-3}]$, Wang got a nonvanishing boundary term [13], and give a theoretical explanation for gravitational action on boundary. In others words, Wang provides a kind of method to study the Kastler-Kalau-Walze type theorem for manifolds with boundary. In [11], Moscovici computed $\mathrm{tr}[c(\omega)e^{-tD^2}]$ for a form ω and the Dirac operator D, and used this formula to derive information about the asymptotic distribution of multiplicities in the quasi-regular representation of semisimple Lie grroup modulo a

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2010 *Mathematics Subject Classification.* 53C40; 53C42.

Key words and phrases. The generalized noncommutative residue; The Dirac operator; The Kastler-Kalau-Walze type theorem.

compact discrete subgroup. In [10], Mickelsson and Paycha computed the index of the Dirac operator by the super noncommutative residue. Motivated by [11] and [10], we define the generalized noncommutative residue for the Dirac operator which is the generalization of the noncommutative residue and the super noncommutative residue.
The motivation of this paper is to compute the generalized noncommutative residue $\widetilde{\mathrm{Wres}}[\pi^+(LD^{-1})\circ\pi^+D^{-1}]$ and $\widetilde{\mathrm{Wres}}[\pi^+(LD^{-2})\circ\pi^+D^{-2}]$, where $L=c(X_1)c(X_2)c(X_3)\cdots c(X_l)$, and prove the Kastler-Kalau-Walze type theorem for the generalized noncommutative residue on 4-dimensional and 6-dimensional compact manifolds.
The paper is organized in the following way. In Section 2, we define the generalized noncommutative residue and get the Kastler-Kalau-Walze type theorem for the generalized noncommutative residue on manifolds without boundary. In Section 3 and in Section 4, we prove the Kastler-Kalau-Walze type theorem for the generalized noncommutative residue on 4-dimensional and 6-dimensional manifolds with boundary respectively.

2. The Dirac operator and its generalized noncommutative residue

Firstly we recall the definition of Dirac operator. Let M be an n-dimensional ($n\geq 3$) oriented compact Riemannian manifold with a Riemannian metric g^M and let ∇^L be the Levi-Civita connection about g^M. In the fixed orthonormal frame $\{e_1,\cdots,e_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(e_1,\cdots,e_n)=(e_1,\cdots,e_n)(\omega_{s,t}). \tag{2.1}$$

Let $\epsilon(e_j^*)$, $\iota(e_j^*)$ be the exterior and interior multiplications respectively, where $e_j^*=g^{TM}(e_j,\cdot)$. Write

$$\widehat{c}(e_j)=\epsilon(e_j^*)+\iota(e_j^*);\ \ c(e_j)=\epsilon(e_j^*)-\iota(e_j^*), \tag{2.2}$$

which satisfies

$$\begin{aligned}
&\widehat{c}(e_i)\widehat{c}(e_j)+\widehat{c}(e_j)\widehat{c}(e_i)=2g^M(e_i,e_j);\\
&c(e_i)c(e_j)+c(e_j)c(e_i)=-2g^M(e_i,e_j);\\
&c(e_i)\widehat{c}(e_j)+\widehat{c}(e_j)c(e_i)=0.
\end{aligned} \tag{2.3}$$

By [19], we have the Dirac operator

$$D=\sum_{i=1}^n c(e_i)[e_i-\frac{1}{4}\sum_{s,t}\omega_{s,t}(e_i)c(e_s)c(e_t)]. \tag{2.4}$$

Set $L=c(X_1)c(X_2)c(X_3)\cdots c(X_l)$, where $X_j=\sum_{\alpha=1}^n a_{j\alpha}e_\alpha$, $1\leq j\leq l$, is a smooth vector field. We define the generalized noncommutative residue of $(D^2)^{-\frac{n-2}{2}}$ by $\mathrm{Wres}[L(D^2)^{-\frac{n-2}{2}}]$. When $L=c(e_1)\cdots c(e_n)$, we get the super noncommutative residue of $(D^2)^{-\frac{n-2}{2}}$. When $X_1=X_2$, and $l=2$, $|X_1|=1$, we get $\mathrm{Wres}[(D^2)^{-\frac{n-2}{2}}]$.
By (4.30) in [8], we have

$$\int_{|\xi|=1}\sigma_{-n}(D^{-n+2})(x,\xi)d\xi=\frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!}(-\frac{1}{12}s),$$

(2.5)

where s is the scalar curvature. Then we have the following theorem

Theorem 2.1. *If M is an n-dimensional compact oriented manifolds without boundary, and n is even, then we get the following equality:*

$$\mathrm{Wres}[L(D^2)^{-\frac{n-2}{2}}]=\frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!}\int_M\Big(-\frac{1}{12}\mathrm{tr}[c(X_1)c(X_2)\cdots c(X_l)]s\Big)d\mathrm{Vol_M}. \quad (2.6)$$

Now, we need to compute $\mathrm{tr}[c(X_1)c(X_2)\cdots c(X_l)]$ by case. Obviously, we know when l is odd, $\mathrm{tr}[c(X_1)c(X_2)\cdots c(X_l)]$=0. In the following, we compute $\mathrm{tr}[c(X_1)c(X_2)]$, $\mathrm{tr}[c(X_1)c(X_2)c(X_3)c(X_4)]$ and $\mathrm{tr}[c(X_1)c(X_2)c(X_3)c(X_4)c(X_5)c(X_6)]$.
case$(\mathbf{1})$
By $X_j=\sum_{\alpha=1}^n a_{j\alpha}e_\alpha$ and (2.3), we have

$$\mathrm{tr}[c(X_1)c(X_2)]=\sum_{\alpha,\beta=1}^n a_{1\alpha}a_{2\beta}\mathrm{tr}[c(e_\alpha)c(e_\beta)]=-\sum_{\alpha=1}^n a_{1\alpha}a_{2\alpha}\mathrm{tr}[\mathtt{id}]=-g(X_1,X_2)\mathrm{tr}[\mathtt{id}]. \quad (2.7)$$

case$(\mathbf{2})$

$$\mathrm{tr}[c(X_1)c(X_2)c(X_3)c(X_4)]=\sum_{\alpha,\beta,\gamma,\mu=1}^n a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)]. \quad (2.8)$$

case$(\mathbf{2-a})$ When $\alpha=\beta,\gamma=\mu$.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu=1}^n a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)]=\sum_{\alpha,\gamma=1}^n a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\gamma}\mathrm{tr}[\mathtt{id}]. \quad (2.9)$$

case$(\mathbf{2-b})$ When $\alpha\neq\beta,\alpha=\gamma,\beta=\mu$.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu=1}^n a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)]=-\sum_{\alpha\neq\beta=1}^n a_{1\alpha}a_{2\beta}a_{3\alpha}a_{4\beta}\mathrm{tr}[\mathtt{id}]. \quad (2.10)$$

case$(\mathbf{2-c})$ When $\alpha\neq\beta,\alpha=\mu,\beta=\gamma$.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu=1}^n a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)]=\sum_{\alpha\neq\beta=1}^n a_{1\alpha}a_{2\beta}a_{3\beta}a_{4\alpha}\mathrm{tr}[\mathtt{id}]. \quad (2.11)$$

case$(\mathbf{2-d})$ Other cases.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)]=0. \tag{2.12}$$

Therefore

$$\begin{aligned}
&\mathrm{tr}[c(X_1)c(X_2)c(X_3)c(X_4)]\\
&=(\sum_{\alpha,\gamma=1}^{n} a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\gamma}-\sum_{\alpha\neq\beta=1}^{n} a_{1\alpha}a_{2\beta}a_{3\alpha}a_{4\beta}+\sum_{\alpha\neq\beta=1}^{n} a_{1\alpha}a_{2\beta}a_{3\beta}a_{4\alpha})\mathrm{tr}[\mathtt{id}]\\
&=(\sum_{\alpha,\gamma=1}^{n} a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\gamma}-\sum_{\alpha,\beta=1}^{n} a_{1\alpha}a_{2\beta}a_{3\alpha}a_{4\beta}+\sum_{\alpha=\beta=1}^{n} a_{1\alpha}a_{2\beta}a_{3\alpha}a_{4\beta}\\
&+\sum_{\alpha,\beta=1}^{n} a_{1\alpha}a_{2\beta}a_{3\beta}a_{4\alpha}-\sum_{\alpha=\beta=1}^{n} a_{1\alpha}a_{2\beta}a_{3\alpha}a_{4\beta})\mathrm{tr}[\mathtt{id}]\\
&=(\sum_{\alpha,\gamma=1}^{n} a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\gamma}-\sum_{\alpha,\beta=1}^{n} a_{1\alpha}a_{2\beta}a_{3\alpha}a_{4\beta}+\sum_{\alpha,\beta=1}^{n} a_{1\alpha}a_{2\beta}a_{3\beta}a_{4\alpha})\mathrm{tr}[\mathtt{id}]\\
&=[g(X_1,X_2)g(X_3,X_4)-g(X_1,X_3)g(X_2,X_4)+g(X_1,X_4)g(X_2,X_3)]\mathrm{tr}[\mathtt{id}].
\end{aligned} \tag{2.13}$$

case$(\mathbf{3})$

$$\begin{aligned}
&\mathrm{tr}[c(X_1)c(X_2)c(X_3)c(X_4)c(X_5)c(X_6)]\\
&=\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)].
\end{aligned} \tag{2.14}$$

case$(\mathbf{3-a})$ When $\alpha=\beta,\gamma=\mu,\delta=\nu$.
By (2.3), we have

$$\begin{aligned}
&\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]\\
&=-\sum_{\alpha,\gamma,\delta=1}^{n} a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\gamma}a_{5\delta}a_{6\delta}\mathrm{tr}[\mathtt{id}].
\end{aligned} \tag{2.15}$$

case$(\mathbf{3-b})$ When $\alpha=\beta,\gamma\neq\mu,\gamma=\delta,\mu=\nu$.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]$$

$$= \sum_{\alpha,\gamma\neq\mu=1}^{n} a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\mu}a_{5\gamma}a_{6\mu}\mathrm{tr}[\mathtt{id}]. \tag{2.16}$$

case($\mathbf{3-c}$) When $\alpha=\beta,\gamma\neq\mu,\gamma=\nu,\delta=\mu$.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]$$
$$= -\sum_{\alpha,\gamma\neq\mu=1}^{n} a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\mu}a_{5\mu}a_{6\gamma}\mathrm{tr}[\mathtt{id}]. \tag{2.17}$$

case($\mathbf{3-d}$) When $\alpha\neq\beta,\alpha=\gamma,\beta=\mu,\delta=\nu$.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]$$
$$= \sum_{\alpha\neq\beta,\delta=1}^{n} a_{1\alpha}a_{2\beta}a_{3\alpha}a_{4\beta}a_{5\delta}a_{6\delta}\mathrm{tr}[\mathtt{id}]. \tag{2.18}$$

case($\mathbf{3-e}$) When $\alpha\neq\beta,\alpha=\gamma,\beta\neq\mu,\beta=\delta,\mu=\nu$.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]$$
$$= -\sum_{\alpha\neq\beta,\beta\neq\mu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\alpha}a_{4\mu}a_{5\beta}a_{6\mu}\mathrm{tr}[\mathtt{id}]. \tag{2.19}$$

case($\mathbf{3-f}$) When $\alpha\neq\beta,\alpha=\gamma,\beta\neq\mu,\beta=\nu,\mu=\delta$.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]$$
$$= \sum_{\alpha\neq\beta,\beta\neq\mu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\alpha}a_{4\mu}a_{5\mu}a_{6\beta}\mathrm{tr}[\mathtt{id}]. \tag{2.20}$$

case($\mathbf{3-g}$) When $\alpha\neq\beta,\alpha=\mu,\beta=\gamma,\delta=\nu$.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]$$

$$= -\sum_{\alpha\neq\beta,\delta=1}^{n} a_{1\alpha}a_{2\beta}a_{3\beta}a_{4\alpha}a_{5\delta}a_{6\delta}\mathrm{tr}[\mathtt{id}]. \tag{2.21}$$

case$(\mathbf{3-h})$ When $\alpha\neq\beta\neq\gamma,\alpha=\mu,\beta=\delta,\gamma=\nu$.
By (2.3), we have

$$\begin{aligned}&\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]\\&=\sum_{\alpha\neq\beta\neq\gamma=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\alpha}a_{5\beta}a_{6\gamma}\mathrm{tr}[\mathtt{id}].\end{aligned} \tag{2.22}$$

case$(\mathbf{3-i})$ When $\alpha\neq\beta\neq\gamma,\alpha=\mu,\beta=\nu,\delta=\gamma$.
By (2.3), we have

$$\begin{aligned}&\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]\\&=-\sum_{\alpha\neq\beta\neq\gamma=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\alpha}a_{5\gamma}a_{6\beta}\mathrm{tr}[\mathtt{id}].\end{aligned} \tag{2.23}$$

case$(\mathbf{3-j})$ When $\alpha\neq\beta,\alpha\neq\mu,\alpha=\delta,\beta=\gamma,\mu=\nu$.
By (2.3), we have

$$\begin{aligned}&\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]\\&=\sum_{\alpha\neq\beta,\alpha\neq\mu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\beta}a_{4\mu}a_{5\alpha}a_{6\mu}\mathrm{tr}[\mathtt{id}].\end{aligned} \tag{2.24}$$

case$(\mathbf{3-k})$ When $\alpha\neq\beta\neq\gamma,\alpha=\delta,\beta=\mu,\gamma=\nu$.
By (2.3), we have

$$\begin{aligned}&\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]\\&=-\sum_{\alpha\neq\beta\neq\gamma=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\beta}a_{5\alpha}a_{6\gamma}\mathrm{tr}[\mathtt{id}].\end{aligned} \tag{2.25}$$

case$(\mathbf{3-l})$ When $\alpha\neq\beta\neq\gamma,\alpha=\delta,\beta=\nu,\mu=\gamma$.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]$$

$$= \sum_{\alpha\neq\beta\neq\gamma=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\gamma}a_{5\alpha}a_{6\beta}\mathrm{tr}[\mathtt{id}]. \tag{2.26}$$

case(3 − m) When $\alpha\neq\beta,\alpha\neq\mu,\alpha=\nu,\beta=\gamma,\mu=\delta$.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]$$
$$= -\sum_{\alpha\neq\beta,\alpha\neq\mu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\beta}a_{4\mu}a_{5\mu}a_{6\alpha}\mathrm{tr}[\mathtt{id}]. \tag{2.27}$$

case(3 − n) When $\alpha\neq\beta\neq\gamma,\alpha=\nu,\beta=\mu,\gamma=\delta$.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]$$
$$= \sum_{\alpha\neq\beta\neq\gamma=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\beta}a_{5\gamma}a_{6\alpha}\mathrm{tr}[\mathtt{id}]. \tag{2.28}$$

case(3 − o) When $\alpha\neq\beta\neq\gamma,\alpha=\nu,\beta=\delta,\gamma=\mu$.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)]$$
$$= -\sum_{\alpha\neq\beta\neq\gamma=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\gamma}a_{5\beta}a_{6\alpha}\mathrm{tr}[\mathtt{id}]. \tag{2.29}$$

case(3 − p) Other cases.
By (2.3), we have

$$\sum_{\alpha,\beta,\gamma,\mu,\delta,\nu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\mu}a_{5\delta}a_{6\nu}\mathrm{tr}[c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(e_\delta)c(e_\nu)] = 0. \tag{2.30}$$

Similar to (2.13), we have

$$-\sum_{\alpha,\gamma,\delta=1}^{n} a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\gamma}a_{5\delta}a_{6\delta}\mathrm{tr}[\mathtt{id}] = -g(X_1,X_2)g(X_3,X_4)g(X_5,X_6)\mathrm{tr}[\mathtt{id}], \tag{2.31}$$

$$\sum_{\alpha,\gamma\neq\mu=1}^{n} a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\mu}a_{5\gamma}a_{6\mu}\mathrm{tr}[\mathtt{id}] - \sum_{\alpha,\gamma\neq\mu=1}^{n} a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\mu}a_{5\mu}a_{6\gamma}\mathrm{tr}[\mathtt{id}]$$

$$
\begin{aligned}
&= \sum_{\alpha,\gamma,\mu=1}^{n} a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\mu}a_{5\gamma}a_{6\mu}\mathrm{tr}[\mathtt{id}] - \sum_{\alpha,\gamma=\mu=1}^{n} a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\gamma}a_{5\gamma}a_{6\gamma}\mathrm{tr}[\mathtt{id}] \\
&- \sum_{\alpha,\gamma,\mu=1}^{n} a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\mu}a_{5\mu}a_{6\gamma}\mathrm{tr}[\mathtt{id}] + \sum_{\alpha,\gamma=\mu=1}^{n} a_{1\alpha}a_{2\alpha}a_{3\gamma}a_{4\gamma}a_{5\gamma}a_{6\gamma}\mathrm{tr}[\mathtt{id}] \\
&= g(X_1,X_2)[g(X_3,X_5)g(X_4,X_6) - g(X_3,X_6)g(X_4,X_5)]\mathrm{tr}[\mathtt{id}],
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
&\sum_{\alpha\neq\beta,\delta=1}^{n} a_{1\alpha}a_{2\beta}a_{3\alpha}a_{4\beta}a_{5\delta}a_{6\delta}\mathrm{tr}[\mathtt{id}] - \sum_{\alpha\neq\beta,\delta=1}^{n} a_{1\alpha}a_{2\beta}a_{3\beta}a_{4\alpha}a_{5\delta}a_{6\delta}\mathrm{tr}[\mathtt{id}] \\
&\quad= [g(X_1,X_3)g(X_2,X_4) - g(X_1,X_4)g(X_2,X_3)]g(X_5,X_6)\mathrm{tr}[\mathtt{id}],
\end{aligned}
\tag{2.33}
$$

$$
\begin{aligned}
&\sum_{\alpha\neq\beta,\beta\neq\mu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\alpha}a_{4\mu}a_{5\mu}a_{6\beta}\mathrm{tr}[\mathtt{id}] - \sum_{\alpha\neq\beta,\alpha\neq\mu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\beta}a_{4\mu}a_{5\mu}a_{6\alpha}\mathrm{tr}[\mathtt{id}] \\
&+ \sum_{\alpha\neq\beta,\alpha\neq\mu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\beta}a_{4\mu}a_{5\alpha}a_{6\mu}\mathrm{tr}[\mathtt{id}] - \sum_{\alpha\neq\beta,\beta\neq\mu=1}^{n} a_{1\alpha}a_{2\beta}a_{3\alpha}a_{4\mu}a_{5\beta}a_{6\mu}\mathrm{tr}[\mathtt{id}] \\
&= \Big\{g(X_1,X_3)[g(X_2,X_6)g(X_4,X_5) - g(X_2,X_5)g(X_4,X_6)] + g(X_2,X_3)[g(X_1,X_5)g(X_4,X_6) \\
&- g(X_1,X_6)g(X_4,X_5)]\Big\}\mathrm{tr}[\mathtt{id}],
\end{aligned}
\tag{2.34}
$$

and

$$
\begin{aligned}
&\sum_{\alpha\neq\beta\neq\gamma=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\alpha}a_{5\beta}a_{6\gamma}\mathrm{tr}[\mathtt{id}] - \sum_{\alpha\neq\beta\neq\gamma=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\alpha}a_{5\gamma}a_{6\beta}\mathrm{tr}[\mathtt{id}] \\
&- \sum_{\alpha\neq\beta\neq\gamma=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\beta}a_{5\alpha}a_{6\gamma}\mathrm{tr}[\mathtt{id}] + \sum_{\alpha\neq\beta\neq\gamma=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\gamma}a_{5\alpha}a_{6\beta}\mathrm{tr}[\mathtt{id}] \\
&+ \sum_{\alpha\neq\beta\neq\gamma=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\beta}a_{5\gamma}a_{6\alpha}\mathrm{tr}[\mathtt{id}] - \sum_{\alpha\neq\beta\neq\gamma=1}^{n} a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\gamma}a_{5\beta}a_{6\alpha}\mathrm{tr}[\mathtt{id}] \\
&= \Big\{g(X_1,X_4)[g(X_2,X_5)g(X_3,X_6) - g(X_2,X_6)g(X_3,X_5)] + g(X_1,X_6)[g(X_2,X_4)g(X_3,X_5) \\
&- g(X_2,X_5)g(X_3,X_4)] + g(X_1,X_5)[g(X_2,X_6)g(X_3,X_4) - g(X_2,X_4)g(X_3,X_6)]\Big\}\mathrm{tr}[\mathtt{id}].
\end{aligned}
\tag{2.35}
$$

Therefore

$$
\begin{aligned}
&\mathrm{tr}[c(X_1)c(X_2)c(X_3)c(X_4)c(X_5)c(X_6)] \\
&\quad= \Big\{g(X_1,X_2)[g(X_3,X_5)g(X_4,X_6) - g(X_3,X_6)g(X_4,X_5) - g(X_3,X_4)g(X_5,X_6)]
\end{aligned}
$$

$$
\begin{aligned}
&+g(X_1,X_3)[g(X_2,X_4)g(X_5,X_6)-g(X_2,X_6)g(X_4,X_5)-g(X_2,X_5)g(X_3,X_6)]\\
&+g(X_1,X_4)[g(X_2,X_5)g(X_3,X_6)-g(X_2,X_6)g(X_3,X_5)-g(X_2,X_3)g(X_5,X_6)]\\
&+g(X_1,X_5)[g(X_2,X_6)g(X_3,X_4)-g(X_2,X_4)g(X_3,X_6)-g(X_2,X_3)g(X_4,X_6)]\\
&+g(X_1,X_6)[g(X_2,X_4)g(X_3,X_5)-g(X_2,X_5)g(X_3,X_4)-g(X_2,X_3)g(X_4,X_5)]\Big\}\mathrm{tr}[\mathtt{id}].
\end{aligned}
\tag{2.36}
$$

When $X_j=e_j$, $l=n$, $\mathrm{tr}[c(e_1)\cdots c(e_n)]=\mathrm{str}[1]=0$. So we have

Theorem 2.2. *We have the following equalities*

$\mathrm{Swres}[(D^2)^{-\frac{n-2}{2}}]=0;$

$\mathrm{Wres}[c(X_1)c(X_2)(D^2)^{-\frac{n-2}{2}}]$

$$
=\frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!}\int_M\frac{1}{12}g(X_1,X_2)\mathrm{str}[id]d\mathrm{Vol_M};
$$

$\mathrm{Wres}[c(X_1)c(X_2)c(X_3)c(X_4)(D^2)^{-\frac{n-2}{2}}]$

$$
=\frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!}\int_M-\frac{1}{12}[g(X_1,X_2)g(X_3,X_4)-g(X_1,X_3)g(X_2,X_4)+g(X_1,X_4)g(X_2,X_3)]
$$

$\mathrm{str}[id]d\mathrm{Vol_M};$

$\mathrm{Wres}[c(X_1)c(X_2)c(X_3)c(X_4)c(X_5)c(X_6)(D^2)^{-\frac{n-2}{2}}]$

$$
\begin{aligned}
&=\frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!}\int_M-\frac{1}{12}\Big\{g(X_1,X_2)[g(X_3,X_5)g(X_4,X_6)-g(X_3,X_6)g(X_4,X_5)-g(X_3,X_4)\\
&g(X_5,X_6)]+g(X_1,X_3)[g(X_2,X_4)g(X_5,X_6)-g(X_2,X_6)g(X_4,X_5)-g(X_2,X_5)g(X_3,X_6)]\\
&+g(X_1,X_4)[g(X_2,X_5)g(X_3,X_6)-g(X_2,X_6)g(X_3,X_5)-g(X_2,X_3)g(X_5,X_6)]\\
&+g(X_1,X_5)[g(X_2,X_6)g(X_3,X_4)-g(X_2,X_4)g(X_3,X_6)-g(X_2,X_3)g(X_4,X_6)]\\
&+g(X_1,X_6)[g(X_2,X_4)g(X_3,X_5)-g(X_2,X_5)g(X_3,X_4)-g(X_2,X_3)g(X_4,X_5)]\Big\}\mathrm{str}[id]d\mathrm{Vol_M}.
\end{aligned}
\tag{2.37}
$$

3. A KASTLER-KALAU-WALZE TYPE THEOREM FOR 4-DIMENSIONAL MANIFOLDS WITH BOUNDARY

In this section, we prove the Kastler-Kalau-Walze type theorem for the generalized noncommutative residue $\widetilde{\mathrm{Wres}}[\pi^+(LD^{-1})\circ\pi^+(D^{-1})]$ on 4-dimensional oriented compact manifolds with boundary. We firstly recall some basic facts and formulas about Boutet de Monvel's calculus and the definition of the noncommutative residue for manifolds with boundary which will be used in the following. For more details, see Section 2 in [16].

Let M be a 4-dimensional compact oriented manifold with boundary ∂M. We assume that the metric g^M on M has the following form near the boundary,

$$
g^M=\frac{1}{h(x_n)}g^{\partial M}+dx_n^2,
\tag{3.1}
$$

where $g^{\partial M}$ is the metric on ∂M and $h(x_n) \in C^\infty([0,1)) := \{\widehat{h}|_{[0,1)}|\widehat{h} \in C^\infty((-\varepsilon,1))\}$ for some $\varepsilon > 0$ and $h(x_n)$ satisfies $h(x_n) > 0$, $h(0) = 1$, where x_n denotes the normal directional coordinate.
Then similar to [16], we can compute the generalized noncommutative residue

$$\widetilde{\mathrm{Wres}}[\pi^+(LD^{-1}) \circ \pi^+(D^{-1})] = \int_M \int_{|\xi|=1} \mathrm{trace}_{\wedge^* T^*M \otimes \mathbb{C}}[\sigma_{-4}(LD^{-2})]\sigma(\xi)dx + \int_{\partial M} \Phi, \tag{3.2}$$

where

$$\Phi = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \mathrm{trace}_{\wedge^* T^*M \otimes \mathbb{C}}[\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+(LD^{-1})(x',0,\xi',\xi_n)$$
$$\times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(D^{-1})(x',0,\xi',\xi_n)]d\xi_n \sigma(\xi')dx', \tag{3.3}$$

and the sum is taken over $r + l - k - j - |\alpha| = -3,\ \ r \le -1,\ \ l \le -1$.

By Theorem 2.1, we can compute the interior of $\widetilde{\mathrm{Wres}}[\pi^+(LD^{-1}) \circ \pi^+(D^{-1})]$,
(1)when $l = 2$, we get

$$\int_M \int_{|\xi|=1} \mathrm{trace}_{\wedge^* T^*M}[\sigma_{-4}(LD^{-2})]\sigma(\xi)dx = 32\pi^2 \int_M \left(\frac{1}{3} g(X_1, X_2) s\right) d\mathrm{Vol_M}. \tag{3.4}$$

(2)when $l = 4$, we get

$$\int_M \int_{|\xi|=1} \mathrm{trace}_{\wedge^* T^*M}[\sigma_{-4}(LD^{-2})]\sigma(\xi)dx$$
$$= 32\pi^2 \int_M \left(-\frac{1}{3}[g(X_1, X_2)g(X_3, X_4) - g(X_1, X_3)g(X_2, X_4) + g(X_1, X_4)g(X_2, X_3)]s\right) d\mathrm{Vol_M}. \tag{3.5}$$

(3)when $l = 1$ *or* 3, we get

$$\int_M \int_{|\xi|=1} \mathrm{trace}_{\wedge^* T^*M}[\sigma_{-4}(LD^{-2})]\sigma(\xi)dx = 0. \tag{3.6}$$

Now we need to compute $\int_{\partial M} \Phi$. By [16], we get the following symbols.

Lemma 3.1. *The following identities hold:*

$$\sigma_1(D) = ic(\xi);$$
$$\sigma_0(D) = -\frac{1}{4} \sum_{i,s,t} \omega_{s,t}(e_i)c(e_i)c(e_s)c(e_t). \tag{3.7}$$

By $\sigma(LD^{-1}) = \sigma(L)\sigma(D^{-1})$, we have

Lemma 3.2. *The following identities hold:*

$$\sigma_{-1}(D^{-1}) = \frac{ic(\xi)}{|\xi|^2};$$
$$\sigma_{-1}(LD^{-1}) = \frac{iLc(\xi)}{|\xi|^2};$$

$$\sigma_{-2}(D^{-1})=\frac{c(\xi)\sigma_0(D)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)\Big[\partial_{x_j}(c(\xi))|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)\Big];$$
$$\sigma_{-2}(LD^{-1})=L\left\{\frac{c(\xi)\sigma_0(D)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)\Big[\partial_{x_j}(c(\xi))|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)\Big]\right\}. \tag{3.8}$$

When $n=4$, then $\mathrm{tr}_{S(TM)}[\mathtt{id}]=\dim(\wedge^*(\mathbb{R}^2))=4$, the sum is taken over $r+\ell-k-j-|\alpha|=-3,\ r\leq -1,\ \ell\leq -1$, then we have the following five cases:
case a) I) $r=-1,\ \ell=-1,\ k=j=0,\ |\alpha|=1$.
By (3.3), we get

$$\Phi_1=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(LD^{-1})\times\partial^\alpha_{x'}\partial_{\xi_n}\sigma_{-1}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.9}$$

By Lemma 2.2 in [16], for $i<n$, then

$$\partial_{x_i}\left(\frac{ic(\xi)}{|\xi|^2}\right)(x_0)=\frac{i\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2}-\frac{ic(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4}=0, \tag{3.10}$$

so when l is an integer between 1 and 4, we have $\Phi_1=0$.
case a) II) $r=-1,\ \ell=-1,\ k=|\alpha|=0,\ j=1$.
By (3.3), we get

$$\Phi_2=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(LD^{-1})\times\partial^2_{\xi_n}\sigma_{-1}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.11}$$

By Lemma 3.2, we have

$$\partial^2_{\xi_n}\sigma_{-1}(D^{-1})(x_0)=i\left(-\frac{6\xi_nc(dx_n)+2c(\xi')}{|\xi|^4}+\frac{8\xi_n^2c(\xi)}{|\xi|^6}\right); \tag{3.12}$$

When $l=2$, we have

$$\begin{aligned}&\partial_{x_n}\sigma_{-1}(LD^{-1})(x_0)\\&=\partial_{x_n}\sigma_{-1}(c(X_1)c(X_2)D^{-1})(x_0)\\&=\frac{i\sum_{\alpha,\beta=1}^4\partial_{x_n}(a_{1\alpha})a_{2\beta}c(e_\alpha)c(e_\beta)c(\xi)}{|\xi|^2}+\frac{i\sum_{\alpha,\beta=1}^4 a_{1\alpha}\partial_{x_n}(a_{2\beta})c(e_\alpha)c(e_\beta)c(\xi)}{|\xi|^2}\\&+\frac{i\sum_{\alpha,\beta=1}^4 a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)\partial_{x_n}c(\xi')(x_0)}{|\xi|^2}-\frac{i\sum_{\alpha,\beta=1}^4 a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)c(\xi)|\xi'|^2h'(0)}{|\xi|^4}.\end{aligned} \tag{3.13}$$

By (2.1.1), (2.1.2) in [16] and the Cauchy integral formula, we have

$$\begin{aligned}&\pi^+_{\xi_n}\left[\frac{i\sum_{\alpha,\beta=1}^4\partial_{x_n}(a_{1\alpha})a_{2\beta}c(e_\alpha)c(e_\beta)c(\xi)}{|\xi|^2}\right](x_0)|_{|\xi'|=1}\\&=i\sum_{\alpha,\beta=1}^4\partial_{x_n}(a_{1\alpha})a_{2\beta}c(e_\alpha)c(e_\beta)\pi^+_{\xi_n}\left[\frac{c(\xi)}{|\xi|^2}\right](x_0)|_{|\xi'|=1}\end{aligned}$$

$$= \sum_{\alpha,\beta=1}^{4} \partial_{x_n}(a_{1\alpha})a_{2\beta}c(e_\alpha)c(e_\beta)\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)}. \tag{3.14}$$

Similarly, we have

$$\pi^+_{\xi_n}\left[\frac{i\sum_{\alpha,\beta=1}^{4} a_{1\alpha}\partial_{x_n}(a_{2\beta})c(e_\alpha)c(e_\beta)c(\xi)}{|\xi|^2}\right](x_0)|_{|\xi'|=1}$$
$$= \frac{\sum_{\alpha,\beta=1}^{4} a_{1\alpha}\partial_{x_n}(a_{2\beta})c(e_\alpha)c(e_\beta)[c(\xi')+ic(dx_n)]}{2(\xi_n-i)}. \tag{3.15}$$

$$\pi^+_{\xi_n}\left[\frac{i\sum_{\alpha,\beta=1}^{4} a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)\partial_{x_n}c(\xi')}{|\xi|^2}\right](x_0)|_{|\xi'|=1}$$
$$= \frac{\sum_{\alpha,\beta=1}^{4} a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n-i)}. \tag{3.16}$$

$$\pi^+_{\xi_n}\left[\frac{i\sum_{\alpha,\beta=1}^{4} a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)c(\xi)|\xi'|^2h'(0)}{|\xi|^4}\right](x_0)|_{|\xi'|=1}$$
$$= -i\sum_{\alpha,\beta=1}^{4} a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)\left[\frac{(i\xi_n+2)c(\xi')+ic(dx_n)}{4(\xi_n-i)^2}\right]. \tag{3.17}$$

By the relation of the Clifford action and $\mathrm{tr}ab = \mathrm{tr}ba$, we have the equalities:

$$\mathrm{tr}[c(\xi')c(dx_n)] = 0;\ \ \mathrm{tr}[c(dx_n)^2] = -4;\ \ \mathrm{tr}[c(\xi')^2](x_0)|_{|\xi'|=1} = -4;\ \ \mathrm{tr}[\partial_{x_n}c(\xi')c(dx_n)] = 0;$$
$$\mathrm{tr}[\partial_{x_n}c(\xi')c(\xi')](x_0)|_{|\xi'|=1} = -2h'(0);\ \ \sum_{\alpha,\beta=1}^{4}\partial_{x_n}(a_{1\alpha})(a_{2\alpha}) + \sum_{\alpha,\beta=1}^{4} a_{1\alpha}\partial_{x_n}(a_{2\alpha}) = \partial_{x_n}[g(X_1,X_2)];$$
$$\mathrm{tr}[\sum_{\alpha,\beta=1}^{4} a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)c(\xi')c(\xi')] = \mathrm{tr}[\sum_{\alpha,\beta=1}^{4} a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)c(dx_n)c(dx_n)] = 4g(X_1,X_2);$$
$$\mathrm{tr}[\sum_{\alpha,\beta=1}^{4}\partial_{x_n}(a_{1\alpha})a_{2\beta}c(e_\alpha)c(e_\beta)c(\xi')c(\xi')] = \mathrm{tr}[\sum_{\alpha,\beta=1}^{4}\partial_{x_n}(a_{1\alpha})a_{2\beta}c(e_\alpha)c(e_\beta)c(dx_n)c(dx_n)]$$
$$= 4\sum_{\alpha,\beta=1}^{4}\partial_{x_n}(a_{1\alpha})a_{2\alpha}. \tag{3.18}$$

We note that $i < n$, $\int_{|\xi'|=1}\{\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\}\sigma(\xi') = 0$, so $\mathrm{tr}[c(e_\alpha)c(e_\beta)c(\xi')c(dx_n)]$ has no contribution for computing **case a) II)**.

Then, we have

$$\begin{aligned}
&\mathrm{trace}[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(c(X_1)c(X_2)D^{-1})\times\partial^2_{\xi_n}\sigma_{-1}(D^{-1})](x_0)\\
&=4\frac{3\xi_n^2 i-\xi_n^3-3\xi_n+i}{(\xi_n-i)^4(\xi_n+i)^3}\sum_{\alpha=1}^{4}\partial_{x_n}[g(X_1,X_2)]+2h'(0)\frac{8\xi_n^3 i+5\xi_n i+3+11\xi_n^2-\xi_n^3}{(\xi_n-i)^5(\xi_n+i)^3}g(X_1,X_2).
\end{aligned}\tag{3.19}$$

Then

$$\begin{aligned}
\Phi_2=&-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\Big\{4\frac{3\xi_n^2 i-\xi_n^3-3\xi_n+i}{(\xi_n-i)^4(\xi_n+i)^3}\partial_{x_n}[g(X_1,X_2)]\\
&+2\frac{8\xi_n^3 i+5\xi_n i+3+11\xi_n^2-\xi_n^3}{(\xi_n-i)^5(\xi_n+i)^3}g(X_1,X_2)\Big\}d\xi_n\sigma(\xi')dx'\\
=&-2\partial_{x_n}[g(X_1,X_2)]\Omega_3\int_{\Gamma^+}\frac{3\xi_n^2 i-\xi_n^3-3\xi_n+i}{(\xi_n-i)^4(\xi_n+i)^3}d\xi_n dx'\\
&-g(X_1,X_2)\Omega_3\int_{\Gamma^+}\frac{8\xi_n^3 i+5\xi_n i+3+11\xi_n^2-\xi_n^3}{(\xi_n-i)^5(\xi_n+i)^3}d\xi_n dx'\\
=&-2\partial_{x_n}[g(X_1,X_2)]\Omega_3\frac{2\pi i}{3!}\left[\frac{3\xi_n^2 i-\xi_n^3-3\xi_n+i}{(\xi_n+i)^3}\right]^{(3)}\Bigg|_{\xi_n=i}dx'\\
&-g(X_1,X_2)\Omega_3\frac{2\pi i}{4!}\left[\frac{8\xi_n^3 i+5\xi_n i+3+11\xi_n^2-\xi_n^3}{(\xi_n+i)^3}\right]^{(4)}\Bigg|_{\xi_n=i}dx'\\
=&\Big\{-\frac{\partial_{x_n}[g(X_1,X_2)]}{2}-\frac{3h'(0)g(X_1,X_2)}{8}\Big\}\pi\Omega_3dx',
\end{aligned}\tag{3.20}$$

where Ω_3 is the canonical volume of S^3.
Similar to (2.8)-(2.13), we get

$$\begin{aligned}
&\mathrm{tr}[\sum_{\alpha,\beta,\gamma,\mu=1}^{4}\partial_{x_n}(a_{1\alpha})a_{2\beta}a_{3\gamma}a_{4\mu}c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(\xi')c(\xi')]\\
&=\mathrm{tr}[\sum_{\alpha,\beta,\gamma,\mu=1}^{4}\partial_{x_n}(a_{1\alpha})a_{2\beta}a_{3\gamma}a_{4\mu}c(e_\alpha)c(e_\beta)c(e_\gamma)c(e_\mu)c(dx_n)c(dx_n)]\\
&=4(\sum_{\alpha,\beta=1}^{n}\partial_{x_n}(a_{1\alpha})a_{2\beta}a_{3\alpha}a_{4\beta}-\sum_{\alpha,\gamma=1}^{n}\partial_{x_n}(a_{1\alpha})a_{2\alpha}a_{3\gamma}a_{4\gamma}-\sum_{\alpha,\beta=1}^{n}\partial_{x_n}(a_{1\alpha})a_{2\beta}a_{3\beta}a_{4\alpha}).
\end{aligned}\tag{3.21}$$

Set

$$A^1=\sum_{\alpha,\beta=1}^{n}\partial_{x_n}(a_{1\alpha})a_{2\beta}a_{3\alpha}a_{4\beta}-\sum_{\alpha,\gamma=1}^{n}\partial_{x_n}(a_{1\alpha})a_{2\alpha}a_{3\gamma}a_{4\gamma}-\sum_{\alpha,\beta=1}^{n}\partial_{x_n}(a_{1\alpha})a_{2\beta}a_{3\beta}a_{4\alpha};$$

$$A^2=\sum_{\alpha,\beta=1}^{n}a_{1\alpha}\partial_{x_n}(a_{2\beta})a_{3\alpha}a_{4\beta}-\sum_{\alpha,\gamma=1}^{n}a_{1\alpha}\partial_{x_n}(a_{2\alpha})a_{3\gamma}a_{4\gamma}-\sum_{\alpha,\beta=1}^{n}a_{1\alpha}\partial_{x_n}(a_{2\beta})a_{3\beta}a_{4\alpha};$$
$$A^3=\sum_{\alpha,\beta=1}^{n}a_{1\alpha}a_{2\beta}\partial_{x_n}(a_{3\alpha})a_{4\beta}-\sum_{\alpha,\gamma=1}^{n}a_{1\alpha}a_{2\alpha}\partial_{x_n}(a_{3\gamma})a_{4\gamma}-\sum_{\alpha,\beta=1}^{n}a_{1\alpha}a_{2\beta}\partial_{x_n}(a_{3\beta})a_{4\alpha};$$
$$A^4=\sum_{\alpha,\beta=1}^{n}a_{1\alpha}a_{2\beta}a_{3\alpha}\partial_{x_n}(a_{4\beta})-\sum_{\alpha,\gamma=1}^{n}a_{1\alpha}a_{2\alpha}a_{3\gamma}\partial_{x_n}(a_{4\gamma})-\sum_{\alpha,\beta=1}^{n}a_{1\alpha}a_{2\beta}a_{3\beta}\partial_{x_n}(a_{4\alpha}). \tag{3.22}$$

By

$$\begin{aligned}
&\sum_{\alpha,\gamma=1}^{n}[\partial_{x_n}(a_{1\alpha})a_{2\alpha}a_{3\gamma}a_{4\gamma}+a_{1\alpha}\partial_{x_n}(a_{2\alpha})a_{3\gamma}a_{4\gamma}+a_{1\alpha}a_{2\alpha}\partial_{x_n}(a_{3\gamma})a_{4\gamma}+a_{1\alpha}a_{2\alpha}a_{3\gamma}\partial_{x_n}(a_{4\gamma})]\\
&=\sum_{\alpha,\gamma=1}^{n}\partial_{x_n}(a_{1\alpha}a_{2\alpha})a_{3\gamma}a_{4\gamma}+\sum_{\alpha,\gamma=1}^{n}a_{1\alpha}a_{2\alpha}\partial_{x_n}(a_{3\gamma}a_{4\gamma})\\
&=\partial_{x_n}[g(X_1,X_2)g(X_3,X_4)];\\
&\sum_{\alpha,\beta=1}^{n}[\partial_{x_n}(a_{1\alpha})a_{2\beta}a_{3\alpha}a_{4\beta}+a_{1\alpha}\partial_{x_n}(a_{2\beta})a_{3\alpha}a_{4\beta}+a_{1\alpha}a_{2\beta}\partial_{x_n}(a_{3\alpha})a_{4\beta}+a_{1\alpha}a_{2\beta}a_{3\alpha}\partial_{x_n}(a_{4\beta})]\\
&=\sum_{\alpha,\beta=1}^{n}\partial_{x_n}(a_{1\alpha}a_{3\alpha})a_{2\beta}a_{4\beta}+\sum_{\alpha,\beta=1}^{n}a_{1\alpha}a_{3\alpha}\partial_{x_n}(a_{2\beta}a_{4\beta})\\
&=\partial_{x_n}[g(X_1,X_3)g(X_2,X_4)];\\
&\sum_{\alpha,\beta=1}^{n}\partial_{x_n}[(a_{1\alpha})a_{2\beta}a_{3\beta}a_{4\alpha}+a_{1\alpha}\partial_{x_n}(a_{2\beta})a_{3\beta}a_{4\alpha}+a_{1\alpha}a_{2\beta}\partial_{x_n}(a_{3\beta})a_{4\alpha}+a_{1\alpha}a_{2\beta}a_{3\beta}\partial_{x_n}(a_{4\alpha})]\\
&=\sum_{\alpha,\beta=1}^{n}\partial_{x_n}(a_{1\alpha}a_{4\alpha})a_{2\beta}a_{3\beta}+\sum_{\alpha,\beta=1}^{n}a_{1\alpha}a_{4\alpha}\partial_{x_n}(a_{2\beta}a_{3\beta})\\
&=\partial_{x_n}[g(X_1,X_4)g(X_2,X_3)].
\end{aligned} \tag{3.23}$$

Then, we have

$$A^1+A^2+A^3+A^4=\partial_{x_n}[g(X_1,X_3)g(X_2,X_4)-g(X_1,X_4)g(X_2,X_3)-g(X_1,X_2)g(X_3,X_4)]. \tag{3.24}$$

So when $l=4$, we get

$$\begin{aligned}\Phi_2=\Big\{&-\frac{1}{2}\partial_{x_n}[g(X_1,X_3)g(X_2,X_4)-g(X_1,X_4)g(X_2,X_3)-g(X_1,X_2)g(X_3,X_4)]+\frac{3h'(0)}{8}\\
&[g(X_1,X_4)g(X_2,X_3)-g(X_1,X_2)g(X_3,X_4)+g(X_1,X_3)g(X_2,X_4)]\Big\}\pi\Omega_3dx'.\end{aligned} \tag{3.25}$$

case a) III) $r=-1,\ \ell=-1,\ j=|\alpha|=0,\ k=1.$
By (3.3), we get

$$\Phi_3=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(LD^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-1}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \quad (3.26)$$

By Lemma 3.2, we have

$$\begin{aligned}&\partial_{\xi_n}\partial_{x_n}\sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1}\\ &=-ih'(0)\left[\tfrac{c(dx_n)}{|\xi|^4}-4\xi_n\tfrac{c(\xi')+\xi_nc(dx_n)}{|\xi|^6}\right]-\tfrac{2\xi_ni\partial_{x_n}[c(\xi')](x_0)}{|\xi|^4};\end{aligned} \quad (3.27)$$

$$\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(LD^{-1})(x_0)|_{|\xi'|=1}=-\frac{\sum_{\alpha,\beta=1}^4 a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)[c(\xi')+ic(dx_n)]}{2(\xi_n-i)^2}. \quad (3.28)$$

Similar to case a) II), when $l=2$, we have

$$\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(LD^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-1}(D^{-1})](x_0)=2h'(0)\frac{-5i\xi_n+3\xi_n^2+\xi_n^3i+1}{(\xi_n-i)^5(\xi_n+i)^3}g(X_1,X_2). \quad (3.29)$$

So we have

$$\begin{aligned}\Phi_3&=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}h'(0)\frac{-5i\xi_n+3\xi_n^2+\xi_n^3i+1}{(\xi_n-i)^5(\xi_n+i)^3}g(X_1,X_2)d\xi_n\sigma(\xi')dx'\\ &=-h'(0)g(X_1,X_2)\Omega_3\int_{\Gamma^+}\frac{-5i\xi_n+3\xi_n^2+\xi_n^3i+1}{(\xi_n-i)^5(\xi_n+i)^3}d\xi_ndx'\\ &=-h'(0)g(X_1,X_2)\Omega_3\frac{2\pi i}{4!}\left[\frac{-5i\xi_n+3\xi_n^2+\xi_n^3i+1}{(\xi_n+i)^3}\right]^{(4)}\Bigg|_{\xi_n=i}dx'\\ &=-\frac{3h'(0)}{8}g(X_1,X_2)\pi\Omega_3dx'.\end{aligned} \quad (3.30)$$

Similarly, when $l=4$, we get

$$\Phi_3=\frac{3h'(0)}{8}[g(X_1,X_2)g(X_3,X_4)-g(X_1,X_3)g(X_2,X_4)+g(X_1,X_4)g(X_2,X_3)]\pi\Omega_3dx'. \quad (3.31)$$

case b) $r=-2,\ \ell=-1,\ k=j=|\alpha|=0.$
By (3.3), we get

$$\Phi_4=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^+_{\xi_n}\sigma_{-2}(LD^{-1})\times\partial_{\xi_n}\sigma_{-1}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \quad (3.32)$$

By Lemma 3.2, we have

$$\sigma_{-2}(LD^{-1})(x_0)=L\left\{\frac{c(\xi)\sigma_0(D)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)\left[\partial_{x_j}(c(\xi))|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)\right]\right\}, \quad (3.33)$$

where

$$\sigma_0(D)(x_0) = -\frac{1}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)c(e_s)c(e_t). \tag{3.34}$$

We denote

$$H(x_0) = -\frac{1}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)c(e_s)c(e_t), \tag{3.35}$$

where $H(x_0) = c_0c(dx_n)$ and $c_0 = -\frac{3}{4}h'(0)$.
Then

$$\begin{aligned}&\pi^+_{\xi_n}\sigma_{-2}(LD^{-1}(x_0))|_{|\xi'|=1}\\ &= \pi^+_{\xi_n}\Big[\frac{Lc(\xi)H(x_0)c(\xi) + Lc(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1+\xi_n^2)^2} - h'(0)\frac{Lc(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^3}\Big].\end{aligned} \tag{3.36}$$

Since

$$\partial_{\xi_n}\sigma_{-1}(D^{-1}) = i\left[\frac{c(dx_n)}{1+\xi_n^2} - \frac{2\xi_nc(\xi') + 2\xi_n^2c(dx_n)}{(1+\xi_n^2)^2}\right]. \tag{3.37}$$

By computations, we have

$$\begin{aligned}&\pi^+_{\xi_n}\Big[\frac{Lc(\xi)H(x_0)c(\xi)+Lc(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1+\xi_n^2)^2}\Big] - h'(0)\pi^+_{\xi_n}\Big[L\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n)^3}\Big]\\ &:= E_1 - E_2,\end{aligned} \tag{3.38}$$

where

$$\begin{aligned}E_1 = &\frac{-1}{4(\xi_n - i)^2}[(2+i\xi_n)Lc(\xi')H(x_0)c(\xi') + i\xi_nLc(dx_n)H(x_0)c(dx_n) + (2+i\xi_n)Lc(\xi')\\ &c(dx_n)\partial_{x_n}c(\xi') + iLc(dx_n)H(x_0)c(\xi') + iLc(\xi')H(x_0)c(dx_n) - iL\partial_{x_n}c(\xi')]\end{aligned} \tag{3.39}$$

and

$$E_2 = \frac{h'(0)}{2}L\left[\frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3}[ic(\xi') - c(dx_n)]\right]. \tag{3.40}$$

By (3.37) and (3.40), when $l = 2$, we have

$$\mathrm{tr}[E_2\times\partial_{\xi_n}\sigma_{-1}(D^{-1})]|_{|\xi'|=1} = -2ih'(0)\frac{-i\xi_n^2-\xi_n+4i}{4(\xi_n-i)^3(\xi_n+i)^2}g(X_1,X_2). \tag{3.41}$$

Then,

$$\begin{aligned}&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[E_2\times\partial_{\xi_n}\sigma_{-1}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\ &= -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty} -2ih'(0)\frac{-i\xi_n^2 - \xi_n + 4i}{4(\xi_n - i)^3(\xi_n + i)^2}g(X_1,X_2)d\xi_n\sigma(\xi')dx'\\ &= -\frac{1}{2}h'(0)g(X_1,X_2)\Omega_3\int_{\Gamma^+}\frac{-i\xi_n^2 - \xi_n + 4i}{4(\xi_n - i)^3(\xi_n + i)^2}d\xi_ndx'\\ &= -\frac{1}{2}h'(0)g(X_1,X_2)\Omega_3\frac{2\pi i}{3!}\left[\frac{-i\xi_n^2 - \xi_n + 4i}{(\xi_n + i)^2}\right]^{(3)}\Bigg|_{\xi_n=i}dx'\end{aligned}$$

$$= \frac{3h'(0)}{4} g(X_1, X_2)\pi\Omega_3 dx'. \tag{3.42}$$

By (3.37) and (3.39), when $l = 2$, we have

$$\mathrm{tr}[E_1 \times \partial_{\xi_n}\sigma_{-1}(D^{-1})]|_{|\xi'|=1} = \frac{-3h'(0)i}{2(\xi_n - i)^2(\xi_n + i)^2} g(X_1, X_2) - \frac{(\xi_n^2 - i\xi_n - 2)h'(0)}{2(\xi_n - i)^3(\xi_n + i)^2} g(X_1, X_2). \tag{3.43}$$

By (3.43), we have

$$\begin{aligned}
& -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \mathrm{trace}[E_1 \times \partial_{\xi_n}\sigma_{-1}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx' \\
&= -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \frac{-3h'(0)i}{2(\xi_n - i)^2(\xi_n + i)^2} g(X_1, X_2)d\xi_n\sigma(\xi')dx' \\
&- i\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \frac{(\xi_n^2 - i\xi_n - 2)h'(0)}{2(\xi_n - i)^3(\xi_n + i)^2} g(X_1, X_2)d\xi_n\sigma(\xi')dx' \\
&= -\frac{3}{2}h'(0)g(X_1, X_2)\Omega_3 \int_{\Gamma^+} \frac{1}{(\xi_n - i)^2(\xi_n + i)^2} d\xi_n dx' \\
&- \frac{i}{2}h'(0)g(X_1, X_2)\Omega_3 \int_{\Gamma^+} \frac{\xi_n^2 - i\xi_n - 2}{(\xi_n - i)^3(\xi_n + i)^2} d\xi_n dx' \\
&= -\frac{3}{2}h'(0)g(X_1, X_2)\Omega_3 \frac{2\pi i}{1!}\left[\frac{1}{(\xi_n + i)^2}\right]^{(1)}\Bigg|_{\xi_n = i} dx' \\
&- \frac{i}{2}h'(0)g(X_1, X_2)\Omega_3 \frac{2\pi i}{3!}\left[\frac{\xi_n^2 - i\xi_n - 2}{(\xi_n + i)^2}\right]^{(2)}\Bigg|_{\xi_n = i} dx' \\
&= -\frac{15h'(0)}{8} g(X_1, X_2)\pi\Omega_3 dx'.
\end{aligned} \tag{3.44}$$

Then, when $l = 2$, we have

$$\Phi_4 = -\frac{15h'(0)}{8} g(X_1, X_2)\pi\Omega_3 dx' + \frac{3h'(0)}{4} g(X_1, X_2)\pi\Omega_3 dx' = -\frac{9h'(0)}{8} g(X_1, X_2)\pi\Omega_3 dx'. \tag{3.45}$$

Similarly, when $l = 4$, we get

$$\Phi_4 = \frac{9h'(0)}{8}[g(X_1, X_2)g(X_3, X_4) - g(X_1, X_3)g(X_2, X_4) + g(X_1, X_4)g(X_2, X_3)]\pi\Omega_3 dx'. \tag{3.46}$$

case c) $r = -1,\ \ell = -2,\ k = j = |\alpha| = 0.$
By (3.3), we get

$$\Phi_5 = -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \mathrm{trace}[\pi_{\xi_n}^+\sigma_{-1}(LD^{-1}) \times \partial_{\xi_n}\sigma_{-2}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.47}$$

By Lemma 3.2, we have

$$\pi^+_{\xi_n}\sigma_{-1}(LD^{-1})|_{|\xi'|=1}=\frac{L[c(\xi')+ic(dx_n)]}{2(\xi_n-i)}. \tag{3.48}$$

Since

$$\sigma_{-2}(D^{-1})(x_0)=\frac{c(\xi)\sigma_0(D)(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)\left[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)|\xi|^2_{\partial_M}\right], \tag{3.49}$$

where

$$\sigma_0(D)(x_0)=-\frac{1}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)c(e_s)c(e_t). \tag{3.50}$$

By computations, we have

$$\begin{aligned}
&\partial_{\xi_n}\sigma_{-2}(D^{-1})(x_0)|_{|\xi'|=1}=\partial_{\xi_n}\left\{\frac{c(\xi)H(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)]\right\}\\
&=\frac{1}{(1+\xi_n^2)^3}\Big[(2\xi_n-2\xi_n^3)c(dx_n)Hc(dx_n)+(1-3\xi_n^2)c(dx_n)Hc(\xi')\\
&+(1-3\xi_n^2)c(\xi')Hc(dx_n)-4\xi_nc(\xi')Hc(\xi')+(3\xi_n^2-1)\partial_{x_n}c(\xi')\\
&-4\xi_nc(\xi')c(dx_n)\partial_{x_n}c(\xi')+2h'(0)c(\xi')+2h'(0)\xi_nc(dx_n)\Big]+6\xi_nh'(0)\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^4}.
\end{aligned} \tag{3.51}$$

By (3.48) and (3.51), we have

$$\begin{aligned}
&\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-1}(LD^{-1})\times\partial_{\xi_n}\sigma_{-2}(D^{-1})](x_0)|_{|\xi'|=1}\\
&=-\frac{12h'(0)i\xi_n}{(\xi_n-i)^3(\xi_n+i)^4}g(X_1,X_2)-\frac{3h'(0)(i\xi_n^2+\xi_n-2i)}{(\xi_n-i)^3(\xi_n+i)^3}g(X_1,X_2).
\end{aligned} \tag{3.52}$$

So, we have

$$\begin{aligned}
\Phi_5&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-1}(LD^{-1})\times\partial_{\xi_n}\sigma_{-2}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\
&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}-\frac{12h'(0)i\xi_n}{(\xi_n-i)^3(\xi_n+i)^4}g(X_1,X_2)d\xi_n\sigma(\xi')dx'\\
&+i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{3h'(0)(i\xi_n^2+\xi_n-2i)}{(\xi_n-i)^3(\xi_n+i)^3}g(X_1,X_2)d\xi_n\sigma(\xi')dx'\\
&=-12g(X_1,X_2)\Omega_3\int_{\Gamma^+}\frac{\xi_n}{(\xi-i)^3(\xi+i)^4}d\xi_ndx'+3ih'(0)g(X_1,X_2)\Omega_3\int_{\Gamma^+}\frac{i\xi_n^2+\xi_n-2i}{(\xi-i)^3(\xi+i)^3}d\xi_ndx'\\
&=-\frac{ih'(0)}{2}g(X_1,X_2)\Omega_3\frac{2\pi i}{2!}\left[\frac{\xi_n}{(\xi+i)^4}\right]^{(2)}\Bigg|_{\xi_n=i}dx'\\
&+3ih'(0)g(X_1,X_2)\Omega_3\frac{2\pi i}{2!}\left[\frac{i\xi_n^2+\xi_n-2i}{(\xi+i)^3}\right]^{(2)}\Bigg|_{\xi_n=i}dx'
\end{aligned}$$

$$= \frac{9h'(0)}{8} g(X_1, X_2)\pi\Omega_3 dx'. \tag{3.53}$$

Then, when $l = 2$, we have

$$\Phi_5 = \frac{9h'(0)}{8} g(X_1, X_2)\pi\Omega_3 dx'. \tag{3.54}$$

Similarly, when $l = 4$, we get

$$\Phi_5 = -\frac{9h'(0)}{8}[g(X_1, X_2)g(X_3, X_4) - g(X_1, X_3)g(X_2, X_4) + g(X_1, X_4)g(X_2, X_3)]\pi\Omega_3 dx'. \tag{3.55}$$

Now Φ is the sum of the cases (a), (b) and (c), therefore, when $l = 2$, we have

$$\Phi = \sum_{i=1}^{5} \Phi_i = -\frac{\partial_{x_n}[g(X_1, X_2)]}{2}\pi\Omega_3 dx'. \tag{3.56}$$

Similarly, when $l = 4$, we get

$$\Phi = \sum_{i=1}^{5} \Phi_i = \frac{\partial_{x_n}[g(X_1, X_2)g(X_3, X_4) - g(X_1, X_3)g(X_2, X_4) + g(X_1, X_4)g(X_2, X_3)]}{2}\pi\Omega_3 dx'. \tag{3.57}$$

Obviously, when $l = 1$ *or* 3, we get $\Phi_1 = \Phi_2 = \Phi_3 = \Phi_4 = \Phi_5 = 0$.
Then, by (3.3)-(3.6), we obtain following theorem

Theorem 3.1. *Let M be a 4-dimensional oriented compact manifold with boundary ∂M and the metric g^M be defined as (3.1), then the following is the generalized noncommutative residue of the Dirac operator*
(1)when $l = 2$, we get

$$\begin{aligned}&\widetilde{\mathrm{Wres}}[\pi^+(LD^{-1}) \circ \pi^+ D^{-1}]\\ &= 32\pi^2 \int_M \left(\frac{1}{3} g(X_1, X_2)s\right) d\mathrm{Vol_M} - \int_{\partial M} \left(\frac{1}{2}\partial_{x_n}[g(X_1, X_2)]\right)\pi\Omega_3 d\mathrm{Vol_M},\end{aligned} \tag{3.58}$$

(2)when $l = 4$, we get

$$\begin{aligned}&\widetilde{\mathrm{Wres}}[\pi^+(LD^{-1}) \circ \pi^+ D^{-1}]\\ &= 32\pi^2 \int_M \left(-\frac{1}{3}[g(X_1, X_2)g(X_3, X_4) - g(X_1, X_3)g(X_2, X_4) + g(X_1, X_4)g(X_2, X_3)]s\right) d\mathrm{Vol_M}\\ &+ \int_{\partial M} \left(\frac{1}{2}\partial_{x_n}[g(X_1, X_2)g(X_3, X_4) - g(X_1, X_3)g(X_2, X_4) + g(X_1, X_4)g(X_2, X_3)]\right)\pi\Omega_3 d\mathrm{Vol_M},\end{aligned} \tag{3.59}$$

(3)when $l = 1$ or 3, we get

$$\widetilde{\mathrm{Wres}}[\pi^+(LD^{-1}) \circ \pi^+ D^{-1}] = 0. \tag{3.60}$$

4. A Kastler-Kalau-Walze type theorem for 6-dimensional manifolds with boundary

Firstly, we prove the Kastler-Kalau-Walze type theorems for the generalized noncommutative residue $\widetilde{\mathrm{Wres}}[\pi^+(LD^{-2})\circ\pi^+(D^{-2})]$ on 6-dimensional manifolds with boundary. From [13], we know that

$$\widetilde{\mathrm{Wres}}[\pi^+(LD^{-2})\circ\pi^+(D^{-2})]=\int_M\int_{|\xi'|=1}\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\sigma_{-6}(LD^{-4})]\sigma(\xi)dx+\int_{\partial M}\Psi, \tag{4.1}$$

where

$$\Psi=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\times\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\partial^j_{x_n}\partial^\alpha_{\xi'}\partial^k_{\xi_n}\sigma^+_r(LD^{-2})(x',0,\xi',\xi_n)$$
$$\times\partial^\alpha_{x'}\partial^{j+1}_{\xi_n}\partial^k_{x_n}\sigma_l(D^{-2})(x',0,\xi',\xi_n)]d\xi_n\sigma(\xi')dx', \tag{4.2}$$

and the sum is taken over $r+\ell-k-j-|\alpha|-1=-6,\quad r\leq -1,\ \ \ell\leq -3.$
By Theorem 2.1, we can compute the interior of $\widetilde{\mathrm{Wres}}[\pi^+(LD^{-2})\circ\pi^+(D^{-2})]$,
(1)when $l=2$, we get

$$\int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M}[\sigma_{-6}(LD^{-4})]\sigma(\xi)dx=128\pi^2\int_M\left(\frac{2}{3}g(X_1,X_2)s\right)d\mathrm{Vol_M}, \tag{4.3}$$

(2)when $l=4$, we get

$$\int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M}[\sigma_{-6}(LD^{-4})]\sigma(\xi)dx$$
$$=128\pi^2\int_M\left(-\frac{2}{3}[g(X_1,X_2)g(X_3,X_4)-g(X_1,X_3)g(X_2,X_4)+g(X_1,X_4)g(X_2,X_3)]s\right)d\mathrm{Vol_M}, \tag{4.4}$$

(3)when $l=6$, we get

$$\int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M}[\sigma_{-6}(LD^{-4})]\sigma(\xi)dx$$
$$=128\pi^2\int_M\Bigg\{-\frac{2}{3}\Bigg(g(X_1,X_2)[g(X_3,X_5)g(X_4,X_6)-g(X_3,X_6)g(X_4,X_5)]-g(X_1,X_2)$$
$$g(X_3,X_4)g(X_5,X_6)+g(X_1,X_3)[g(X_2,X_4)g(X_5,X_6)-g(X_2,X_6)g(X_4,X_5)-g(X_2,X_5)$$
$$g(X_3,X_6)]+g(X_1,X_4)[g(X_2,X_5)g(X_3,X_6)-g(X_2,X_6)g(X_3,X_5)-g(X_2,X_3)g(X_5,X_6)]$$
$$+g(X_1,X_5)[g(X_2,X_6)g(X_3,X_4)-g(X_2,X_4)g(X_3,X_6)-g(X_2,X_3)g(X_4,X_6)]+g(X_1,X_6)$$
$$[g(X_2,X_4)g(X_3,X_5)-g(X_2,X_5)g(X_3,X_4)-g(X_2,X_3)g(X_4,X_5)]\Bigg)s\Bigg\}d\mathrm{Vol_M}, \tag{4.5}$$

(4)when $l=1$ *or* 3 *or* 5, we get

$$\int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M}[\sigma_{-6}(LD^{-4})]\sigma(\xi)dx=0. \tag{4.6}$$

Let the cotangent vector $\xi = \sum_j \xi_j dx_j$, $\xi^j = g^{ij}\xi_i$, $\delta_i = -\frac{1}{4}\sum_{s,t}\omega_{s,t}(e_i)c(e_s)c(e_t)$, $g^{ij} = g(dx_i, dx_j)$ and $\nabla^L_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k$, $\Gamma^k = g^{ij}\Gamma^k_{ij}$, $\delta^j = g^{ij}\delta_i$. Then, by [9], we obtain

Lemma 4.1. *The following identities hold:*

$$\sigma_{-2}(LD^{-2}) = L|\xi|^{-2};$$
$$\sigma_{-3}(LD^{-2}) = -\sqrt{-1}|\xi|^{-4}L\xi_k(\Gamma^k - 2\delta^k) - 2\sqrt{-1}|\xi|^{-6}L\xi^j\xi_\alpha\xi_\beta\partial_j g^{\alpha\beta}. \tag{4.7}$$

When $n = 6$, then $\mathrm{tr}_{S(TM)}[\mathtt{id}] = 8$. Since the sum is taken over $-r - \ell + k + j + |\alpha| - 1 = -6$, $r, \ell \leq -2$, then we have the $\int_{\partial_M} \Psi$ is the sum of the following five cases:
case (a) (I) $r = -2$, $\ell = -2$, $j = k = 0$, $|\alpha| = 1$.
By (4.2), we get

$$\Psi_1 = -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}\Big[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(LD^{-2}) \times \partial^\alpha_{x'}\partial_{\xi_n}\sigma_{-2}(D^{-2})\Big](x_0)d\xi_n\sigma(\xi')dx'. \tag{4.8}$$

By Lemma 4.1, for $i < n$, we have

$$\begin{aligned}\partial_{x_i}\sigma_{-2}(D^{-2})(x_0) &= \partial_{x_i}(|\xi|^{-2})(x_0)\\ &= -\frac{\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4}\\ &= 0,\end{aligned} \tag{4.9}$$

so when l is an integer between 1 and 6, we have $\Psi_1 = 0$.

case (a) (II) $r = -2$, $\ell = -2$, $|\alpha| = k = 0$, $j = 1$.
By (4.2), we have

$$\Psi_2 = -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-2}(LD^{-2}) \times \partial^2_{\xi_n}\sigma_{-2}(D^{-2})\Big](x_0)d\xi_n\sigma(\xi')dx'. \tag{4.10}$$

By computations, we have

$$\partial^2_{\xi_n}\sigma_{-2}(D^{-2})(x_0) = \partial^2_{\xi_n}(|\xi|^{-2})(x_0) = 2\frac{3\xi_n^2 - 1}{(1+\xi_n^2)^3}. \tag{4.11}$$

$$\begin{aligned}\partial_{x_n}\sigma_{-2}(LD^{-2})(x_0)|_{|\xi'|=1} &= \frac{\partial_{x_n}[\sum_{\alpha,\beta=1}^6 a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)]}{|\xi|^2} - \frac{\sum_{\alpha,\beta=1}^6 a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)h'(0)}{|\xi|^4}\\ &= \frac{\sum_{\alpha,\beta=1}^6 \partial_{x_n}(a_{1\alpha})a_{2\beta}c(e_\alpha)c(e_\beta)}{|\xi|^2} + \frac{\sum_{\alpha,\beta=1}^6 a_{1\alpha}\partial_{x_n}(a_{2\beta})c(e_\alpha)c(e_\beta)}{|\xi|^2}\\ &\quad - \frac{\sum_{\alpha,\beta=1}^6 a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)h'(0)}{|\xi|^4}.\end{aligned} \tag{4.12}$$

$$
\begin{aligned}
&\pi^{+}_{\xi_n}\partial_{x_n}\sigma_{-2}(LD^{-2})(x_0)|_{|\xi'|=1}\\
&=-\frac{i}{2(\xi_n-i)}\sum_{\alpha,\beta=1}^{6}\partial_{x_n}(a_{1\alpha})a_{2\beta}c(e_\alpha)c(e_\beta)-\frac{i}{2(\xi_n-i)}\sum_{\alpha,\beta=1}^{6}a_{1\alpha}\partial_{x_n}(a_{2\beta})c(e_\alpha)c(e_\beta)\\
&+\frac{(i\xi_n+2)h'(0)}{4(\xi_n-i)^2}\sum_{\alpha,\beta=1}^{6}a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta).
\end{aligned}
\tag{4.13}
$$

Since $n=6$, $\mathrm{tr}[-\mathtt{id}]=-8$. By the relation of the Clifford action and $\mathrm{tr}ab=\mathrm{tr}ba$, then

$$
\begin{aligned}
&\mathrm{tr}[c(\xi')c(dx_n)]=0;\ \ \mathrm{tr}[c(dx_n)^2]=-8;\ \ \mathrm{tr}[c(\xi')^2](x_0)|_{|\xi'|=1}=-8;\ \ \mathrm{tr}[\partial_{x_n}c(\xi')c(dx_n)]=0;\\
&\mathrm{tr}[\partial_{x_n}c(\xi')c(\xi')](x_0)|_{|\xi'|=1}=-4h'(0);\ \ \sum_{\alpha,\beta=1}^{6}\partial_{x_n}(a_{1\alpha})(a_{2\alpha})+\sum_{\alpha,\beta=1}^{6}a_{1\alpha}\partial_{x_n}(a_{2\alpha})=\partial_{x_n}[g(X_1,X_2)];\\
&\mathrm{tr}[\sum_{\alpha,\beta=1}^{6}a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)c(\xi')c(\xi')]=\mathrm{tr}[\sum_{\alpha,\beta=1}^{6}a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)c(dx_n)c(dx_n)]=8g(X_1,X_2);\\
&\mathrm{tr}[\sum_{\alpha,\beta=1}^{6}\partial_{x_n}(a_{1\alpha})a_{2\beta}c(e_\alpha)c(e_\beta)c(\xi')c(\xi')]=\mathrm{tr}[\sum_{\alpha,\beta=1}^{6}\partial_{x_n}(a_{1\alpha})a_{2\beta}c(e_\alpha)c(e_\beta)c(dx_n)c(dx_n)]\\
&=8\sum_{\alpha,\beta=1}^{6}\partial_{x_n}(a_{1\alpha})a_{2\alpha}.
\end{aligned}
\tag{4.14}
$$

When $l=2$, we have

$$
\begin{aligned}
&\mathrm{trace}\Big[\partial_{x_n}\pi^{+}_{\xi_n}\sigma_{-2}(LD^{-2})\times\partial^2_{\xi_n}\sigma_{-2}(D^{-2})\Big](x_0)\\
&=8\frac{i(3\xi_n^2-1)}{(\xi_n-i)^4(\xi_n+i)^3}\partial_{x_n}[g(X_1,X_2)]-4h'(0)\frac{(i\xi_n+2)(3\xi_n^2-1)}{(\xi_n-i)^5(\xi_n+i)^3}g(X_1,X_2).
\end{aligned}
\tag{4.15}
$$

Then, we obtain

$$
\begin{aligned}
\Psi_2=&-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\bigg\{8\frac{i(3\xi_n^2-1)}{(\xi_n-i)^4(\xi_n+i)^3}\partial_{x_n}[g(X_1,X_2)]\\
&-4h'(0)\frac{(i\xi_n+2)(3\xi_n^2-1)}{(\xi_n-i)^5(\xi_n+i)^3}g(X_1,X_2)\bigg\}d\xi_n\sigma(\xi')dx'\\
=&-4i\partial_{x_n}[g(X_1,X_2)]\Omega_4\int_{\Gamma^+}\frac{3\xi_n^2-1}{(\xi_n-i)^4(\xi_n+i)^3}d\xi_ndx'\\
&+2h'(0)g(X_1,X_2)\Omega_4\int_{\Gamma^+}\frac{(i\xi_n+2)(3\xi_n^2-1)}{(\xi_n-i)^5(\xi_n+i)^3}d\xi_ndx'\\
=&-4i\partial_{x_n}[g(X_1,X_2)]\Omega_4\frac{2\pi i}{3!}\left[\frac{3\xi_n^2-1}{(\xi_n+i)^3}\right]^{(3)}\bigg|_{|\xi'|=1}dx'\\
&+2h'(0)g(X_1,X_2)\Omega_4\frac{2\pi i}{4!}\left[\frac{(i\xi_n+2)(3\xi_n^2-1)}{(\xi_n+i)^3}\right]^{(4)}\bigg|_{|\xi'|=1}dx'
\end{aligned}
$$

$$= \left(\frac{5h'(0)}{8} g(X_1, X_2) - \partial_{x_n}[g(X_1, X_2)] \right) \pi\Omega_4 dx', \tag{4.16}$$

where ■$_4$ is the canonical volume of S^4.
Similarly, when $l = 4$, we get

$$\begin{aligned}\Psi_2 = \Bigg(&- \frac{5h'(0)}{8} [g(X_1, X_2)g(X_3, X_4) - g(X_1, X_3)g(X_2, X_4) + g(X_1, X_4)g(X_2, X_3)] \\ &+ \partial_{x_n}[g(X_1, X_2)g(X_3, X_4) - g(X_1, X_3)g(X_2, X_4) + g(X_1, X_4)g(X_2, X_3)] \Bigg) \pi\Omega_4 dx'.\end{aligned} \tag{4.17}$$

Similarly, when $l = 6$, we get

$$\begin{aligned}\Psi_2 = \Bigg\{ &- \frac{5h'(0)}{8} \Big(g(X_1, X_2)[g(X_3, X_5)g(X_4, X_6) - g(X_3, X_6)g(X_4, X_5) - g(X_3, X_4) \\ & g(X_5, X_6)] + g(X_1, X_3)[g(X_2, X_4)g(X_5, X_6) - g(X_2, X_6)g(X_4, X_5) - g(X_2, X_5) \\ & g(X_3, X_6)] + g(X_1, X_4)[g(X_2, X_5)g(X_3, X_6) - g(X_2, X_6)g(X_3, X_5) - g(X_2, X_3) \\ & g(X_5, X_6)] + g(X_1, X_5)[g(X_2, X_6)g(X_3, X_4) - g(X_2, X_4)g(X_3, X_6) - g(X_2, X_3) \\ & g(X_4, X_6)] + g(X_1, X_6)[g(X_2, X_4)g(X_3, X_5) - g(X_2, X_5)g(X_3, X_4) - g(X_2, X_3) \\ & g(X_4, X_5)] \Big) + \partial_{x_n} \Big(g(X_1, X_2)[g(X_3, X_5)g(X_4, X_6) - g(X_3, X_6)g(X_4, X_5) \\ & - g(X_3, X_4)g(X_5, X_6)] + g(X_1, X_3)[g(X_2, X_4)g(X_5, X_6) - g(X_2, X_6)g(X_4, X_5) \\ & - g(X_2, X_5)g(X_3, X_6)] + g(X_1, X_4)[g(X_2, X_5)g(X_3, X_6) - g(X_2, X_6)g(X_3, X_5) \\ & - g(X_2, X_3)g(X_5, X_6)] + g(X_1, X_5)[g(X_2, X_6)g(X_3, X_4) - g(X_2, X_4)g(X_3, X_6) \\ & - g(X_2, X_3)g(X_4, X_6)] + g(X_1, X_6)[g(X_2, X_4)g(X_3, X_5) - g(X_2, X_5)g(X_3, X_4) \\ & - g(X_2, X_3)g(X_4, X_5)] \Big) \Bigg\} \pi\Omega_4 dx'.\end{aligned} \tag{4.18}$$

case (a) (III) $r = -2,\ \ell = -2,\ |\alpha| = j = 0,\ k = 1.$
By (4.2), we have

$$\begin{aligned}\Psi_3 &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \mathrm{trace}\Big[\partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-2}(LD^{-2}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-2}(D^{-2})\Big](x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \mathrm{trace}\Big[\partial^2_{\xi_n} \pi^+_{\xi_n} \sigma_{-2}(LD^{-2}) \times \partial_{x_n} \sigma_{-2}(D^{-2})\Big](x_0) d\xi_n \sigma(\xi') dx'\end{aligned} \tag{4.19}$$

By computations, when $l = 2$, we have

$$\partial_{x_n} \sigma_{-2}(D^{-2})(x_0)|_{|\xi'|=1} = -\frac{h'(0)}{(1+\xi_n^2)^2}. \tag{4.20}$$

$$\partial^2_{\xi_n} \pi^+_{\xi_n} \sigma_{-2}(LD^{-2})(x_0)|_{|\xi'|=1} = -\frac{\sum_{\alpha,\beta=1}^{6} a_{1\alpha} a_{2\beta} c(e_\alpha) c(e_\beta) i}{(\xi_n - i)^3}. \tag{4.21}$$

Combining (4.13) and (4.20), we have

$$\mathrm{trace}\left[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(LD^{-2})\times\partial_{x_n}\sigma_{-2}(D^{-2})\right](x_0)(x_0)|_{|\xi'|=1}=-\frac{8h'(0)i}{(\xi_n-i)^5(\xi+i)^2}g(X_1,X_2). \tag{4.22}$$

Then

$$\begin{aligned}\Psi_3&=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}-\frac{8h'(0)i}{(\xi_n-i)^5(\xi+i)^2}g(X_1,X_2)d\xi_n\sigma(\xi')dx'\\&=-4h'(0)ig(X_1,X_2)\Omega_4\int_{\Gamma^+}\frac{1}{(\xi_n-i)^5(\xi+i)^2}d\xi_n dx'\\&=-4h'(0)ig(X_1,X_2)\Omega_4\frac{2\pi i}{4!}\left[\frac{1}{(\xi+i)^2}\right]^{(4)}\Bigg|_{|\xi'|=1}dx'\\&=\frac{-5h'(0)}{8}g(X_1,X_2)\pi\Omega_4dx'.\end{aligned} \tag{4.23}$$

Similarly, when $l=4$, we get

$$\Psi_3=\frac{5h'(0)}{8}[g(X_1,X_2)g(X_3,X_4)-g(X_1,X_3)g(X_2,X_4)+g(X_1,X_4)g(X_2,X_3)]\pi\Omega_4dx'. \tag{4.24}$$

When $l=6$, we get

$$\begin{aligned}\Psi_3=\frac{5h'(0)}{8}\Big\{&g(X_1,X_2)[g(X_3,X_5)g(X_4,X_6)-g(X_3,X_6)g(X_4,X_5)-g(X_3,X_4)g(X_5,X_6)]\\&+g(X_1,X_3)[g(X_2,X_4)g(X_5,X_6)-g(X_2,X_6)g(X_4,X_5)-g(X_2,X_5)g(X_3,X_6)]\\&+g(X_1,X_4)[g(X_2,X_5)g(X_3,X_6)-g(X_2,X_6)g(X_3,X_5)-g(X_2,X_3)g(X_5,X_6)]\\&+g(X_1,X_5)[g(X_2,X_6)g(X_3,X_4)-g(X_2,X_4)g(X_3,X_6)-g(X_2,X_3)g(X_4,X_6)]\\&+g(X_1,X_6)[g(X_2,X_4)g(X_3,X_5)-g(X_2,X_5)g(X_3,X_4)-g(X_2,X_3)g(X_4,X_5)]\Big\}\pi\Omega_4dx'.\end{aligned} \tag{4.25}$$

case (b) $r=-2,\ \ell=-3,\ |\alpha|=j=k=0.$
By (4.2), we have

$$\begin{aligned}\Psi_4&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left[\pi^+_{\xi_n}\sigma_{-2}(LD^{-2})\times\partial_{\xi_n}\sigma_{-3}(D^{-2})\right](x_0)d\xi_n\sigma(\xi')dx'\\&=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(LD^{-2})\times\sigma_{-3}(D^{-2})](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned} \tag{4.26}$$

When $l=2$, we have

$$\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(LD^{-2})(x_0)|_{|\xi'|=1}=\frac{\sum_{\alpha,\beta=1}^{6}a_{1\alpha}c(e_\alpha)a_{2\beta}c(e_\beta)i}{2(\xi_n-i)^2}. \tag{4.27}$$

In the normal coordinate, $g^{ij}(x_0)=\delta_i^j$ and $\partial_{x_j}(g^{\alpha\beta})(x_0)=0$, if $j<n$; $\partial_{x_j}(g^{\alpha\beta})(x_0)=h'(0)\delta^\alpha_\beta$, if $j=n$. So by [16], when $k<n$, we have $\Gamma^n(x_0)=\frac{5}{2}h'(0)$, $\Gamma^k(x_0)=0$, $\delta^n(x_0)=0$ and $\delta^k=\frac{1}{4}h'(0)c(e_k)c(e_n)$. Then, we obtain

$$\begin{aligned}
&\sigma_{-3}(D^{-2})(x_0)|_{|\xi'|=1}\\
&=-\sqrt{-1}|\xi|^{-4}\xi_k(\Gamma^k-2\delta^k)(x_0)|_{|\xi'|=1}-\sqrt{-1}|\xi|^{-6}2\xi^j\xi_\alpha\xi_\beta\partial_jg^{\alpha\beta}(x_0)|_{|\xi'|=1}\\
&=\frac{-i}{(1+\xi_n^2)^2}\Big(-\frac{1}{2}h'(0)\sum_{k<n}\xi_k\widetilde{c}(e_k)\widetilde{c}(e_n)+\frac{5}{2}h'(0)\xi_n\Big)-\frac{2ih'(0)\xi_n}{(1+\xi_n^2)^3}.
\end{aligned}\tag{4.28}$$

We note that $i<n$, $\int_{|\xi'|=1}\{\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\}\sigma(\xi')=0$, so the first term in (4.28) has no contribution for computing **case (b)**.
By (4.27) and (4.28), we have

$$\begin{aligned}
&\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(LD^{-2})\times\sigma_{-3}(D^{-2})](x_0)|_{|\xi'|=1}\\
&=-\frac{2h'(0)\xi_n(5\xi_n^2-1)}{(\xi_n-i)^5(\xi_n+i)^3}g(X_1,X_2).
\end{aligned}\tag{4.29}$$

So when $l=2$, we have

$$\begin{aligned}
\Psi_4&=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}-\frac{2h'(0)\xi_n(5\xi_n^2-1)}{(\xi_n-i)^5(\xi_n+i)^3}g(X_1,X_2)d\xi_n\sigma(\xi')dx'\\
&=-2ih'(0)g(X_1,X_2)\Omega_4\int_{\Gamma^+}\frac{\xi_n(5\xi_n^2-1)}{(\xi_n-i)^5(\xi_n+i)^3}d\xi_ndx'\\
&=-2ih'(0)g(X_1,X_2)\Omega_4\frac{2\pi i}{4!}\left[\frac{\xi_n(5\xi_n^2-1)}{(\xi_n+i)^3}\right]^{(4)}\Bigg|_{|\xi'|=1}dx'\\
&=\frac{15h'(0)}{8}g(X_1,X_2)\pi\Omega_4dx'.
\end{aligned}\tag{4.30}$$

Similarly, when $l=4$, we get

$$\Psi_4=-\frac{15h'(0)}{8}[g(X_1,X_2)g(X_3,X_4)-g(X_1,X_3)g(X_2,X_4)+g(X_1,X_4)g(X_2,X_3)]\pi\Omega_4dx'.\tag{4.31}$$

When $l=6$, we get

$$\begin{aligned}
\Psi_4=-\frac{15h'(0)}{8}\Big\{&g(X_1,X_2)[g(X_3,X_5)g(X_4,X_6)-g(X_3,X_6)g(X_4,X_5)-g(X_3,X_4)g(X_5,X_6)]\\
&+g(X_1,X_3)[g(X_2,X_4)g(X_5,X_6)-g(X_2,X_6)g(X_4,X_5)-g(X_2,X_5)g(X_3,X_6)]\\
&+g(X_1,X_4)[g(X_2,X_5)g(X_3,X_6)-g(X_2,X_6)g(X_3,X_5)-g(X_2,X_3)g(X_5,X_6)]\\
&+g(X_1,X_5)[g(X_2,X_6)g(X_3,X_4)-g(X_2,X_4)g(X_3,X_6)-g(X_2,X_3)g(X_4,X_6)]\\
&+g(X_1,X_6)[g(X_2,X_4)g(X_3,X_5)-g(X_2,X_5)g(X_3,X_4)-g(X_2,X_3)g(X_4,X_5)]\Big\}\pi\Omega_4dx'.
\end{aligned}\tag{4.32}$$

case (c) $r=-3,\ \ell=-2,\ |\alpha|=j=k=0.$
By (4.2), we have

$$\begin{aligned}\Psi_5&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_n}\sigma_{-3}(LD^{-2})\times\partial_{\xi_n}\sigma_{-2}(D^{-2})\Big](x_0)d\xi_n\sigma(\xi')dx'\\&=\mathbf{case(b)}-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_n}\partial_{\xi_n}\sigma_{-2}(D^{-2})\times\sigma_{-3}(LD^{-2})\Big](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned}\tag{4.33}$$

Then, we only need to compute

$$-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_n}\partial_{\xi_n}\sigma_{-2}(D^{-2})\times\sigma_{-3}(LD^{-2})\Big](x_0)d\xi_n\sigma(\xi')dx'.\tag{4.34}$$

By Lemma 4.1, when $l=2$, we have

$$\partial_{\xi_n}\sigma_{-2}(D^{-2})(x_0)=-\frac{2\xi_n}{(1+\xi_n^2)^2}.\tag{4.35}$$

$$\begin{aligned}&\sigma_{-3}(LD^{-2})\\&=\sum_{\alpha,\beta=1}^{6}a_{1\alpha}a_{2\beta}c(e_\alpha)c(e_\beta)\Big(\frac{-i}{(1+\xi_n^2)^2}\Big(-\frac{1}{2}h'(0)\sum_{k<n}\xi_k\widetilde{c}(e_k)\widetilde{c}(e_n)+\frac{5}{2}h'(0)\xi_n\Big)-\frac{2ih'(0)\xi_n}{(1+\xi_n^2)^3}\Big).\end{aligned}\tag{4.36}$$

We note that $i<n$, $\int_{|\xi'|=1}\{\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\}\sigma(\xi')=0$, so the first term in (4.36) has no contribution for computing **case (c)**.
On the other hand,

$$\mathrm{trace}\Big[\pi^+_{\xi_n}\partial_{\xi_n}\sigma_{-2}(D^{-2})\times\sigma_{-3}(LD^{-2})\Big](x_0)=-8\frac{ih'(0)\xi_n^2(9+5\xi_n^2)}{(1+\xi_n^2)^5}g(X_1,X_2).\tag{4.37}$$

Then, when $l=2$, we have

$$\begin{aligned}\Psi_5&=\frac{15h'(0)}{8}g(X_1,X_2)\pi\Omega_4dx'-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}-8\frac{ih'(0)\xi_n^2(9+5\xi_n^2)}{(1+\xi_n^2)^5}g(X_1,X_2)d\xi_n\sigma(\xi')dx'\\&=\frac{15h'(0)}{8}g(X_1,X_2)\pi\Omega_4dx'-8h'(0)g(X_1,X_2)\Omega_4\int_{\Gamma^+}\frac{\xi_n^2(9+5\xi_n^2)}{(1+\xi_n^2)^5}d\xi_ndx'\\&=\frac{15h'(0)}{8}g(X_1,X_2)\pi\Omega_4dx'-8h'(0)g(X_1,X_2)\Omega_4\frac{2\pi i}{4!}\left[\frac{\xi_n^2(9+5\xi_n^2)}{(\xi_n+i)^5}\right]^{(4)}\Bigg|_{|\xi'|=1}dx'\\&=-\frac{15h'(0)}{8}g(X_1,X_2)\pi\Omega_4dx'.\end{aligned}\tag{4.38}$$

Similarly, when $l=4$, we get

$$\Psi_5=\frac{15h'(0)}{8}[g(X_1,X_2)g(X_3,X_4)-g(X_1,X_3)g(X_2,X_4)+g(X_1,X_4)g(X_2,X_3)]\pi\Omega_4dx'. \tag{4.39}$$

When $l=6$, we get

$$\begin{aligned}\Psi_5=\frac{15h'(0)}{8}\Big\{&g(X_1,X_2)[g(X_3,X_5)g(X_4,X_6)-g(X_3,X_6)g(X_4,X_5)-g(X_3,X_4)g(X_5,X_6)]\\&+g(X_1,X_3)[g(X_2,X_4)g(X_5,X_6)-g(X_2,X_6)g(X_4,X_5)-g(X_2,X_5)g(X_3,X_6)]\\&+g(X_1,X_4)[g(X_2,X_5)g(X_3,X_6)-g(X_2,X_6)g(X_3,X_5)-g(X_2,X_3)g(X_5,X_6)]\\&+g(X_1,X_5)[g(X_2,X_6)g(X_3,X_4)-g(X_2,X_4)g(X_3,X_6)-g(X_2,X_3)g(X_4,X_6)]\\&+g(X_1,X_6)[g(X_2,X_4)g(X_3,X_5)-g(X_2,X_5)g(X_3,X_4)-g(X_2,X_3)g(X_4,X_5)]\Big\}\pi\Omega_4dx'.\end{aligned} \tag{4.40}$$

Now Ψ is the sum of the cases (a), (b) and (c), then when $l=2$, we get

$$\Psi=\sum_{i=1}^{5}\Psi_i=-\partial_{x_n}[g(X_1,X_2)]\pi\Omega_4dx'. \tag{4.41}$$

Similarly, when $l=4$, we get

$$\Psi=\sum_{i=1}^{5}\Psi_i=\partial_{x_n}[g(X_1,X_2)g(X_3,X_4)-g(X_1,X_3)g(X_2,X_4)+g(X_1,X_4)g(X_2,X_3)]\pi\Omega_4dx'. \tag{4.42}$$

Similarly, when $l=6$, we get

$$\begin{aligned}\Psi=\sum_{i=1}^{5}\Psi_i=\partial_{x_n}\Big(&g(X_1,X_2)[g(X_3,X_5)g(X_4,X_6)-g(X_3,X_6)g(X_4,X_5)-g(X_3,X_4)\\&g(X_5,X_6)]+g(X_1,X_3)[g(X_2,X_4)g(X_5,X_6)-g(X_2,X_6)g(X_4,X_5)-g(X_2,X_5)\\&g(X_3,X_6)]+g(X_1,X_4)[g(X_2,X_5)(X_3,X_6)-g(X_2,X_6)g(X_3,X_5)-g(X_2,X_3)\\&g(X_5,X_6)]+g(X_1,X_5)[g(X_2,X_6)g(X_3,X_4)-g(X_2,X_4)g(X_3,X_6)-g(X_2,X_3)\\&g(X_4,X_6)]+g(X_1,X_6)[g(X_2,X_4)g(X_3,X_5)-g(X_2,X_5)g(X_3,X_4)-g(X_2,X_3)\\&g(X_4,X_5)]\Big)\pi\Omega_4dx'.\end{aligned} \tag{4.43}$$

Obviously, when $l=1$ *or* 3 *or* 5, we get $\Psi_1=\Psi_2=\Psi_3=\Psi_4=\Psi_5=0$.
By (4.1)-(4.6), we obtain following theorem

Theorem 4.1. *Let M be a 6-dimensional oriented compact manifold with boundary ∂M and the metric g^M be defined as (3.1), then the following are the generalized noncommutative residue of the Dirac operator*
(1)when $l=2$, we get

$$\widetilde{\mathrm{Wres}}[\pi^+(LD^{-2})\circ\pi^+(D^{-2})]$$

$$= 128\pi^2 \int_M \left(\frac{2}{3} g(X_1, X_2) s\right) d\mathrm{Vol_M} + \int_{\partial M} \left(- \partial_{x_n} [g(X_1, X_2)]\right) \pi\Omega_4 d\mathrm{Vol_M}. \quad (4.44)$$

(2)when $l = 4$, we get

$$\widetilde{\mathrm{Wres}}[\pi^+(LD^{-2}) \circ \pi^+(D^{-2})]$$
$$= 128\pi^2 \int_M \left(- \frac{2}{3} [g(X_1, X_2) g(X_3, X_4) - g(X_1, X_3) g(X_2, X_4) + g(X_1, X_4) g(X_2, X_3)] s\right) d\mathrm{Vol_M}$$
$$+ \int_{\partial M} \left(\partial_{x_n} [g(X_1, X_2) g(X_3, X_4) - g(X_1, X_3) g(X_2, X_4) + g(X_1, X_4) g(X_2, X_3)]\right) \pi\Omega_4 d\mathrm{Vol_M}. \quad (4.45)$$

(3)when $l = 6$, we get

$$\widetilde{\mathrm{Wres}}[\pi^+(LD^{-2}) \circ \pi^+(D^{-2})]$$
$$= 128\pi^2 \int_M \Big\{ - \frac{2}{3} \Big(g(X_1, X_2)[g(X_3, X_5) g(X_4, X_6) - g(X_3, X_6) g(X_4, X_5) - g(X_3, X_4)$$
$$g(X_5, X_6)] + g(X_1, X_3)[g(X_2, X_4) g(X_5, X_6) - g(X_2, X_6) g(X_4, X_5) - g(X_2, X_5) g(X_3, X_6)]$$
$$+ g(X_1, X_4)[g(X_2, X_5) g(X_3, X_6) - g(X_2, X_6) g(X_3, X_5) - g(X_2, X_3) g(X_5, X_6)] + g(X_1, X_5)$$
$$[g(X_2, X_6) g(X_3, X_4) - g(X_2, X_4) g(X_3, X_6) - g(X_2, X_3) g(X_4, X_6)] + g(X_1, X_6)[g(X_2, X_4)$$
$$g(X_3, X_5) - g(X_2, X_5) g(X_3, X_4) - g(X_2, X_3) g(X_4, X_5)] \Big) s \Big\} d\mathrm{Vol_M}$$
$$+ \int_{\partial M} \Big\{ \partial_{x_n} \Big(g(X_1, X_2)[g(X_3, X_5) g(X_4, X_6) - g(X_3, X_6) g(X_4, X_5) - g(X_3, X_4) g(X_5, X_6)]$$
$$+ g(X_1, X_3)[g(X_2, X_4) g(X_5, X_6) - g(X_2, X_6) g(X_4, X_5) - g(X_2, X_5) g(X_3, X_6)] + g(X_1, X_4)$$
$$[g(X_2, X_5) g(X_3, X_6) - g(X_2, X_6) g(X_3, X_5) - g(X_2, X_3) g(X_5, X_6)] + g(X_1, X_5)[g(X_2, X_6)$$
$$g(X_3, X_4) - g(X_2, X_4) g(X_3, X_6) - g(X_2, X_3) g(X_4, X_6)] + g(X_1, X_6)[g(X_2, X_4) g(X_3, X_5)$$
$$- g(X_2, X_5) g(X_3, X_4) - g(X_2, X_3) g(X_4, X_5)] \Big) \Big\} \pi\Omega_4 d\mathrm{Vol_M}. \quad (4.46)$$

(4)when $l = 1$ or 3 or 5, we get

$$\widetilde{\mathrm{Wres}}[\pi^+(LD^{-2}) \circ \pi^+(D^{-2})] = 0. \quad (4.47)$$

Acknowledgments. The second author was supported in part by NSFC No.11771070. The authors thank the referee for his (or her) careful reading and helpful comments.

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