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SUPER WARPED PRODUCTS WITH A SEMI-SYMMETRIC METRIC CONNECTION

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ABSTRACT. In this paper, we define the semi-symmetric metric connection on super Riemannian manifolds. We compute the semi-symmetric metric connection and its curvature tensor and its Ricci tensor on super warped product spaces. We introduce two kind of super warped product spaces with the semi-symmetric metric connection and give the conditions under which these two super warped product spaces with the semi-symmetric metric connection are the Einstein super spaces with the semi-symmetric metric connection.

1. INTRODUCTION

The (singly) warped product $B \times_h F$ of two pseudo-Riemannian manifolds (B, g_B) and (F, g_F) with a smooth function $h : B \to (0, \infty)$ is the product manifold $B \times F$ with the metric tensor $g = g_B \oplus h^2 g_F$. Here, (B, g_B) is called the base manifold and (F, g_F) is called as the fiber manifold and h is called as the warping function. Generalized Robertson-Walker space-times and standard static space-times are two well-known warped product spaces.The concept of warped products was first introduced by Bishop and ONeil (see [2]) to construct examples of Riemannian manifolds with negative curvature. In Riemannian geometry, warped product manifolds and their generic forms have been used to construct new examples with interesting curvature properties since then. In [4], F. Dobarro and E. Dozo had studied from the viewpoint of partial differential equations and variational methods, the problem of showing when a Riemannian metric of constant scalar curvature can be produced on a product manifolds by a warped product construction. In [5], Ehrlich, Jung and Kim got explicit solutions to warping function to have a constant scalar curvature for generalized Robertson-Walker space-times. In [1], explicit solutions were also obtained for the warping function to make the space-time as Einstein when the fiber is also Einstein.
The definition of a semi-symmetric metric connection was given by H. Hayden in [8]. In 1970, K. Yano [11] considered a semi-symmetric metric connection and studied some of its properties. He proved that a Riemannian manifold admitting the semi-symmetric metric connection has vanishing curvature tensor if and only if it is conformally flat. Motivated by the Yano' result, in [9], Sular and Özgur studied warped product manifolds

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with a semi-symmetric metric connection, they computed curvature of semi-symmetric metric connection and considered Einstein warped product manifolds with a semi-symmetric metric connection. In [10], we extended the results of Sular and Özgur to multiply twisted products with a semi-symmetric metric connection.
On the other hand, in [3], the definition of super warped product spaces was given. In [6], several new super warped product spaces were given and the authors also studied the Einstein equations with cosmological constant in these new super warped product spaces. Our motivation is to extend the results of Sular and Özgur to super warped product spaces with a semi-symmetric metric connection.
In Section 2, we state some definitions of super manifolds and super Riemannian metric. We also define the semi-symmetric metric connection on super Riemannian manifolds and prove that there is a unique semi-symmetric metric connection on super Riemannian manifolds which is metric and has the semi-symmetric torsion. In Section 3, we compute the semi-symmetric metric connection and its curvature tensor and its Ricci tensor on super warped product spaces. In Section 4, we introduce two kind of super warped product spaces with the semi-symmetric metric connection and give the conditions under which these two super warped product spaces with the semi-symmetric metric connection are the Einstein super spaces with the semi-symmetric metric connection.

2. A semi-symmetric metric connection on super Riemannian manifolds

Firstly we introduce some notations on Riemannian supergeometry.

Definition 2.1. *A locally $\mathbb{Z}_2$-ringed space is a pair $S := (|S|, \mathcal{O}_S)$ where $|S|$ is a second-countable Hausdorff space, and a $\mathcal{O}_S$ is a sheaf of $\mathbb{Z}_2$-graded $\mathbb{Z}_2$-commutative associative unital $\mathbb{R}$-algebras, such that the stalks $\mathcal{O}_{S,p}$, $p \in |S|$ are local rings.*

In this context, $\mathbb{Z}_2$-commutative means that any two sections $s, t \in \mathcal{O}_S(|U|)$, $|U| \subset |S|$ open, of homogeneous degree $|s| \in \mathbb{Z}_2$ and $|t| \in \mathbb{Z}_2$ commute up to the sign rule $st = (-1)^{|s||t|}ts$. $\mathbb{Z}_2$-ring space $U^{m|n} := (U, C^\infty_{U^m} \otimes \wedge\mathbb{R}^n)$, is called standard superdomain where $C^\infty_{U^m}$ is the sheaf of smooth functions on U and $\wedge\mathbb{R}^n$ is the exterior algebra of $\mathbb{R}^n$. We can employ (natural) coordinates $x^I := (x^a, \xi^A)$ on any $\mathbb{Z}_2$-domain, where x^a form a coordinate system on U and the ξ^A are formal coordinates.

Definition 2.2. *A supermanifold of dimension $m|n$ is a super ringed space $M = (|M|, \mathcal{O}_M)$ that is locally isomorphic to $\mathbb{R}^{m|n}$ and $|M|$ is a second countable and Hausdorff topological space.*

The tangent sheaf $\mathcal{T}M$ of a $\mathbb{Z}_2$-manifold M is defined as the sheaf of derivations of sections of the structure sheaf, i.e., $\mathcal{T}M(|U|) := \mathrm{Der}(\mathcal{O}_M(|U|))$, for arbitrary open set $|U| \subset |M|$. Naturally, this is a sheaf of locally free $\mathcal{O}_M$-modules. Global sections of the tangent sheaf are referred to as *vector fields*. We denote the $\mathcal{O}_M(|M|)$-module of vector fields as $\mathrm{Vect}(M)$. The dual of the tangent sheaf is the *cotangent sheaf*, which we denote as $\mathcal{T}^*M$. This is also a sheaf of locally free $\mathcal{O}_M$-modules. Global section of the tangent sheaf we will refer to as *one-forms* and we denote the $\mathcal{O}_M(|M|)$-module of one-forms as $\Omega^1(M)$.

Definition 2.3. *A Riemannian metric on a $\mathbb{Z}_2$-manifold M is a $\mathbb{Z}_2$-homogeneous, $\mathbb{Z}_2$-symmetric, non-degenerate, $\mathcal{O}_M$-linear morphisms of sheaves $\langle -,- \rangle_g \; : \quad \mathcal{T}M \otimes \mathcal{T}M \to \mathcal{O}_M$. A $\mathbb{Z}_2$-manifold equipped with a Riemannian metric is referred to as a Riemannian $\mathbb{Z}_2$-manifold.*

We will insist that the Riemannian metric is homogeneous with respect to the $\mathbb{Z}_2$-degree, and we will denote the degree of the metric as $|g| \in \mathbb{Z}_2$. Explicitly, a Riemannian metric has the following properties:
(1)$|\langle X,Y \rangle_g| = |X| + |Y| + |g|$,
(2)$\langle X,Y \rangle_g = (-1)^{|X||Y|} \langle Y,X \rangle_g$,
(3) If $\langle X,Y \rangle_g = 0$ for all $Y \in Vect(M)$, then $X = 0$,
(4) $\langle fX+Y,Z \rangle_g = f\langle X,Z \rangle_g + \langle Y,Z \rangle_g$,
for arbitrary (homogeneous) $X,Y,Z \in \mathrm{Vect}(M)$ and $f \in C^\infty(M)$. We will say that a Riemannian metric is even if and only if it has degree zero. Similarly, we will say that a Riemannian metric is odd if and only if it has degree one. Any Riemannian metric we consider will be either even or odd as we will only be considering homogeneous metrics. Now we recall the definition of the warped product of Riemannian $\mathbb{Z}_2$-manifolds. For details, see the section 2.3 in [3]. Let $M_1 \times M_2$ be the product of two $\mathbb{Z}_2$-manifolds M_1 and M_2. Let $(M_i, g_i)(i = 1,2)$ be Riemannian $\mathbb{Z}_2$-manifolds whose Riemannian metric are of the same $\mathbb{Z}_2$-degree. Let $\mu \in C^\infty(M_1)$ be a degree 0 invertible global functions that is strictly positive, i.e. $\varepsilon_{M_1}(\mu)$ a strictly positive function on $|M_1|$ where ε is simply "throwing away" the formal coordinates. Then the warped product is defined as

$$M_1 \times_\mu M_2 := (M_1 \times M_2, g := \pi_1^* g_1 + (\pi_1^*\mu)\pi_2^* g_2),$$

where $\pi_i : M_1 \times M_2 \to M_i \quad (i = 1,2)$ is the projection. By Proposition 4 in [3], the warped product $M_1 \times_\mu M_2$ is a Riemannian $\mathbb{Z}_2$-manifold.

Definition 2.4. *(Definition 9 in* [3]*) An affine connection on a $\mathbb{Z}_2$-manifold is a $\mathbb{Z}_2$-degree preserving map*

$$\nabla : \;\; \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M); \;\; (X,Y) \mapsto \nabla_X Y,$$

that satisfies the following
1) Bi-linearity

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z; \;\; \nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z,$$

2)$C^\infty(M)$-linearrity in the first argument

$$\nabla_{fX} Y = f \nabla_X Y,$$

3)The Leibniz rule

$$\nabla_X(fY) = X(f)Y + (-1)^{|X||f|} f \nabla_X Y,$$

for all homogeneous $X,Y,Z \in \mathrm{Vect}(M)$ and $f \in C^\infty(M)$.

Definition 2.5. *(Definition 10 in* [3]*) The torsion tensor of an affine connection*
$T_\nabla : \;\; \mathrm{Vect}(M) \otimes_{C^\infty(M)} \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ *is defined as*

$$T_\nabla(X,Y) := \nabla_X Y - (-1)^{|X||Y|} \nabla_Y X - [X,Y],$$

for any (homogeneous) $X,Y \in \mathrm{Vect}(M)$. An affine connection is said to be symmetric if the torsion vanishes.

Definition 2.6. *(Definition 11 in* [3]*) An affine connection on a Riemannian $\mathbb{Z}_2$-manifold (M,g) is said to be metric compatible if and only if*

$$X\langle Y,Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + (-1)^{|X||Y|}\langle Y, \nabla_X Z\rangle_g,$$

for any $X,Y,Z \in \mathrm{Vect}(M)$.

Theorem 2.1. *(Theorem 1 in* [3]*)There is a unique symmetric (torsionless) and metric compatible affine connection ∇^L on a Riemannian $\mathbb{Z}_2$-manifold (M,g) which satisfies the Koszul formula*

$$\begin{aligned} 2\left\langle \nabla^L_X Y, Z\right\rangle_g &= X\langle Y,Z\rangle_g + \langle [X,Y],Z\rangle_g \\ &+ (-1)^{|X|(|Y|+|Z|)}(Y\langle Z,X\rangle_g - \langle [Y,Z],X\rangle_g) \\ &- (-1)^{|Z|(|X|+|Y|)}(Z\langle X,Y\rangle_g - \langle [Z,X],Y\rangle_g), \end{aligned} \tag{2.1}$$

for all homogeneous $X,Y,Z \in \mathrm{Vect}(M)$.

Definition 2.7. *(Definition 13 in* [3]*) The Riemannian curvature tensor of an affine connection*

$$R_\nabla: \ \mathrm{Vect}(M)\otimes_{C^\infty(M)}\mathrm{Vect}(M)\otimes_{C^\infty(M)}\mathrm{Vect}(M)\to \mathrm{Vect}(M)$$

is defined as

$$R_\nabla(X,Y)Z = \nabla_X\nabla_Y - (-1)^{|X||Y|}\nabla_Y\nabla_X - \nabla_{[X,Y]}Z, \tag{2.2}$$

for all X,Y and $Z \in \mathrm{Vect}(M)$.

Directly from the definition it is clear that

$$R_\nabla(X,Y)Z = -(-1)^{|X||Y|}R_\nabla(Y,X)Z, \tag{2.3}$$

for all X,Y and $Z \in \mathrm{Vect}(M)$.

Definition 2.8. *(Definition 14 in* [3]*) The Ricci curvature tensor of an affine connection is the symmetric rank-2 covariant tensor defined as*

$$Ric_\nabla(X,Y) := (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|X|+|Y|)}\frac{1}{2}\left[R_\nabla(\partial_{x^I},X)Y + (-1)^{|X||Y|}R_\nabla(\partial_{x^I},Y)X\right]^I, \tag{2.4}$$

where $X,Y \in \mathrm{Vect}(M)$ and $[\quad]^I$ denotes the coefficient of ∂_{x^I} and ∂_{x^I} is the natural frame of $\mathcal{T}M$.

Definition 2.9. *(Definition 16 in* [3]*) Let $f \in C^\infty(M)$ be an arbitrary function on a Riemannian $\mathbb{Z}_2$-manifold (M,g). The gradient of f is the unique vector field $\mathrm{grad}_g f$ such that*

$$X(f) = (-1)^{|f||g|}\left\langle X, \mathrm{grad}_g f\right\rangle_g, \tag{2.5}$$

for all $X \in \mathrm{Vect}(M)$.

Definition 2.10. *(Definition 17 in* [3]*) Let (M,g) be a Riemannian $\mathbb{Z}_2$-manifold and let ∇^L be the associated Levi-Civita connection. The covariant divergence is the map $\mathrm{Div}_L : \mathrm{Vect}(M) \to C^\infty(M)$, given by*

$$\mathrm{Div}_L(X) = (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|X|)}(\nabla_{\partial_{x^I}}X)^I, \tag{2.6}$$

for any arbitrary $X \in \mathrm{Vect}(M)$.

Definition 2.11. *(Definition 18 in* [3]*) Let (M, g) be a Riemannian $\mathbb{Z}_2$-manifold and let ∇^L be the associated Levi-Civita connection. The connection Laplacian (acting on functions) is the differential operator of $\mathbb{Z}_2$-degree $|g|$ defined as*

$$\triangle_g(f) = \mathrm{Div}_L(\mathrm{grad}_g f), \tag{2.7}$$

for any and all $f \in C^\infty(M)$.

Definition 2.12. *Let (M, g) be a Riemannian $\mathbb{Z}_2$-manifold and $P \in \mathrm{Vect}(M)$ which satisfied $|g| + |P| = 0$ and the semi-symmetric metric connection ∇ on (M, g) is given by*

$$\nabla_X Y = \nabla^L_X Y + X \cdot g(Y, P) - g(X, Y)P = \nabla^L_X Y + (-1)^{|X||Y|} g(Y, P)X - g(X, Y)P, \tag{2.8}$$

for any homogenous $X, Y \in \mathrm{Vect}(M)$ and we define $\nabla_{X_1+X_2} Y+ = \nabla_{X_1} Y + \nabla_{X_2} Y$; $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$, for the general $X = X_1 + X_2$ and $Y = Y_1 + Y_2$. Here $X \cdot f = (-1)^{|X||f|} fX$ for $f \in C^\infty(M)$.

We can verify that $\nabla_X Y$ satisfies Definition 2.4, then $\nabla_X Y$ is an affine connection. By Definition 2.5, we get

$$T_\nabla(X, Y) = X \cdot g(Y, P) - (-1)^{|X||Y|} Y \cdot g(X, P), \tag{2.9}$$

In this case, we call that $\nabla_X Y$ is a semi-symmetric connection. By Definition 2.6 and (2.8) and ∇^L preserving the metric and $|g| + |P| = 0$, we get

$$\begin{aligned}
&\langle \nabla_X Y, Z \rangle_g + (-1)^{|X||Y|} \langle Y, \nabla_X Z \rangle_g \qquad (2.10)\\
&= X \langle Y, Z \rangle_g + \langle X \cdot g(Y, P), Z \rangle_g - \langle g(X, Y)P, Z \rangle_g \\
&+ (-1)^{|X||Y|} [\langle Y, X \cdot g(Z, P) \rangle_g - \langle Y, g(X, Z)P \rangle_g] \\
&= X \langle Y, Z \rangle_g + (-1)^{|g(Y,P)||X|} g(Y, P) g(X, Z) - g(X, Y) g(P, Z) \\
&+ (-1)^{|X||Y|} (-1)^{|g(Z,P)||X|} g(Y, g(Z, P)X) - (-1)^{|X||Y|} (-1)^{|g(X,Z)||Y|} g(X, Z) g(Y, P), \\
&= X \langle Y, Z \rangle_g .
\end{aligned}$$

So ∇ preserves the metric.

Theorem 2.2. *There is a unique metric compatible affine connection ∇ on a Riemannian $\mathbb{Z}_2$-manifold (M, g) which satisfies (2.9).*

Proof. By (2.9) and (2.10), we know that the semi-symmetric metric connection ∇ satisfies the conditions in Theorem 2.2. We only prove the uniqueness. Let ∇^1 be a connection which satisfies the conditions in Theorem 2.14. Let $\nabla^1_X Y = \nabla^L_X Y + H(X, Y)$. then

$$H(fX, Y) = fH(X, Y), \;\; H(X, fY) = (-1)^{|f||X|} H(X, Y). \tag{2.11}$$

By ∇^1 and ∇^L preserving the metric, we get

$$g(H(X, Y), Z) + (-1)^{|X||Y|} g(Y, H(X, Z)) = 0. \tag{2.12}$$

By ∇^L having no torsion, we have

$$T_{\nabla^1}(X, Y) = H(X, Y) - (-1)^{|X||Y|} H(Y, X). \tag{2.13}$$

By (2.12) and (2.13) and $|H| = 0$, we have

$$g(T_{\nabla^1}(X, Y), Z) + (-1)^{|Z|(|X|+|Y|)} g(T_{\nabla^1}(Z, X), Y) \tag{2.14}$$

$$+(-1)^{|X||Y|}(-1)^{|Z|(|X|+|Y|)}g(T_{\nabla^1}(Z,Y),X)=2g(H(X,Y),Z).$$

By (2.9) and (2.14), we get

$$2g(H(X,Y),Z)=-2g(X,Y)g(Y,Z)+2g(X\cdot g(Y,P),Z). \tag{2.15}$$

Then $H(X,Y)=X\cdot g(Y,P)-g(X,Y)P$ and ∇^1 is ∇ in the definition 2.13. □

Let π be a one form defined by $\pi(Z):=g(Z,P)$, then $|\pi|=0$. We have

Proposition 2.1. *The following equality holds*

$$\begin{aligned}R_\nabla(X,Y)Z&=R^L(X,Y)Z+(-1)^{(|X|+|Y|)|Z|}g(Z,\nabla^L_XP)Y\\&-(-1)^{|X||Y|}(-1)^{(|X|+|Y|)|Z|}g(Z,\nabla^L_YP)X-(-1)^{|g(Y,Z)||X|}g(Y,Z)\nabla^L_XP\\&+(-1)^{|X||Y|}(-1)^{|g(X,Z)||Y|}g(X,Z)\nabla^L_YP+(-1)^{|X|(|Y|+|Z|)}(-1)^{|Y||Z|}\pi(Z)\pi(Y)X\\&-(-1)^{|X|(|Y|+|Z|)}g(Y,Z)\pi(P)X-(-1)^{(|X|+|Y|)|Z|}\pi(Z)\pi(X)Y\\&+(-1)^{|Y||Z|)}g(X,Z)\pi(P)Y+(-1)^{|X||g(Y,Z)|}g(Y,Z)\pi(X)P\\&-(-1)^{|X||Y|}(-1)^{|Y||g(X,Z)|}g(X,Z)\pi(Y)P.\end{aligned} \tag{2.16}$$

Proof. By the Leibniz rule and ∇^L preserving metric, we have

$$\nabla^L_X(\pi(Z)Y)=\pi(\nabla^L_XZ)Y+(-1)^{|X||Z|}g(Z,\nabla^L_XP)Y+(-1)^{|X||Z|}\pi(Z)\nabla^L_XY, \tag{2.17}$$

$$\nabla^L_X(g(Y,Z)P)=g(\nabla^L_XY,Z)P+(-1)^{|X||Y|}g(Y,\nabla^L_XZ)P+(-1)^{|g(Y,Z)||X|}g(Y,Z)\nabla^L_XP. \tag{2.18}$$

By (2.2), (2.8), (2.17) and (2.18) and some computations, we can get Proposition 2.15. □

3. Super warped products with a semi-symmetric metric connection

Let $(M_1=M\times_\mu N, g_\mu=\pi_1^*g_1+\pi_1^*(\mu)\pi_2^*g_2)$ be the super warped product with $|g|=|g_1|=|g_2|$ and $|\mu|=0$. For simplicity, we assume that $\mu=h^2$ with $|h|=0$. Let $\nabla^{L,\mu}$ be the Levi-Civita connection on (M_1,g_μ) and $\nabla^{L,M}$ (resp. $\nabla^{L,N}$) be the Levi-Civita connection on (M,g_1) (resp.(N,g_2)).

Lemma 3.1. *For $X,Y\in\mathrm{Vect}(M)$ and $U,W\in\mathrm{Vect}(N)$, we have*

$$\begin{aligned}&(1)\nabla^{L,\mu}_XY=\nabla^{L,M}_XY,\ \ (2)\nabla^{L,\mu}_XU=\frac{X(h)}{h}U,\\&(3)\nabla^{L,\mu}_UX=(-1)^{|U||X|}\frac{X(h)}{h}U,\ \ (4)\nabla^{L,\mu}_UW=-hg_2(U,W)\mathrm{grad}_{g_1}h+\nabla^{L,N}_UW.\end{aligned} \tag{3.1}$$

Proof. By (2.1) and $[X,V]=0$, we have $g_\mu(\nabla^{L,\mu}_XY,Z)=g_1(\nabla^{L,M}_XY,Z)$ and $g_\mu(\nabla^{L,\mu}_XY,V)=0$, so (1) holds. Similarly, we have $g_\mu(\nabla^{L,\mu}_XU,Y)=0$ and $2g_\mu(\nabla^{L,\mu}_XU,V)=\frac{X(\mu)}{\mu}g_\mu(U,V)$, so (2) holds by $\mu=h^2$. By $\nabla^{L,\mu}$ having no torsion and (2), we have (3). By (2.1) and (2.5), we have

$$\begin{aligned}2g_\mu(\nabla^{L,\mu}_UW,X)&=-(-1)^{|X|(|U|+|W|)}X(\mu)g_2(U,W)\\&=-(-1)^{|X|(|U|+|W|)}g_1(X,\mathrm{grad}_{g_1}(\mu))g_2(U,W)=-g_\mu(g_2(U,W)\mathrm{grad}_{g_1}(\mu),X),\end{aligned} \tag{3.2}$$

and $g_\mu(\nabla^{L,\mu}_UW,U_1)=g_\mu(\nabla^{L,N}_UW,U_1)$ for $U_1\in\mathrm{Vect}(N)$, so (4) holds. □

Let $R^{L,\mu}$ denote the curvature tensor of the Levi-Civita connection on (M_1, g_μ). Let $R^{L,M}$ (resp. $R^{L,N}$) be the curvature tensor of the Levi-Civita connection on (M, g_1) (resp.(N, g_2)). Let $H^h_M(X,Y) := XY(h) - \nabla^{L,M}_X Y(h)$, then $H^h_M(fX,Y) = fH^h_M(X,Y)$ and $H^h_M(X,fY) = (-1)^{|f||X|} f H^h_M(X,Y)$. H^h_M is a $(0,2)$ tensor.

Proposition 3.1. *For $X,Y,Z \in \mathrm{Vect}(M)$ and $U,V,W \in \mathrm{Vect}(N)$, we have*

$$(1) R^{L,\mu}(X,Y)Z = R^{L,M}(X,Y)Z, \quad (2) R^{L,\mu}(V,X)Y = -(-1)^{|V|(|X|+|Y|)} \frac{H^h_M(X,Y)}{h} V, \tag{3.3}$$

$$(3) R^{L,\mu}(X,Y)V = 0, \quad (4) R^{L,\mu}(V,W)X = 0,$$

$$(5) R^{L,\mu}(X,V)W = -(-1)^{|X|(|V|+|W|+|g|)} \frac{g_\mu(V,W)}{h} \nabla^{L,M}_X (\mathrm{grad}_{g_1} h),$$

$$(6) R^{L,\mu}(V,W)U = R^{L,N}(V,W)U - (-1)^{|V|(|W|+|U|)} g_2(W,U)(\mathrm{grad}_{g_1} h)(h)V$$

$$+ (-1)^{|W||U|} g_2(V,U)(\mathrm{grad}_{g_1} h)(h)W.$$

Proof. (1) (1) comes from (1) in Lemma 3.1 and (2.2).
(2)By Lemma 3.1 and the Leibniz rule, we have

$$\nabla^{L,\mu}_V \nabla^{L,\mu}_X Y = (-1)^{|V|(|X|+|Y|)} \frac{\nabla^{L,M}_X Y(h)}{h} V, \quad -\nabla^{L,\mu}_X \nabla^{L,\mu}_V Y = -(-1)^{|V|(|X|+|Y|)} \frac{XY(h)}{h} V \tag{3.4}$$

By (2.2) and (3.4) and $[V,X] = 0$ and the definition of $H^h_M(X,Y)$, we get (2).
(3) By Lemma 3.1, we have $\nabla^{L,\mu}_X \nabla^{L,\mu}_Y V = \frac{XY(h)}{h} V$. So by (2.2) and the definition of $[X,Y]$, we get (3).
(4)By Lemma 3.1, we have

$$\nabla^{L,\mu}_V \nabla^{L,\mu}_W X = (-1)^{|W||X|} [V(\frac{X(h)}{h})W + (-1)^{|V||X|} \frac{X(h)}{h} \nabla^{L,\mu}_V W]. \tag{3.5}$$

By (2.2) and $V(\frac{X(h)}{h}) = 0$ and $\nabla^{L,\mu}$ having no torsion, we get (4).
(5)For $W_1 \in \mathrm{Vect}(N)$, we have by (4.12) in [7] and (4)

$$g_\mu(R^{L,\mu}(X,V)W, W_1) = (-1)^{(|X|+|V|)(|W|+|W_1|)} g_\mu(R^{L,\mu}(W,W_1)X, V) = 0 \tag{3.6}$$

By Proposition 9 in [3] and (2), we have

$$\begin{aligned} g_\mu(R^{L,\mu}(X,V)W, Y) &= -(-1)^{|W||Y|} g_\mu(R^{L,\mu}(X,V)Y, W) \\ &= -(-1)^{(|W|+|V|)|Y|} \frac{H^h_M(X,Y)}{h} g_\mu(V,W). \end{aligned} \tag{3.7}$$

By the definition of $\mathrm{grad}_{g_1}(h)$ and $\nabla^{L,M}$ preserving the metric, we can get

$$g_1(\nabla^{L,M}_X (\mathrm{grad}_{g_1} h), Y) = (-1)^{|Y||g_1|} H^h_M(X,Y). \tag{3.8}$$

So

$$g_\mu(R^{L,\mu}(X,V)W, Y) = -(-1)^{|X|(|V|+|W|+|g|)} g_\mu(\frac{g_\mu(V,W)}{h} \nabla^{L,M}_X (\mathrm{grad}_{g_1} h), Y). \tag{3.9}$$

By (3.6) and (3.9), we get (5).
(6) By (4), we have

$$g_\mu(R^{L,\mu}(V,W)U, X) = -(-1)^{|X||U|} g_\mu(R^{L,\mu}(V,W)X, U) = 0. \tag{3.10}$$

By Lemma 3.1 and the Leibniz rule, we have

$$g_\mu(\nabla_V^{L,\mu}\nabla_W^{L,\mu}U,W_1)=g_\mu(\nabla_V^{L,N}\nabla_W^{L,N}U,W_1)-(-1)^{|V|(|U|+|W|)}g_\mu(W,U)\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}g_\mu(V,W_1). \tag{3.11}$$

Then by (2.2) and (3.11), we have

$$\begin{aligned}g_\mu(R^{L,\mu}(V,W)U,W_1)&=g_\mu(R^{L,N}(V,W)U,W_1)\\&-(-1)^{|V|(|U|+|W|)}g_\mu(W,U)\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}g_\mu(V,W_1)\\&+(-1)^{|U||W|}g_\mu(V,U)\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}g_\mu(W,W_1).\end{aligned} \tag{3.12}$$

By (3.10) and (3.12), we get (6). □

For $\overline{X},\overline{Y},P\in \mathrm{Vect}(M_1)$, we define

$$\nabla^\mu_{\overline{X}}\overline{Y}=\nabla^{L,\mu}_{\overline{X}}\overline{Y}+\overline{X}\cdot g_\mu(\overline{Y},P)-g_\mu(\overline{X},\overline{Y})P. \tag{3.13}$$

For $X,Y,P\in \mathrm{Vect}(M)$, we define

$$\nabla^M_XY=\nabla^{L,M}_XY+X\cdot g_1(Y,P)-g_1(X,Y)P. \tag{3.14}$$

By Lemma 3.1 and (3.13), (3.14), we have

Lemma 3.2. *For $X,Y,P\in \mathrm{Vect}(M)$ and $U,W\in \mathrm{Vect}(N)$ and $\pi(X)=g_1(X,P)$, we have*

$$\begin{aligned}&(1)\nabla^\mu_XY=\nabla^M_XY,\ \ (2)\nabla^\mu_XU=\frac{X(h)}{h}U,\\&(3)\nabla^\mu_UX=(-1)^{|U||X|}[\frac{X(h)}{h}+\pi(X)]U,\\&(4)\nabla^\mu_UW=-hg_2(U,W)\mathrm{grad}_{g_1}h+\nabla^{L,N}_UW-g_\mu(U,W)P.\end{aligned} \tag{3.15}$$

Lemma 3.3. *For $X,Y\in \mathrm{Vect}(M)$ and $U,W,P\in \mathrm{Vect}(N)$, we have*

$$\begin{aligned}&(1)\nabla^\mu_XY=\nabla^{L,M}_XY-g_1(X,Y)P,\ \ (2)\nabla^\mu_XU=\frac{X(h)}{h}U+X\cdot g_\mu(U,P),\\&(3)\nabla^\mu_UX=(-1)^{|U||X|}\frac{X(h)}{h}U,\\&(4)\nabla^\mu_UW=-hg_2(U,W)\mathrm{grad}_{g_1}h+\nabla^{L,N}_UW+U\cdot g_\mu(W,P)-g_\mu(U,W)P.\end{aligned} \tag{3.16}$$

By Proposition 2.1 and Proposition 3.1 and Lemma 3.1, we get by some computations

Proposition 3.2. *For $X,Y,Z,P\in \mathrm{Vect}(M)$ and $U,V,W\in \mathrm{Vect}(N)$, we have*

$$\begin{aligned}&(1)R_{\nabla^\mu}(X,Y)Z=R_{\nabla^M}(X,Y)Z,\\&(2)R_{\nabla^\mu}(V,X)Y=-(-1)^{|V|(|X|+|Y|)}\left[\frac{H^h_M(X,Y)}{h}+(-1)^{|X||Y|}g_1(Y,\nabla^{L,M}_XP)\right.\\&\qquad\left.+g_1(X,Y)\frac{P(h)}{h}+g_1(X,Y)\pi(P)-\pi(X)\pi(Y)\right]V,\\&(3)R_{\nabla^\mu}(X,Y)V=0,\ \ (4)R_{\nabla^\mu}(V,W)X=0,\end{aligned} \tag{3.17}$$

$$(5)R_{\nabla^\mu}(X,V)W = -(-1)^{|X|(|V|+|W|+|g|)}g_\mu(V,W)\left[\frac{\nabla^{L,M}_X(\mathrm{grad}_{g_1}h)}{h}\right.$$
$$\left.+(-1)^{(|X|+|P|)|g|}\frac{P(h)}{h}X+\nabla^{L,M}_X P + X\cdot g_1(P,P)-\pi(X)P\right],$$

When $|g|=|P|=0$, then

$$R_{\nabla^\mu}(X,V)W = -(-1)^{|X|(|V|+|W|)}g_\mu(V,W)\left[\frac{\nabla^{L,M}_X(\mathrm{grad}_{g_1}h)}{h}\right.$$
$$\left.+\frac{P(h)}{h}X+\nabla^{L,M}_X P + X\cdot g_1(P,P)-\pi(X)P\right],$$

$$(6)R_{\nabla^\mu}(U,V)W = R^{L,N}(U,V)W+\left[(-1)^{|g|(|W|+|g|)}\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}\right.$$
$$\left.+(-1)^{|P|(|W|+|g|)}\frac{P(h)}{h}+(-1)^{|P|(|W|+|g|)}\pi(P)\right]$$
$$\cdot\left[(-1)^{|V||W|}(-1)^{|P||U|}g_\mu(U,W)V-(-1)^{|U|(|V|+|W|)}(-1)^{|P||V|}g_\mu(V,W)U\right]$$
$$+(-1)^{|V||W|}g_\mu(U,W)\frac{P(h)}{h}V-(-1)^{|U|(|V|+|W|)}g_\mu(V,W)\frac{P(h)}{h}U,$$

When $|g|=|P|=0$, then

$$R_{\nabla^\mu}(U,V)W = R^{L,N}(U,V)W+\left[\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}+2\frac{P(h)}{h}+\pi(P)\right]$$
$$\cdot\left[(-1)^{|V||W|}g_\mu(U,W)V-(-1)^{|U|(|V|+|W|)}g_\mu(V,W)U\right].$$

Similarly, we have

Proposition 3.3. *For $X,Y,Z\in\mathrm{Vect}(M)$ and $U,V,W,P\in\mathrm{Vect}(N)$, we have*

$$(1)R_{\nabla^\mu}(X,Y)Z = R_{\nabla^{L,M}}(X,Y)Z+(-1)^{|X||Y|}\frac{Y(h)}{h}g_\mu(X,Z)P \tag{3.18}$$
$$-\frac{X(h)}{h}g_\mu(Y,Z)P-(-1)^{|X|(|Y|+|Z|)}g_\mu(Y,Z)\pi(P)X$$
$$+(-1)^{|Y||Z|}g_\mu(X,Z)\pi(P)Y,$$

$$(2)R_{\nabla^\mu}(V,X)Y = -(-1)^{|V|(|X|+|Y|)}\frac{H^h_M(X,Y)}{h}V$$
$$-(-1)^{|X||Y|}hg_2(V,P)g_1(Y,\mathrm{grad}_{g_1}h)X$$
$$-(-1)^{|g(X,Y)||V|}g_1(X,Y)[\nabla^{L,N}_V P-hg_2(V,P)\mathrm{grad}_{g_1}h]$$
$$-(-1)^{|V|(|X|+|Y|)}g_1(X,Y)\pi(P)V+(-1)^{|g(X,Y)||V|}g_1(X,Y)\pi(V)P,$$

$$(3)R_{\nabla^\mu}(X,Y)V = (-1)^{(|X|+|Y|)|V|}\pi(V)[\frac{X(h)}{h}Y-(-1)^{|X||Y|}\frac{Y(h)}{h}X],$$

$$(4)R_{\nabla^\mu}(V,W)X = -(-1)^{|X||W|}hg_2(V,P)g_1(X,\mathrm{grad}_{g_1}h)W$$

$$
\begin{aligned}
&+(-1)^{|V||W|}(-1)^{|X||V|}hg_2(W,P)g_1(X,\mathrm{grad}_{g_1}h)V,\\
(5)R_{\nabla^\mu}(X,V)W=&-(-1)^{|X|(|V|+|W|+|g|)}\frac{g_\mu(V,W)}{h}\nabla^{L,M}_X(\mathrm{grad}_{g_1}h)\\
&+(-1)^{(|X|+|V|)|W|}[(-1)^{|X||W|}\frac{X(h)}{h}g_\mu(W,P)V-(-1)^{|X||V|}g_\mu(W,\nabla^{L,N}_VP)X]\\
&-(-1)^{|g_\mu(V,W)||X|}g_\mu(V,W)\frac{X(h)}{h}P-(-1)^{|X|(|V|+|W|)}g_\mu(V,W)\pi(P)X\\
&+(-1)^{(|X|+|V|)|W|}(-1)^{|X||V|}\pi(W)\pi(V)X,\\
(6)R_{\nabla^\mu}(U,V)W=&R^{L,N}(U,V)W-(-1)^{|U|(|V|+|W|)}g_2(V,W)(\mathrm{grad}_{g_1}h)(h)U\\
&+(-1)^{|V||W|}g_2(U,W)(\mathrm{grad}_{g_1}h)(h)V\\
&+(-1)^{(|U|+|V|)|W|}[g_\mu(W,\nabla^{L,N}_UP)V-(-1)^{|U||V|}g_\mu(W,\nabla^{L,N}_VP)U]\\
&+(-1)^{|U||V|}(-1)^{|g(U,W)||V|}g_\mu(U,W)[\nabla^{L,N}_VP-hg_2(V,P)\mathrm{grad}_{g_1}h]\\
&-(-1)^{|g(V,W)||U|}g_\mu(V,W)[\nabla^{L,N}_UP-hg_2(U,P)\mathrm{grad}_{g_1}h]\\
&-(-1)^{|U|(|V|+|W|)}g_\mu(V,W)\pi(P)U+(-1)^{|V||W|}g_\mu(U,W)\pi(P)V\\
&+(-1)^{|U||g_\mu(V,W)|}g_\mu(V,W)\pi(U)P-(-1)^{|U||V|}(-1)^{|g(U,W)||V|}g_\mu(U,W)\pi(V)P\\
&+(-1)^{(|U|+|V|)|W|}\pi(W)[(-1)^{|U||V|}\pi(V)U-\pi(U)V].
\end{aligned}
$$

In the following, we compute the Ricci tensor of M_1. Let M (resp. N) have the (p,m) (resp. (q,n)) dimension. Let $\partial_{x^I}=\{\partial_{x^a},\partial_{\xi^A}\}$ (resp. $\partial_{y^J}=\{\partial_{y^b},\partial_{\eta^B}\}$) denote the natural tangent frames on M (resp. N). Let $\mathrm{Ric}^{L,\mu}$ (resp. $\mathrm{Ric}^{L,M}$, $\mathrm{Ric}^{L,N}$) denote the Ricci tensor of (M_1,g_μ) (resp. (M,g_1), (N,g_2)). Then by (2.4), (2.7) and (3.3), we have

Proposition 3.4. *The following equalities holds*

$$
\begin{aligned}
&(1)\mathrm{Ric}^{L,\mu}(\partial_{x^I},\partial_{x^K})=\mathrm{Ric}^{L,M}(\partial_{x^I},\partial_{x^K})-\frac{(q-n)}{h}H^h_M(\partial_{x^I},\partial_{x^K}), \qquad (3.19)\\
&(2)\mathrm{Ric}^{L,\mu}(\partial_{x^I},\partial_{y^J})=\mathrm{Ric}^{L,\mu}(\partial_{y^J},\partial_{x^I})=0,\\
&(3)\mathrm{Ric}^{L,\mu}(\partial_{y^L},\partial_{y^J})=\mathrm{Ric}^{L,N}(\partial_{y^L},\partial_{y^J})-g_\mu(\partial_{y^L},\partial_{y^J})[\frac{\triangle^L_{g_1}(h)}{h}+(q-n-1)\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}].
\end{aligned}
$$

Let $\mathrm{Ric}^{\nabla^\mu}$ (resp. Ric^{∇^M}) denote the Ricci tensor of (M_1,∇^μ,g_μ) (resp. (M,g_1,∇^M)). Then by Proposition 3.2 and (2.4), (2.6), we have

Proposition 3.5. *The following equalities holds*

$$
\begin{aligned}
(1)\mathrm{Ric}^{\nabla^\mu}(\partial_{x^I},\partial_{x^K})=&\mathrm{Ric}^{\nabla^M}(\partial_{x^I},\partial_{x^K})-(q-n)\left[\frac{H^h_M(\partial_{x^I},\partial_{x^K})}{h}\right. \qquad (3.20)\\
&+\frac{1}{2}(-1)^{|\partial_{x^I}||\partial_{x^K}|}g_1(\partial_{x^K},\nabla^{L,M}_{\partial_{x^I}}P)+\frac{1}{2}g_1(\partial_{x^I},\nabla^{L,M}_{\partial_{x^K}}P)\\
&\left.+g_1(\partial_{x^I},\partial_{x^K})\frac{P(h)}{h}+g_1(\partial_{x^I},\partial_{x^K})\pi(P)-\pi(\partial_{x^I})\pi(\partial_{x^K})\right],\\
(2)\mathrm{Ric}^{\nabla^\mu}(\partial_{x^I},\partial_{y^J})=&\mathrm{Ric}^{\nabla^\mu}(\partial_{y^J},\partial_{x^I})=0,
\end{aligned}
$$

When $|g| = |P| = 0$, then

$$(3)\mathrm{Ric}^{\nabla^\mu}(\partial_{y^L}, \partial_{y^J}) = \mathrm{Ric}^{L,N}(\partial_{y^L}, \partial_{y^J}) - g_\mu(\partial_{y^L}, \partial_{y^J})[\frac{\triangle^L_{g_1}(h)}{h} + \mathrm{Div}^M_L(P)$$
$$+ (q-n-1)\frac{(\mathrm{grad}_{g_1} h)(h)}{h^2} + (2q+p-m-2n-2)\frac{P(h)}{h}$$
$$+ (p+q-m-n-2)\pi(P)].$$

4. Special super warped products with a semi-symmetric metric connection

In this section, we construct an Einstein super warped product with a semi-symmetric metric connection. Let $(N^{(q,n)}, g_2)$ be a super Riemannian manifold and $\mathbb{R}^{(1,0)}$ be the real line. Let $h(t)$ and $\mu(t) = h(t)^2$ be non-zero functions for $t \in \mathbb{R}$. Let $|g_2| = 0$. We consider the super Riemannian manifold $M_1 = \mathbb{R}^{(1,0)} \times_\mu N^{(q,n)}$ and $g_\mu = -dt \otimes dt + h^2 g_2$. Let $P = \partial_t$. Then $R_{\nabla^{\mathbb{R}}}(\partial_t, \partial_t)\partial_t = 0$ and $\mathrm{Ric}^{\nabla^{\mathbb{R}}}(\partial_t, \partial_t) = 0$. We have $H^h_M(\partial_t, \partial_t) = h''$, $\mathrm{grad}_{g_1}(h) = -h'\partial_t$ and $\triangle^L_{g_1}(h) = -h''$. By (3.20), we have

Proposition 4.1. *The following equalities holds*

$$(1)\mathrm{Ric}^{\nabla^\mu}(\partial_t, \partial_t) = -(q-n)(\frac{h''}{h} - \frac{h'}{h}), \tag{4.1}$$
$$(2)\mathrm{Ric}^{\nabla^\mu}(\partial_t, \partial_{y^J}) = \mathrm{Ric}^{\nabla^\mu}(\partial_{y^J}, \partial_t) = 0,$$
$$(3)\mathrm{Ric}^{\nabla^\mu}(\partial_{y^L}, \partial_{y^J}) = \mathrm{Ric}^{L,N}(\partial_{y^L}, \partial_{y^J}) - g_\mu(\partial_{y^L}, \partial_{y^J})$$
$$\cdot [-\frac{h''}{h} - (q-n-1)\frac{(h')^2}{h^2} + (2q-2n-1)\frac{h'}{h} - (q-n-1)].$$

We call that (M_1, g_μ, ∇^μ) is Einstein if

$$\mathrm{Ric}^{\nabla^\mu}(\overline{X}, \overline{Y}) = \lambda g_\mu(\overline{X}, \overline{Y}) \tag{4.2}$$

for $\overline{X}, \overline{Y} \in \mathrm{Vect}(M_1)$ and a constant λ. As in the ordinary warped product case (see Theorem 15 in [10]), by (4.1) and (4.2), we have

Theorem 4.1. *Let $M_1 = \mathbb{R}^{(1,0)} \times_\mu N^{(q,n)}$ and $g_\mu = -dt \otimes dt + h^2 g_2$ and $P = \partial_t$.Then (M_1, g_μ, ∇^μ) is Einstein with the Einstein constant λ if and only if the following conditions are satisfied*
(1) $(N^{(q,n)}, \nabla^{L,N})$ is Einstein with the Einstein constant c_0.
(2)

$$(q-n)(\frac{h''}{h} - \frac{h'}{h}) = \lambda, \tag{4.3}$$

(3)

$$\lambda h^2 - h''h - (q-n-1)(h')^2 + (2q-2n-1)hh' - (q-n-1)h^2 = c_0. \tag{4.4}$$

By Theorem 4.1, similar to the ordinary warped product case (see Theorem 25 and Theorem 26 in [10]), we have

Theorem 4.2. *Let $M_1 = \mathbb{R}^{(1,0)} \times_\mu N^{(q,n)}$ and $g_\mu = -dt \otimes dt + h^2 g_2$ and $P = \partial_t$. We assume that $q - n = 1$. Then (M_1, g_μ, ∇^μ) is Einstein with the Einstein constant $-\lambda_0$ if and only if the following conditions are satisfied*
(1) $(N^{(q,n)}, \nabla^{L,N})$ is Einstein with the Einstein constant $c_0 = 0$.

(2-1) $\lambda_0 < \frac{1}{4}$, $f(t) = c_1 e^{\frac{1+\sqrt{1-4\lambda_0}}{2}t} + c_2 e^{\frac{1-\sqrt{1-4\lambda_0}}{2}t}$,
(2-2) $\lambda_0 = \frac{1}{4}$, $f(t) = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$,
(2-3) $\lambda_0 > \frac{1}{4}$, $f(t) = c_1 e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{4\lambda_0-1}}{2}t\right) + c_2 e^{\frac{1}{2}t}\sin\left(\frac{\sqrt{4\lambda_0-1}}{2}t\right)$,

Proposition 4.2. *Let $M_1 = \mathbb{R}^{(1,0)} \times_\mu N^{(q,n)}$ and $g_\mu = -dt \otimes dt + h^2 g_2$and $P = \partial_t$. We assume that $q - n = 0$. Then (M_1, g_μ, ∇^μ) is Einstein with the Einstein constant $-\lambda_0$ if and only if the following conditions are satisfied*
(1) $(N^{(q,n)}, \nabla^{L,N})$ is Einstein with the Einstein constant $-c_0$.
(2)$\lambda_0 = 0$,
(3) $c_0 - hh'' + h'^2 + h^2 - hh' = 0$

Theorem 4.3. *Let $M_1 = \mathbb{R}^{(1,0)} \times_\mu N^{(q,n)}$ and $g_\mu = -dt \otimes dt + h^2 g_2$and $P = \partial_t$. We assume that $q - n \neq 0, 1$. Then (M_1, g_μ, ∇^μ) is Einstein with the Einstein constant $-\lambda_0$ if and only if $\lambda_0 = 0$ and $h = c_1 e^t + c_2$ and $(N^{(q,n)}, \nabla^{L,N})$ is Einstein with the Einstein constant $(q-n-1)c_2^2$.*

Nextly, we give another example. Let $M = \mathbb{R}^{(1,2)}$ with coordinates (t, ξ, η) and $|t| = 0$, $|\xi| = |\eta| = 1$. We give a metric $g_1 = -dt \otimes dt + d\xi \otimes d\eta - d\eta \otimes d\xi$ on M i.e.

$$g_1(\partial_t, \partial_t) = -1,\ g_1(\partial_\xi, \partial_\eta) = -1,\ g_1(\partial_\eta, \partial_\xi) = 1,\ g_1(\partial_{x^I}, \partial_{x^K}) = 0, \tag{4.5}$$

for the other pair $(\partial_{x^I}, \partial_{x^K})$. Let $M_2 = \mathbb{R}^{(1,2)} \times_\mu N^{(q,n)}$ and $g_\mu = g_1 + h(t)^2 g_2$ and $P = \partial_t$. By Proposition 7 in [3], we have the Christoffel symbols $\Gamma^L_{JI} = 0$, then

$$\nabla^{L,g_1}_{\partial_{x^J}} \partial_{x^K} = 0,\ R^{L,g_1}(X,Y)Z = 0,\ \mathrm{Ric}^{L,g_1}(X,Y) = 0. \tag{4.6}$$

We have

$$H^h_M(\partial_t, \partial_t) = h'',\ H^h_M(\partial_{x^J}, \partial_{x^K}) = 0,\ \text{for the other pair}\ (\partial_{x^I}, \partial_{x^K}). \tag{4.7}$$

$$\mathrm{grad}_{g_1}(h) = -h'\partial_t,\ \triangle^L_{g_1}(h) = -h''. \tag{4.8}$$

By Proposition 3.4 and the Einstein condition, we have

Theorem 4.4. *Let $M_2 = \mathbb{R}^{(1,2)} \times_\mu N^{(q,n)}$ and $g_\mu = g_1 + h^2 g_2$and $P = \partial_t$. Then $(M_2, g_\mu, \nabla^{L,\mu})$ is Einstein with the Einstein constant λ if and only if one of the following conditions is satisfied*
(1) $\lambda = 0$, $q = n$, $(N^{(q,n)}, \nabla^{L,N})$ is Einstein with the Einstein constant $-c_0$ and $hh'' - h'^2 = c_0$.
(2) $\lambda = 0$, $q - n - 1 = 0$, $(N^{(q,n)}, \nabla^{L,N})$ is Einstein with the Einstein constant 0 and $h = c_1 t + c_2$ where c_1, c_2 are constant.
(3) $\lambda = 0$, $q - n - 1 \neq 0, -1$, $(N^{(q,n)}, \nabla^{L,N})$ is Einstein with the Einstein constant $-c_0$ and $h = \pm\sqrt{\frac{c_0}{q-n-1}}t + c_2$, $\frac{c_0}{q-n-1} \geq 0$.

By (2.16) and (4.6), we can get

$$R^{\nabla^{\mathbb{R}^{(1,2)}}}(\partial_\xi, \partial_\eta)\partial_\xi = R^{\nabla^{\mathbb{R}^{(1,2)}}}(\partial_\eta, \partial_\xi)\partial_\xi = \partial_\xi, \tag{4.9}$$
$$R^{\nabla^{\mathbb{R}^{(1,2)}}}(\partial_\xi, \partial_\eta)\partial_\eta = R^{\nabla^{\mathbb{R}^{(1,2)}}}(\partial_\eta, \partial_\xi)\partial_\eta = -\partial_\eta,$$

$$R^{\nabla^{\mathbb{R}^{(1,2)}}}(\partial_\xi,\partial_\xi)\partial_\eta=-2\partial_\xi,\ \ R^{\nabla^{\mathbb{R}^{(1,2)}}}(\partial_\eta,\partial_\eta)\partial_\xi=2\partial_\eta,$$
$$R^{\nabla^{\mathbb{R}^{(1,2)}}}(\partial_{x^J},\partial_{x^K})\partial_{x^L}=0,$$

for other pairs $(\partial_{x^J},\partial_{x^K},\partial_{x^L})$. By (2.4) and (4.9), we have

$$\mathrm{Ric}^{\nabla^{\mathbb{R}^{(1,2)}}}(\partial_\xi,\partial_\eta)=-\mathrm{Ric}^{\nabla^{\mathbb{R}^{(1,2)}}}(\partial_\eta,\partial_\xi)=3,\ \ \mathrm{Ric}^{\nabla^{\mathbb{R}^{(1,2)}}}(\partial_{x^J},\partial_{x^L})=0,\tag{4.10}$$

for other pairs $(\partial_{x^J},\partial_{x^L})$.
If (M_2,g_μ,∇^μ) is Einstein with the Einstein constant λ, by (3.20) and (4.10), we have

$$(q-n)(\frac{h''}{h}-\frac{h'}{h})=\lambda,\ \ 3-(q-n)(-\frac{h'}{h}+1)=-\lambda.\tag{4.11}$$

Solving (4.11), we get

$$h=c_1e^{\pm\sqrt{1-\frac{3}{q-n}}t},\quad \frac{\lambda+3}{q-n}=1\mp\sqrt{1-\frac{3}{q-n}},\ \ 1-\frac{3}{q-n}\geq 0.\tag{4.12}$$

By (3.20) (3) and the Einstein condition, we get $(N^{(q,n)},\nabla^{L,N})$ is Einstein with the Einstein constant c_0 and

$$\lambda h^2-h''h-(q-n-1)(h')^2+(2q-2n-3)hh'-(q-n-3)h^2=c_0.\tag{4.13}$$

By (4.12) and (4.13), we get $q-n-3=0$ and h is a constant and $\lambda=0$ and $c_0=0$. So we have

Theorem 4.5. *Let $M_2=\mathbb{R}^{(1,2)}\times_\mu N^{(q,n)}$ and $g_\mu=g_1+h^2g_2$and $P=\partial_t$. Then (M_2,g_μ,∇^μ) is Einstein with the Einstein constant λ if and only if $\lambda=0$, $h=c_1$, $q-n-3=0$ and $(N^{(q,n)},\nabla^{L,N})$ is Einstein with the Einstein constant* 0*.*

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References

[1] Alías, L. Romero, A. Sánchez, M. *Spacelike hypersurfaces of constant mean curvature and Clabi-Bernstein type problems*. Tohoku Math. J. 49 (1997): 337-345.
[2] Bishop, R. O'Neill, B. *Manifolds of negative curvature*. Trans. Am. Math. Soc. 145 (1969): 1-49.
[3] Bruce, A. Grabowski, J. *Riemannian structures on $\mathbb{Z}_2^n$-manifolds*. Mathematics. 8 (2020): 1469.
[4] Dobarro, F. Dozo, E. *Scalar curvature and warped products of Riemannian manifolds*. Trans. Am. Math. Soc. 303 (1987): 161-168.
[5] Ehrlich, P. Jung, Y. Kim, S. *Constant scalar curvatures on warped product manifolds*. Tsukuba J. Math. 20 (1996): 239-265.
[6] Gholami, F. Darabi, F. Mohammadi, M. Varsaie, S. Roshande, M. *Einstein equations with cosmological constant in Super Space-Time*. arXiv: 2108.11437.
[7] Goertsches, O. *Riemannian supergeometry*. Math. Z. 260 (2008): 557-593.
[8] Hayden, H. *Subspace of a space with torsion*. Proc. Lond. Math. Soc. 34 (1932): 27-50.
[9] Sular, S. Özgur, C. *Warped products with a semi-symmetric metric connection*. Taiwanese J. Math. 15 (2011): 1701-1719.
[10] Wang, Y. *Multiply warped products with a semisymmetric metric connection*. Abstr. Appl. Anal. (2014): Art. ID 742371, 12 pp.
[11] Yano, K. *On semi-symmetric metric connection*. Rev. Roumaine Math. Pures Appl. 15 (1970): 1579-1586.

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