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$*$-SOLITONS ON LORENTZIAN KENMOTSU SPACE FORM

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ABSTRACT. In this article, we have studied the nature of $*$-Ricci soliton, $*$-conformal Ricci soliton, $*$-conformal η-Ricci soliton, generalized $*$-Ricci soliton, generalized $*$-conformal Ricci soliton, generalized $*$-conformal η-Ricci soliton on Lorentzian Kenmotsu space form with respect to Levi-Civita connection and with respect to generalized Tanaka connection. We obtained the value of non-dynamical scalar field p of $*$-conformal Ricci soliton, $*$-conformal η-Ricci soliton, generalized $*$-conformal Ricci soliton and generalized $*$-conformal η-Ricci soliton on which nature of solitons depend, whether it is shrinking, steady or expanding.

1. INTRODUCTION AND MOTIVATIONS

A class of contact Riemannian manifolds which satisfy some special conditions have been studied by K. Kenmotsu [6] is known as Kenmotsu manifolds. A Kenmotsu manifold together with a Lorentzian metric is called Lorentzian Kenmotsu manifold [12]. If a Lorentzian Kenmotsu manifold has constant φ-holomorphic sectional curvature, then it is called Lorentzian Kenmotsu space form.
Tanaka [10] and, independently Webster [17] defined the canonical affine connection on a nondegenerate, integrable CR manifold. Tanno [15] generalized this connection extending its definition to the general contact metric manifold which called generalized Tanaka–Webster connection or generalized Tanaka connection.
In 1982 Hamilton [13] introduced the concept of Ricci flow and proved its existence. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S \tag{1.1}$$

on a compact Riemannian manifold M with Riemannian metric g, where S is the Ricci tensor. A self-similar solution to the Ricci flow (1.1) is called Ricci soliton which moves under the Ricci flow simply by diffeomorphisms of the initial metric, that is, they are stationary points of the Ricci flow in space of metrics on M. A Ricci soliton is a generalization of an Einstein metric. The Ricci soliton equation is given by

$$\mathcal{L}_X g + 2S = 2\lambda g \tag{1.2}$$

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where $\mathcal{L}$ is the Lie derivative, S is the Ricci tensor, g is Riemannian metric, X is a vector field and λ is a scalar. The Ricci soliton is said to be shrinking, steady, and expanding according as λ is positive, zero and negetive respectively.
Fischer during 2003-2004 developed the concept of conformal Ricci flow [2] which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on M is defined by [2]

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg \tag{1.3}$$

where $R(g) = -1$ and p is a non-dynamical scalar field(time dependent scalar field), $R(g)$ is the scalar curvature of the n-dimessional manifold M.
In 2015, N. Basu and A. Bhattacharyya [1] introduced the notion of conformal Ricci soliton and the equation is as follows

$$\mathcal{L}_X g + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g \tag{1.4}$$

where λ is a scalar.
Cho and Kimura [5] introduced the notion of η-Ricci soliton in 2009, as follows

$$\mathcal{L}_\xi g + 2S = 2\lambda g + 2\mu\eta \otimes \eta, \tag{1.5}$$

for some constants λ and μ, where ξ is a soliton vector field and η is an 1-form on M.
In 2018, Siddiqi [8] introduced the notion of conformal η-Ricci soliton, given by

$$\mathcal{L}_\xi g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0, \tag{1.6}$$

for some constants λ and μ, where ξ is a soliton vector field and η is an 1-form on M. where $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ , p is a scalar non-dynamical field (time dependent scalar field) and n is the dimension of manifold.
Tachibana[14] and Hamada[18] introduced the notion of $*$-Ricci tensor on almost Hermitian manifolds and on real hypersurfaces in non-flat complex space respectively and then in 2014, Kaimakamis and Panagiotidou[4] introduced the notion of $*$-Ricci soliton on non-flat complex space forms and the equation as

$$\mathcal{L}_V g + 2S^* + 2\lambda g = 0, \tag{1.7}$$

where $S^*(X,Y) = \frac{1}{2}\left[trace\{\varphi \circ R(X, \varphi Y)\}\right]$ for all vector fields X, Y on M and φ is a (1,1)-tensor field.
In 2022, the authors[16] have defined the $*$-conformal η-Ricci soliton on a Riemannian manifold as

$$\mathcal{L}_\xi g + 2S^* + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0. \tag{1.8}$$

In 2016, Nurowski and Randall [11] introduced the concept of generalized Ricci soliton as a class of over determined system of equations

$$\mathcal{L}_V g = -2aV^\# \odot V^\# + 2bS + 2\lambda g, \tag{1.9}$$

where $\mathcal{L}_V g$ and $V^\#$ denote, respectively, the Lie derivative of the metric g in the directions of vector field V and the canonical one-form associated to V, and some real constants a, b, and λ. Levy [7] acquired the necessary and sufficient conditions for the existence of

such tensors. In 2018 M.D. Siddiqi [9] have studied generalized Ricci solitons on trans-Sasakian manifolds.
Analogous to above equations, we define the generalized ∗-Ricci soliton, generalized ∗-conformal Ricci soliton and generalized ∗-conformal η-Ricci soliton as follows:

Definition 1.1. *A Riemannian manifold (M, g) of dimension n is said to admit generalized ∗-Ricci soliton if*

$$\mathcal{L}_V g = -2aV^\# \odot V^\# + 2bS^* + 2\lambda g. \tag{1.10}$$

where $V \in \Gamma(TM)$ and $\mathcal{L}_V g$ is the Lie-derivative of g along V and $V^\#$ the canonical one-form associated to V and a, b, λ are some constants.

Definition 1.2. *A Riemannian manifold (M, g) of dimension n is said to admit generalized ∗-conformal Ricci soliton if*

$$\mathcal{L}_V g - \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2aV^\# \odot V^\# + 2bS^* = 0. \tag{1.11}$$

where $V \in \Gamma(TM)$ and $\mathcal{L}_V g$ is the Lie-derivative of g along V and $V^\#$ the canonical one-form associated to V and a, b, λ are some constants and p is a scalar non-dynamical field (time dependent scalar field).

Definition 1.3. *A Riemannian manifold (M, g) of dimension n is said to admit generalized ∗-conformal η-Ricci soliton if*

$$\mathcal{L}_V g - \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2aV^\# \odot V^\# + 2bS^* + 2\mu\eta(X)\eta(Y) = 0. \tag{1.12}$$

where $V \in \Gamma(TM)$ and $\mathcal{L}_V g$ is the Lie-derivative of g along V and $V^\#$ the canonical one-form associated to V and a, b, λ, μ are some constants and p is a scalar non-dynamical field (time dependent scalar field).

In this article, we first prove some results on Lorentzian Kenmotsu manifold and derive its curvature tensor and Ricci tensor on Lorentzian Kenmotsu space form with respect to Levi-Civita connection in section-3. In section-4, we consider generalized Tanaka connection on Lorentzian Kenmotsu space form and proved some results on its curvature tensor and Ricci tensor. We have studied the nature of ∗-Ricci soliton, ∗-conformal Ricci soliton, ∗-conformal η-Ricci soliton, generalized ∗-Ricci soliton, generalized ∗-conformal Ricci soliton, generalized ∗-conformal η-Ricci soliton on Lorentzian Kenmotsu space form with respect to Levi-Civita connection in section-5. In section-6, all the above mentioned solitons in section-5 have been studied on Lorentzian Kenmotsu space form with respect to generalized Tanaka connection and obtained the conditions for which the above solitons are expanding or steady or shrinking.
Before proving the main results the following properties are required for the next section. Let M be a $(2n+1)$ dimensional (denoted by M^{2n+1}) manifold having almost contact structure (φ, ξ, η, g) i.e.,

$$\eta(\xi) = 1,\ \varphi^2 = -I + \eta \otimes \xi,\ \varphi(\xi) = 0,\ \eta \circ \varphi = 0. \tag{1.13}$$

where φ is a (1, 1)-tensor field, ξ a contravariant vector field, η a covariant vector field.
A Lorentzian metric g is said to be compatible with the structure (φ, ξ, η, g) if

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y). \tag{1.14}$$

If the manifold M^{2n+1} equipped with an almost contact structure (φ, ξ, η, g) and a compatible Lorentzian metric g, is called Lorentzian almost contact metric manifold.
Note that equations (1.13) and (1.14) imply

$$g(X, \xi) = -\eta(X) \quad \text{and} \quad g(\xi, \xi) = -1. \tag{1.15}$$

Also, equations (1.14) implies

$$g(X, \varphi Y) = -g(\varphi X, Y). \tag{1.16}$$

Recall the four tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ in almost contact manifold, which are defined by

$$\begin{aligned} N^{(1)}(X, Y) &= [\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi, \\ N^{(2)}(X, Y) &= (\mathcal{L}_{\varphi X}\eta)Y - (\mathcal{L}_{\varphi Y}\eta)X, \\ N^{(3)}(X) &= (\mathcal{L}_{\xi}\varphi)X, \\ N^{(4)}(X) &= (\mathcal{L}_{\xi}\eta)X. \end{aligned}$$

An almost contact manifold is normal if and only if $N^{(1)} = 0$.

Proposition 1.1. [3] *For an almost contact manifold $N^{(1)} = 0$ implies $N^{(2)} = N^{(3)} = N^{(4)} = 0$.*

In almost contact Lorentzian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$, the fundamental 2-form Φ is defined as

$$\Phi(X, Y) = g(X, \varphi Y) \quad \text{for all} \quad X, Y \in \mathfrak{X}(M).$$

An almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is Kenmotsu [3] if and only if it is normal and

$$d\Phi = 2\eta \wedge \Phi, \quad d\eta = 0. \tag{1.17}$$

2. MAIN RESULTS

In this section we present the main results related to the paper.

2.1. Lorentzian Kenmotsu space forms with respect to Levi-Civita connection.

In this subsection we proved some results on Lorentzian Kenmotsu manifold and derived its curvature tensor on Lorentzian Kenmotsu space forms with respect to Levi-Civita connection.

Proposition 2.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a Lorentzian almost contact metric manifold and ∇ being Levi-Civita connection, then*

$$\begin{aligned} 2g\left((\nabla_X\varphi)Y, Z\right) = {} & 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) + g\left(N^{(1)}(Y, Z), \varphi X\right) \\ & -N^{(2)}(Y, Z)\eta(X) - 2d\eta(\varphi Y, X)\eta(Z) + 2d\eta(\varphi Z, X)\eta(Y). \end{aligned} \tag{2.1}$$

Proof. The proof follows for almost contact Riemannian manifold (see [3]), using the properties (1.14), (1.15) and (1.16) for Lorentzian manifold. □

Theorem 2.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a Lorentzian Kenmotsu manifold with Levi-Civita connection ∇, then*

$$(\nabla_X\varphi)Y = -\eta(Y)\varphi X + g(X, \varphi Y)\xi. \tag{2.2}$$

Proof. Using equations (1.13), (1.14), (1.15), (1.16), (1.17) and the normality condition in the equation (2.1), we get the result. □

From (1.13) and (2.2), we have

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.3}$$

and from (1.13), (2.3)

$$(\nabla_X \eta)Y = -g(\varphi X, \varphi Y). \tag{2.4}$$

Lemma 2.1. *Let (M, g) be a Lorentzian Kenmotsu manifold with Levi-Civita connection ∇ and R its curvature tensor. Then*

(1) $R(X,Y)\varphi Z - \varphi R(X,Y)Z$
$$= g(X,Z)\varphi Y - g(Y,Z)\varphi X + g(\varphi X, Z)Y - g(\varphi Y, Z)X. \tag{2.5}$$

(2) $R(\varphi X, \varphi Y)Z - R(X,Y)Z$
$$= g(X,Z)Y - g(Y,Z)X + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y. \tag{2.6}$$

Proof. (1) The Ricci identity gives us:

$$\nabla_X \nabla_Y \varphi Z - \nabla_Y \nabla_X \varphi Z - \nabla_{[X,Y]} \varphi Z = R(X,Y)\varphi Z.$$

This implies

$$R(X,Y)\varphi Z - \varphi R(X,Y)Z$$
$$= \nabla_X[(\nabla_Y \varphi)Z] - \nabla_Y[(\nabla_X \varphi)Z] + (\nabla_X \varphi)\nabla_Y Z - (\nabla_Y \varphi)\nabla_X Z - (\nabla_{[X,Y]}\varphi)Z$$

Using (2.2), (2.3) and (2.4), we get

$$R(X,Y)\varphi Z - \varphi R(X,Y)Z$$
$$= g(X,Z)\varphi Y - g(Y,Z)\varphi X + g(\varphi X, Z)Y - g(\varphi Y, Z)X.$$

(2) Taking inner product of the equation (2.5) with φW and then by standard calculation we get the result (2.6).

□

Theorem 2.2. *Let (M, g) be a Lorentzian Kenmotsu manifold with Levi-Civita connection ∇. Then M has constant φ-holomorphic sectional curvature c if and only if*

$$\begin{aligned} R(X,Y)Z = {} & \frac{c+3}{4}[g(X,Z)Y - g(Y,Z)X] \\ & + \frac{c-1}{4}\{[\eta(X)Y - \eta(Y)X]\eta(Z) - [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi \\ & - g(\varphi Y, Z)\varphi X + g(\varphi X, Z)\varphi Y + 2g(\varphi X, Y)\varphi Z\}. \end{aligned} \tag{2.7}$$

Proof. From (2.5), we have

$$g(R(X,\varphi Y)X, \varphi Y) - g(R(X,\varphi Y)Y, \varphi X) = g(X,Y)^2 - g(\varphi X, Y)^2 + g(X,X)g(Y,Y). \tag{2.8}$$

Which implies

$$g(R(X,\varphi X)X, \varphi Y) = g(R(X,\varphi X)Y, \varphi X). \tag{2.9}$$

For any vector field X and Y in $\mathfrak{X}(M)$, we have

$$g(R(X,\varphi X)X, \varphi X) = -cg(X,X)^2. \tag{2.10}$$

Replacing X by $X+Y$ in (2.10), we see

$$-c[2g(X,Y)^2+2g(X,X)g(Y,Y)+2g(Y,Y)g(X,Y)+g(X,X)g(Y,Y)]$$
$$=\frac{1}{2}g(R(X+Y,\varphi X+\varphi Y)(X+Y),\varphi X+\varphi Y)+\frac{c}{2}[g(X,X)^2+g(Y,Y)^2].$$

Using equations (2.5), (2.6), (2.8), (2.9), Bianchi identity and after along and rigorous calculations, we prove the theorem. □

From (2.7), we have the Ricci tensor:

$$S(X,Y)=-\frac{n(c+3)-4}{2}g(X,Y)-\frac{n(c-1)}{2}\eta(X)\eta(Y). \tag{2.11}$$

A Lorentzian Kenmotsu manifold M^{2n+1} having constant φ-holomorphic sectional curvature c is called a Lorentzian Kenmotsu space form and denoted by $M^{2n+1}(c)$.

2.2. **Lorentzian Kenmotsu space form with respect to generalized Tanaka connection.** For an $(2n+1)$-dimensional Lorentzian Kenmotsu manifold M with almost contact structure (φ,ξ,η,g), the relation between generalized Tanaka connection $\mathring{\nabla}$ and Levi-Civita connection ∇ is given by

$$\mathring{\nabla}_X Y=\nabla_X Y+\eta(X)\varphi Y+(\nabla_X\eta)(Y)\xi-\eta(Y)\nabla_X\xi \tag{2.12}$$

By (2.3) and (2.4),

$$\mathring{\nabla}_X Y=\nabla_X Y+\eta(X)\varphi Y-g(\varphi X,\varphi Y)\xi+\eta(Y)\varphi^2 X \tag{2.13}$$

Putting $Y=\xi$ in (2.13),

$$\mathring{\nabla}_X\xi=\nabla_X\xi+\eta(X)\varphi\xi+g(X,\varphi\xi)\xi-\eta(\xi)\nabla_X\xi$$

By (2.3),

$$\mathring{\nabla}_X\xi=0. \tag{2.14}$$

Now,

$$(\mathring{\nabla}_X\eta)Y=\mathring{\nabla}_X\eta(Y)-\eta(\mathring{\nabla}_X Y)$$

Using (2.4) and (2.13),

$$(\mathring{\nabla}_X\eta)Y=0. \tag{2.15}$$

And

$$(\mathring{\nabla}_X g)(Y,Z)=g(\mathring{\nabla}_X Y,Z)+g(Y,\mathring{\nabla}_X Z).$$

By (2.13),

$$(\mathring{\nabla}_X g)(Y,Z)=0. \tag{2.16}$$

Thus we can sate the following theorem

Theorem 2.3. *In a Lorentzian Kenmotsu manifold, the structural vector field, contact 1-form and metric are parallel with respect to the generalized Tanaka connection.*

Using (2.13),

$$(\mathring{\nabla}_X\varphi)Y=\mathring{\nabla}_X\varphi Y-\varphi(\mathring{\nabla}_X Y)=0.$$

The curvature tensor of Lorentzian Kenmotsu manifold with respect to the generalized Tanaka connection is given by

$$\mathring{R}(X,Y)Z=\mathring{\nabla}_X\mathring{\nabla}_Y Z-\mathring{\nabla}_Y\mathring{\nabla}_X Z-\mathring{\nabla}_{[X,Y]}Z=R(X,Y)Z+g(X,Z)Y-g(Y,Z)X \tag{2.17}$$

and

$$\mathring{S}(X,Y) = S(X,Y) - 2(n-1)g(X,Y)$$

If M has constant φ-holomorphic sectional curvature c, then by (2.7) and (2.17) we get

$$\mathring{R}(X,Y)Z = \frac{c+3}{4}[g(X,Z)Y - g(Y,Z)X] + \frac{c-1}{4}\{[\eta(X)Y - \eta(Y)X]\eta(Z) - [g(X,Z)\eta(Y)$$

$$-g(Y,Z)\eta(X)]\xi - g(\varphi Y,Z)\varphi X + g(\varphi X,Z)\varphi Y + 2g(\varphi X,Y)\varphi Z\} + g(X,Z)Y - g(Y,Z)X \tag{2.18}$$

and

$$\mathring{S}(X,Y) = -\frac{n(c-1)}{2}g(\varphi X,\varphi Y) - 4(n-1)g(X,Y). \tag{2.19}$$

2.3. **∗-Ricci tensor on Lorentzian Kenmotsu space form with respect to Levi-Civita connection.** In this subsection we derived the ∗-Ricci tensor in Lorentzion Kenmotsu space form with respect to Levi-Civita connection.

Theorem 2.4. *In a Lorentzion Kenmotsu space form with respect to Levi-Civita connection, the ∗-Ricci tensor*

$$S^*(X,Y) = \frac{n(c-1)+2}{4}g(\varphi X,\varphi Y). \tag{2.20}$$

Proof. Putting $Z = \varphi Z$ in (2.7),

$$R(X,Y)\varphi Z = \frac{c+3}{4}[g(X,\varphi Z)Y - g(Y,\varphi Z)X]$$
$$+\frac{c-1}{4}\{[\eta(X)Y - \eta(Y)X]\eta(\varphi Z) - [g(X,\varphi Z)\eta(Y) - g(Y,\varphi Z)\eta(X)]\xi$$
$$-g(\varphi Y,\varphi Z)\varphi X + g(\varphi X,\varphi Z)\varphi Y + 2g(\varphi X,Y)\varphi^2 Z\}.$$

Taking inner product of above equation with φW and contracting X and W, and then using definition we get the result. □

Corollary 2.1. *In a Lorentzion Kenmotsu space form with respect to Levi-Civita connection,*

$$S^*(X,\xi) = 0. \tag{2.21}$$

Now we can define ∗-η-Einstein manifold as follows:

Definition 2.1. *A manifold M is said to be ∗-η-Einstein if there exists certain $\sigma,\delta \in C^\infty(M)$ such that*

$$S^*(X,Y) = \sigma g(X,Y) + \delta\eta(X)\eta(Y), \tag{2.22}$$

for all $X,Y \in \mathfrak{X}(M)$.

Lemma 2.2. *If a Lorentzian Kenmotsu space form (M,η,φ,ξ,g) with respect to Levi-Civita connection ∇ is ξ-Ricci semi-symmetric i.e. $R(\xi,X)\cdot S = 0$, then $g(X,Y) = -\eta(X)\eta(Y)$ where S is Ricci tensor.*

Proof. From $R(\xi, X) \cdot S = 0$, we have

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0.$$

$$-\frac{n(c+3)-4}{2} g(R(\xi, X)Y, Z) - \frac{n(c-1)}{2} \eta(R(\xi, X)Y)\eta(Z)$$

$$-\frac{n(c+3)-4}{2} g(Y, R(\xi, X)Z) - \frac{n(c-1)}{2} \eta(Y)\eta(R(\xi, X)Z) = 0.$$

$$Or, \ -\frac{n(c-1)}{2} [\eta(Y)\eta(X)\eta(Z) + g(X, Y)\eta(Z) + \eta(Z)\eta(Y)\eta(X) + \eta(Y)g(X, Z)] = 0$$

Putting $Z = \xi$,

$$g(X, Y) = -\eta(Y)\eta(X).$$

□

2.4. **$*$-Ricci soliton with respect to Levi-Civita connection.**

Theorem 2.5. *A Lorentzion Knemotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to Levi-Civita connection ∇ admitting $*$-Ricci soliton is $*$-η-Einstein.*

Proof. By the definition of $*$-Ricci soliton, we get

$$(\mathcal{L}_\xi g)(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) = 0,$$

$$\text{or,} \quad S^*(X, Y) = -\lambda g(X, Y) - g(X, Y) - \eta(X)\eta(Y),$$

$$\text{or,} \quad S^*(X, Y) = -(\lambda + 1)g(X, Y) - \eta(X)\eta(Y).$$

This shows that the manifold is $^*\eta$-Einstein. □

Theorem 2.6. *If a Lorentzian Kenmotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to Levi-Civita connection ∇ admits *-Ricci soliton and is ξ-Ricci semi-symmetric, then the soliton is steady.*

Proof. From the equation of $*$-Ricci soliton, we have

$$\mathcal{L}_\xi g(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) = 0$$

By (2.3),

$$S^*(X, Y) + \lambda g(X, Y) = 0$$

From(2.20),we get

$$\frac{n(c-1)+2}{4} g [g(X, Y) + \eta(X)\eta(Y)] + \lambda g(X, Y) = 0$$

Applaying Lemma (2.2),

$$\lambda = 0.$$

Thus, the soliton is steady. □

2.5. **∗-Conformal Ricci soliton with respect to Levi-Civita connection.**

Theorem 2.7. *A Lorentzion Kenmotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to Levi-Civita connection admitting ∗-conformal Ricci soliton is ∗η-Einstein.*

Proof. The equation of ∗-conformal Ricci soliton given by

$$(\mathcal{L}_V g)(X,Y) + 2S^*(X,Y) + \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X,Y) = 0$$

Since $(M, \eta, \varphi, \xi, g)$ is a ∗-conformal Ricci soliton, we have

$$g(\nabla_\xi X, Y) + g(X, \nabla_\xi Y) + 2S^*(X,Y) + \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X,Y) = 0$$

By (2.3),

$$2g(X,Y) + 2\eta(X)\eta(Y) + 2S^*(X,Y) + \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X,Y) = 0.$$

$$\text{Or,} \quad S^*(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y)$$

Where $A = \frac{1}{2}(p + \frac{2}{2n+1}) - 1 - \lambda$ and $B = -1$, and hence the space form is ∗η-Einstein. □

Theorem 2.8. *If a Lorentzian Kenmotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to Levi-Civita connection admits *-conformal Ricci soliton and is ξ-Ricci semi-symmetric. Then the soliton is expanding or steady or shrinking according as $p < -\frac{2}{2n+1}$ or $p = -\frac{2}{2n+1}$ or $p > -\frac{2}{2n+1}$.*

Proof. From the equation of ∗-conformal Ricci soliton equation, we have

$$(\mathcal{L}_\xi g)(X,Y) + 2S^*(X,Y) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) = 0.$$

$$\text{or, } g(X,Y) + \eta(X)\eta(Y) + S^*(X,Y) + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right] g(X,Y) = 0.$$

From(2.20),we get

$$g(X,Y) + \eta(X)\eta(Y) + \frac{n(c-1)+2}{4}\left[g(X,Y) + \eta(X)\eta(Y)\right] + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right] g(X,Y) = 0.$$

Applying Lemma (2.2),

$$\left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right] g(X,Y) = 0$$

$$\text{or, } \lambda = \frac{1}{2}\left(p + \frac{2}{2n+1}\right)$$

Thus, the soliton is expanding or steady or shrinking according as $p < -\frac{2}{2n+1}$ or $p = -\frac{2}{2n+1}$ or $p > -\frac{2}{2n+1}$. □

2.6. **$*$-Conformal η-Ricci soliton with respect to Levi-Civita connection.**

Theorem 2.9. *A Lorentzion Kenmotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to Levi-Civita connection admitting $*$-conformal η-Ricci soliton is $*\eta$-Einstein.*

Proof. By the definition of $*$-conformal η-Ricci soliton ,we get

$$(\mathcal{L}_\xi g)(X,Y) + 2S^*(X,Y) + \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0$$

By (2.3),

$$2g(X,Y) + 2\eta(X)\eta(Y) + 2S^*(X,Y) + \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

$$\text{Or,} \quad S^*(X,Y) = Cg(X,Y) + D\eta(X)\eta(Y)$$

Where $C = \frac{1}{2}(p + \frac{2}{2n+1}) - 1 - \lambda$ and $D = -1 - \mu$, and hence the space form is $*\eta$-Einstein. □

Theorem 2.10. *If a Lorentzian Kenmotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to Levi-Civita connection admits *-conformal η-Ricci soliton and is ξ-Ricci semi-symmetric. Then the soliton is expanding or steady or shrinking according as $p < -\frac{2}{2n+1} - 2\mu$ or $p = -\frac{2}{2n+1} - 2\mu$ or $p > -\frac{2}{2n+1} - 2\mu$.*

Proof. From the equation of $*$-conformal η-Ricci soliton equation, we have

$$(\mathcal{L}_\xi g)(X,Y) + 2S^*(X,Y) + \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0$$

$$\text{or, } g(X,Y) + \eta(X)\eta(Y) + S^*(X,Y) + \left[\lambda - \frac{1}{2}(p + \frac{2}{2n+1})\right] g(X,Y) + \mu\eta(X)\eta(Y) = 0$$

From(2.20),we get

$$g(X,Y) + \eta(X)\eta(Y) + \frac{n(c-1)+2}{4}\left[g(X,Y) + \eta(X)\eta(Y)\right]$$
$$+ \left[\lambda - \frac{1}{2}(p + \frac{2}{2n+1})\right] g(X,Y) + \mu\eta(X)\eta(Y) = 0$$

Applying Lemma (2.2),

$$\left[\lambda - \frac{1}{2}(p + \frac{2}{2n+1}) - \mu\right] g(X,Y) = 0$$

$$\text{or,} \lambda = \mu + \frac{1}{2}\left(p + \frac{2}{2n+1}\right)$$

Thus, the soliton is expanding or steady or shrinking according as $p < -\frac{2}{2n+1} - 2\mu$ or $p = -\frac{2}{2n+1} - 2\mu$ or $p > -\frac{2}{2n+1} - 2\mu$. □

2.7. **Generalized $*-$Ricci soliton with respect to Levi-Civita connection.**

Theorem 2.11. *If a Lorentzian Knemotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to Levi-Civita connection admits generalized $*$-Ricci soliton, then the soliton is is expanding or steady or shrinking according as $a > 0$ or $a = 0$ or $a < 0$.*

Proof. In the equation (1.10), taking $V^{\#}(X) = g(X, V)$. Then it becomes

$$\mathcal{L}_V g(X,Y) = -2ag(X,V)g(Y,V) + 2bS^*(X,Y) + 2\lambda g(X,Y). \tag{2.23}$$

This implies

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = -2a\eta(X)\eta(Y) + 2bS^*(X,Y) + 2\lambda g(X,Y).$$

By (2.3),

$$g(\varphi X, \varphi Y) + a\eta(X)\eta(Y) - bS^*(X,Y) - \lambda g(X,Y) = 0. \tag{2.24}$$

Putting $Y = \xi$, then

$$a\eta(X) - bS(X,\xi) + \lambda\eta(X) = 0.$$

Using (2.21)

$$(a+\lambda)\eta(X) = 0.$$

This implies

$$\lambda = -a. \tag{2.25}$$

The equation (2.25) shows that, the soliton is is expanding or steady or shrinking according as $a > 0$ or $a = 0$ or $a < 0$. □

Theorem 2.12. *If a Lorentzian Knemotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to Levi-Civita connection admits generalized $*$-Ricci soliton and V is is pointwise colinear vector field with ξ. Then V is constant multiple of ξ.*

Proof. Let $V = \sigma\xi$, where σ is a function on the Lorentzian Knemotsu manifold. Then from the equation (1.10), we have

$$g(\nabla_X \sigma\xi, Y) + g(X, \nabla_Y \sigma\xi) = -2a\sigma^2\eta(X)\eta(Y) + 2bS^*(X,Y) + 2\lambda g(X,Y).$$

By (2.3) we get

$$2\sigma g(\varphi X, \varphi Y) - (X\sigma)\eta(Y) - (Y\sigma)\eta(X) = -2a\eta(X)\eta(Y) + 2bS^*(X,Y) + 2\lambda g(X,Y).$$

Putting $Y = \xi$ and using (2.21),

$$-(X\sigma) - (\xi\sigma)\eta(X) = -2a\sigma^2\eta(X) - 2\lambda\eta(X). \tag{2.26}$$

Putting $X = \xi$,

$$\xi\sigma = a\sigma^2 + \lambda. \tag{2.27}$$

From (2.26) and (2.27), we have

$$X\sigma = (a\sigma^2 + \lambda)\eta(X).$$

$$Or,\ d\sigma = (a\sigma^2 + \lambda)\eta.$$

Applying d on above equation,

$$2a\sigma(d\sigma)\eta + (a\sigma^2 + \lambda)d\eta = 0.$$

$$Or,\ 2a\sigma(d\sigma)\eta = 0.$$

This shows that, either $a = 0$ or σ is constant. □

2.8. **Generalized *-conformal Ricci soliton with respect to Levi-Civita connection.**

Theorem 2.13. *If a Lorentzian Knemotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to Levi-Civita connection inherits generalized $*$-conformal Ricci soliton, then the soliton is expanding or steady or shrinking according as $p < 2a - \frac{2}{2n+1}$ or $p = 2a - \frac{2}{2n+1}$ or $p > 2a - \frac{2}{2n+1}$.*

Proof. We Consider $V^{\#}(X) = g(X, V)$ in the equation (1.11). Then we get

$$\mathcal{L}_V g(X,Y) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) + 2aV^{\#}(X) \odot V^{\#}(Y) + 2bS^{*}(X,Y). \quad (2.28)$$

This implies

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) + 2a\eta(X)\eta(Y) + 2bS^{*}(X,Y) = 0.$$

By (2.3),

$$2g(\varphi X, \varphi Y) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) + 2a\eta(X)\eta(Y) + 2bS^{*}(X,Y) = 0. \quad (2.29)$$

Putting $Y = \xi$ and using (2.21), we get

$$\lambda = \frac{1}{2}\left(p + \frac{2}{2n+1}\right) - a. \quad (2.30)$$

□

Thus, the soliton is expanding or steady or shrinking according as $p < 2a - \frac{2}{2n+1}$ or $p = 2a - \frac{2}{2n+1}$ or $p > 2a - \frac{2}{2n+1}$.

Theorem 2.14. *If a Lorentzian Knemotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to Levi-Civita connection inherits generalized $*$-conformal Ricci soliton and V is is pointwise colinear vector field with ξ. Then V is constant multiple of ξ.*

Proof. Let $V = \sigma\xi$, where σ is a function on M. Then equation (1.11) implies

$$g(\nabla_X(\sigma\xi), Y) + g(X, \nabla_Y(\sigma\xi)) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) + 2a\sigma^2\eta(X)\eta(Y) + 2bS^{*}(X,Y) = 0.$$

Using (2.3),

$$2\sigma g(\varphi X, \varphi Y) - (X\sigma)\eta(Y) - (Y\sigma)\eta(X) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) + 2a\sigma^2\eta(X)\eta(Y) + 2bS^{*}(X,Y) = 0.$$

Replacing Y by ξ and using equation (2.21), we get

$$-(X\sigma) - (\xi\sigma)\eta(X) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] \eta(X) + 2a\sigma^2\eta(X) = 0. \quad (2.31)$$

Putting $X = \xi$ in (2.31),

$$\xi\sigma = a\sigma^2 + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]. \quad (2.32)$$

Using (2.32) in (2.31),

$$X\sigma = \left[a\sigma^2 + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right]\eta(X).$$

$$Or,\ d\sigma = \left[a\sigma^2 + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right]\eta.$$

Applying d on the above equation we get,

$$d\left[a\sigma^2 + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right]\eta + \left[a\sigma^2 + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right]d\eta = 0.$$

Since $d\eta = 0$, we have

$$Or,\ 2a\sigma(d\sigma)\eta = 0.$$

This shows that, either $a = 0$ or σ is constant.

□

2.9. **Generalized ∗-conformal η-Ricci soliton with respect to Levi-Civita connection.**

Theorem 2.15. *If a Lorentzian Knemotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to Levi-Civita connection admits generalized ∗-conformal η-Ricci soliton, then the soliton is expanding or steady or shrinking according as $p < 2a + 2\mu - \frac{2}{2n+1}$ or $p = 2a + 2\mu - \frac{2}{2n+1}$ or $p > 2a + 2\mu - \frac{2}{2n+1}$.*

Proof. Taking $V^\#(X) = g(V, X)$ in (1.12), then

$$\mathcal{L}_V g(X,Y) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]g(X,Y)$$
$$+2aV^\#(X) \odot V^\#(Y) + 2bS^*(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{2.33}$$

The equation (2.33) implies that

$$g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]g(X,Y)$$
$$+2a\eta(X)\eta(Y) + 2bS^*(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

By (2.3),

$$2g(\varphi X, \varphi Y) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]g(X,Y)$$
$$+2a\eta(X)\eta(Y) + 2b\mathring{S}^*(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{2.34}$$

Putting $Y = \xi$ and using (2.21) we get,

$$\lambda = \frac{1}{2}\left(p + \frac{2}{2n+1}\right) - a - \mu. \tag{2.35}$$

□

Thus, the soliton is expanding or steady or shrinking according as $p < 2a + 2\mu - \frac{2}{2n+1}$ or $p = 2a + 2\mu - \frac{2}{2n+1}$ or $p > 2a + 2\mu - \frac{2}{2n+1}$.

Theorem 2.16. *If a Lorentzian Knemotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to Levi-Civita connection inherits generalized ∗-conformal η-Ricci soliton and V is is pointwise colinear vector field with ξ. Then V is constant multiple of ξ.*

Proof. Let $V = \sigma\xi$ in (1.12), where σ is a function on M. Then have

$$g(\nabla_X(\sigma\xi), Y) + g(X, \nabla_Y(\sigma\xi)) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y)$$
$$+2a\sigma^2\eta(X)\eta(Y) + 2bS^*(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Using (2.3),

$$2\sigma g(\varphi X, \varphi Y) - (X\sigma)\eta(Y) - (Y\sigma)\eta(X) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y)$$
$$+2a\sigma^2\eta(X)\eta(Y) + 2b\mathring{S}^*(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Replacing Y by ξ and using equation (2.21), we get

$$-(X\sigma) - (\xi\sigma)\eta(X) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] \eta(X) + 2a\sigma^2\eta(X) + 2\mu\eta(X) = 0. \quad (2.36)$$

Putting $X = \xi$ in (2.36),

$$\xi\sigma = a\sigma^2 + \mu + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]. \quad (2.37)$$

Using (2.37) in (2.36),

$$X\sigma = \left[a\sigma^2 + \mu + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right] \eta(X).$$
$$Or,\ d\sigma = \left[a\sigma^2 + \mu + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right] \eta.$$

Applying d on the above equation we get,

$$d\left[a\sigma^2 + \mu + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right] \eta + \left[a\sigma^2 + \mu + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right] d\eta = 0.$$

Since $d\eta = 0$, we have

$$Or,\ 2a\sigma(d\sigma)\eta = 0.$$

This shows that, either $a = 0$ or σ is constant.

□

2.10. **$*$-Ricci tensor on Lorentzian Kenmotsu space form with respect to the generalized Tanaka connection.** In this section we derived the $*$-Ricci tensor in Lorentzion Kenmotsu space form with respect to generalized Tanaka connection.

Theorem 2.17. *In a Lorentzion Kenmotsu space form with respect to the generalized Tanaka connection, the $*$-Ricci tensor*

$$\mathring{S}^*(X,Y) = \frac{n(c-1)-2}{4} g(\varphi X, \varphi Y). \quad (2.38)$$

Proof. Similar as proof of Theorem 2.4. □

Corollary 2.2. *In a Lorentzion Kenmotsu space form with respect to the generalized Tanaka connection,*

$$\mathring{S}^*(X,\xi) = 0. \quad (2.39)$$

By (2.11), we see that

$$\mathring{S}(X,Y)+2\mathring{S}^*(X,Y)=-(4n-3)g(X,Y)-\eta(X)\eta(Y). \tag{2.40}$$

Definition 2.2. *A manifold M is said to be ∗-Einstein if there exists constant k such that*

$$S^*(X,Y)=kg(X,Y), \tag{2.41}$$

for all $X,Y\in\mathfrak{X}(M)$.

Lemma 2.3. *If a Lorentzian Kenmotsu space form with respect to generalized Tanaka connection $\mathring{\nabla}$ is Ricci semi-symmetric i.e. $\mathring{R}(\xi,X)\cdot\mathring{S}=0$, then $\mathring{S}^*(X,Y)=0$.*

Proof. From $R(\xi,X)\cdot\mathring{S}=0$,

$$\mathring{S}(\mathring{R}(\xi,X)Y,Z)+\mathring{S}(Y,\mathring{R}(\xi,X)Z)=0.$$

$$\text{Or, } -\frac{n(c-1)}{2}\left[g(\varphi R(\xi,X)Y,\varphi Z)+g(\varphi Y,\varphi R(\xi,X)Z)\right]=0.$$

$$\text{Or, } \eta(Y)g(\varphi X,\varphi Z)+\eta(Z)g(\varphi X,\varphi Y)=0.$$

Putting $Z=\xi$,

$$g(\varphi X,\varphi Y)=0. \tag{2.42}$$

From (2.38),

$$\mathring{S}^*(X,Y)=\frac{n(c-1)-2}{4}g(\varphi X,\varphi Y).$$

So, by the equation (2.42),

$$\mathring{S}^*(X,Y)=0$$

□

2.11. ∗-Ricci soliton with respect to generalized Tanaka connection.

Theorem 2.18. *A Lorentzion Knemotsu space form with respect to generalized Tanaka connection $\mathring{\nabla}$ admitting ∗-Ricci soliton is ∗-Einstein.*

Proof. By the definition of ∗-Ricci soliton , we get

$$(\mathcal{L}_\xi g)(X,Y)+2\mathring{S}^*(X,Y)+2\lambda g(X,Y)=0,$$

$$\text{or,}\quad g(\mathring{\nabla}_\xi X,Y)+g(X,\mathring{\nabla}_\xi Y)+\mathring{S}^*(X,Y)=-\lambda g(X,Y),$$

$$\text{or,}\quad \mathring{S}^*(X,Y)=-\lambda g(X,Y).$$

This shows that the manifold is ∗-Einstein. □

Theorem 2.19. *If a Lorentzian Kenmotsu space form with respect to generalized Tanaka connection $\mathring{\nabla}$ admits ∗-Ricci soliton and is ξ-Ricci semi-symmetric. Then the soliton is steady.*

Proof. From the equation of $*$-Ricci soliton, we have

$$\mathcal{L}_\xi g(X,Y) + 2\mathring{S}^*(X,Y) + 2\lambda g(X,Y) = 0$$

$$g(\mathring{\nabla}_\xi X, Y) + g(X, \mathring{\nabla}_\xi Y) + \mathring{S}^*(X,Y) + \lambda g(X,Y) = 0$$

By (2.14),

$$\mathring{S}^*(X,Y) + \lambda g(X,Y) = 0$$

Applaying Lemma (2.3),

$$\lambda g(X,Y) = 0.$$
$$or, \ \lambda = 0.$$

Thus, the soliton is steady. □

2.12. $*$**-Conformal Ricci soliton with respect to generalized Tanaka connection.**

Theorem 2.20. *A Lorentzion Kenmotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to generalized Tanaka connection admits $*$-conformal Ricci soliton is $*\eta$-Einstein.*

Proof. By the definition of $*$-conformal Ricci soliton ,we get

$$(\mathcal{L}_\xi g)(X,Y) + 2\mathring{S}^*(X,Y) + \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X,Y) = 0$$

By (2.14),

$$2\mathring{S}^*(X,Y) + \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X,Y) = 0.$$
$$\text{Or,} \quad S^*(X,Y) = Eg(X,Y),$$

where $E = \frac{1}{2}(p + \frac{2}{2n+1}) - \lambda$. Hence the space form is $*$-Einstein. □

Theorem 2.21. *If a Lorentzian Kenmotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to generalized Tanaka connection admits *-conformal Ricci soliton and is ξ-Ricci semi-symmetric. Then the soliton is expanding or steady or shrinking according as $p < -\frac{2}{2n+1}$ or $p = -\frac{2}{2n+1}$ or $p > -\frac{2}{2n+1}$.*

Proof. From the equation of $*$-conformal η-Ricci soliton equation, we have

$$(\mathcal{L}_\xi g)(X,Y) + 2\mathring{S}^*(X,Y) + \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X,Y) = 0.$$

By (2.14),

$$\mathring{S}^*(X,Y) + \left[\lambda - \frac{1}{2}(p + \frac{2}{2n+1})\right] g(X,Y) = 0.$$

From(2.38), we get

$$\frac{n(c-1)-2}{4}\left[g(X,Y) + \eta(X)\eta(Y)\right] + \left[\lambda - \frac{1}{2}(p + \frac{2}{2n+1})\right] g(X,Y) = 0.$$

Applying Lemma (2.3),

$$\left[\lambda - \frac{1}{2}(p + \frac{2}{2n+1})\right] g(X,Y) = 0.$$
$$\text{or,} \lambda = \frac{1}{2}\left(p + \frac{2}{2n+1}\right).$$

Thus, the soliton is expanding or steady or shrinking according as $p < -\frac{2}{2n+1}$ or $p = -\frac{2}{2n+1}$ or $p > -\frac{2}{2n+1}$. □

2.13. $*$**-Conformal** η**-Ricci soliton with respect to generalized Tanaka connection.**

Theorem 2.22. *A Lorentzion Kenmotsu space form* $(M, \eta, \varphi, \xi, g)$ *with respect to generalized Tanaka connection admits* $*$*-conformal* η*-Ricci soliton is* $*\eta$*-Einstein.*

Proof. By the definition of $*$-conformal η-Ricci soliton ,we get

$$(\mathcal{L}_\xi g)(X,Y) + 2\mathring{S}^*(X,Y) + \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0$$

By (2.14),

$$2\mathring{S}^*(X,Y) + \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

$$\text{Or,}\quad S^*(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y)$$

Where $A = \frac{1}{2}(p + \frac{2}{2n+1}) - \lambda$ and $B = -\mu$, and hence the space form is $*\eta$-Einstein. □

Theorem 2.23. *If a Lorentzian Kenmotsu space form* $(M, \eta, \varphi, \xi, g)$ *with respect to generalized Tanaka connection admits* **-conformal* η*-Ricci soliton and is* ξ*-Ricci semi-symmetric. Then the soliton is expanding or steady or shrinking according as* $p < -\frac{2}{2n+1} - 2\mu$ *or* $p = -\frac{2}{2n+1} - 2\mu$ *or* $p > -\frac{2}{2n+1} - 2\mu$*.*

Proof. From the equation of $*$-conformal η-Ricci soliton equation, we have

$$(\mathcal{L}_\xi g)(X,Y) + 2\mathring{S}^*(X,Y) + \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0$$

By (2.14),

$$\mathring{S}^*(X,Y) + \left[\lambda - \frac{1}{2}(p + \frac{2}{2n+1})\right] g(X,Y) + \mu\eta(X)\eta(Y) = 0$$

From(2.38), we get

$$\frac{n(c-1)-2}{4}\left[g(X,Y) + \eta(X)\eta(Y)\right] + \left[\lambda - \frac{1}{2}(p + \frac{2}{2n+1})\right] g(X,Y) + \mu\eta(X)\eta(Y) = 0$$

Applying Lemma (2.3),

$$\left[\lambda - \frac{1}{2}(p + \frac{2}{2n+1}) - \mu\right] g(X,Y) = 0$$

$$\text{or,}\, \lambda = \mu + \frac{1}{2}\left(p + \frac{2}{2n+1}\right)$$

Thus, the soliton is expanding or steady or shrinking according as $p < -\frac{2}{2n+1} - 2\mu$ or $p = -\frac{2}{2n+1} - 2\mu$ or $p > -\frac{2}{2n+1} - 2\mu$. □

2.14. Generalized $*-$Ricci soliton with respect to generalized Tanaka connection.

Theorem 2.24. *If a Lorentzian Knemotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to generalized Tanaka connection inherits generalized $*$-Ricci soliton, then the soliton is is expanding or steady or shrinking according as $a > 0$ or $a = 0$ or $a < 0$.*

Proof. In the equation (1.10), taking $V^{\#}(X) = g(X, V)$ and replacing S^* by $\mathring{S}^*$. Then we have

$$\mathcal{L}_V g(X, Y) = -2ag(X, V)g(Y, V) + 2b\mathring{S}^*(X, Y) + 2\lambda g(X, Y). \tag{2.43}$$

Since $(M, \eta, \varphi, \xi, g)$ is a generalized $*$-Ricci soliton, the above equation implies

$$g(\mathring{\nabla}_X \xi, Y) + g(X, \mathring{\nabla}_Y \xi) = -2a\eta(X)\eta(Y) + 2b\mathring{S}^*(X, Y) + 2\lambda g(X, Y).$$

By (2.14),

$$a\eta(X)\eta(Y) - b\mathring{S}^*(X, Y) - \lambda g(X, Y) = 0. \tag{2.44}$$

Putting $Y = \xi$, then

$$a\eta(X) - b\mathring{S}(X, \xi) + \lambda\eta(X) = 0.$$

Using (2.39)

$$(a + \lambda)\eta(X) = 0.$$

This implies

$$\lambda = -a. \tag{2.45}$$

The equation (2.45) shows that, the soliton is is expanding or steady or shrinking according as $a > 0$ or $a = 0$ or $a < 0$. □

Theorem 2.25. *If a Lorentzian Knemotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to generalized Tanaka connection inherits generalized $*$-Ricci soliton and V is is pointwise colinear vector field with ξ. Then V is constant multiple of ξ.*

Proof. Let $V = \sigma\xi$, where σ is a function on the Lorentzian Knemotsu space form. Then from (2.43),

$$g(\mathring{\nabla}_X \sigma\xi, Y) + g(X, \mathring{\nabla}_Y \sigma\xi) = -2a\sigma^2\eta(X)\eta(Y) + 2b\mathring{S}^*(X, Y) + 2\lambda g(X, Y).$$

By (2.14) we get

$$-(X\sigma)\eta(Y) - (Y\sigma)\eta(X) = -2a\eta(X)\eta(Y) + 2b\mathring{S}^*(X, Y) + 2\lambda g(X, Y).$$

Putting $Y = \xi$ and using (2.39),

$$-(X\sigma) - (\xi\sigma)\eta(X) = -2a\sigma^2\eta(X) - 2\lambda\eta(X). \tag{2.46}$$

Putting $X = \xi$,

$$\xi\sigma = a\sigma^2 + \lambda. \tag{2.47}$$

From (2.46) and (2.47), we have

$$X\sigma = (a\sigma^2 + \lambda)\eta(X).$$

$$Or,\ d\sigma = (a\sigma^2 + \lambda)\eta.$$

Applying d on above equation,

$$2a\sigma(d\sigma)\eta + (a\sigma^2 + \lambda)d\eta = 0.$$

$$Or,\ 2a\sigma(d\sigma)\eta = 0.$$

This shows that, either $a = 0$ or σ is constant. □

2.15. **Generalized *-conformal Ricci soliton with respect to generalized Tanaka connection.**

Theorem 2.26. *If a Lorentzian Knemotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to generalized Tanaka connection admits generalized ∗-conformal Ricci soliton, then the soliton is expanding or steady or shrinking according as $p < 2a - \frac{2}{2n+1}$ or $p = 2a - \frac{2}{2n+1}$ or $p > 2a - \frac{2}{2n+1}$.*

Proof. Taking $V^{\#}(X) = g(X, V)$ and replacing S^* by $\mathring{S}^*$ in the equation (1.11). It becomes

$$\mathcal{L}_V g(X,Y) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) + 2ag(X,V)g(Y,V) + 2b\mathring{S}^*(X,Y) = 0. \tag{2.48}$$

If $(M, \eta, \varphi, \xi, g)$ is a generalized ∗-conformal Ricci soliton. Then from equation (2.48) we have

$$g(\mathring{\nabla}_X\xi, Y) + g(X, \mathring{\nabla}_Y\xi) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) + 2a\eta(X)\eta(Y) + 2b\mathring{S}^*(X,Y) = 0.$$

By (2.14),

$$-\left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) + 2a\eta(X)\eta(Y) + 2b\mathring{S}^*(X,Y) = 0. \tag{2.49}$$

Putting $Y = \xi$ and using (2.39), we get

$$\lambda = \frac{1}{2}\left(p + \frac{2}{2n+1}\right) - a. \tag{2.50}$$

Thus, the soliton is expanding or steady or shrinking according as $p < 2a - \frac{2}{2n+1}$ or $p = 2a - \frac{2}{2n+1}$ or $p > 2a - \frac{2}{2n+1}$. □

Theorem 2.27. *If a Lorentzian Knemotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to generalized Tanaka connection inherits generalized ∗-conformal Ricci soliton and V is is pointwise colinear vector field with ξ. Then V is constant multiple of ξ.*

Proof. Let $V = \sigma\xi$, where σ is a function on M. Then the equation (2.48) implies

$$g(\mathring{\nabla}_X(\sigma\xi), Y) + g(X, \mathring{\nabla}_Y(\sigma\xi)) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y)$$
$$+2a\sigma^2\eta(X)\eta(Y) + 2b\mathring{S}^*(X,Y) = 0.$$

Using (2.14),

$$-(X\sigma)\eta(Y) - (Y\sigma)\eta(X) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y)$$
$$+2a\sigma^2\eta(X)\eta(Y) + 2b\mathring{S}^*(X,Y) = 0.$$

Replacing Y by ξ and using equation (2.39), we get

$$-(X\sigma) - (\xi\sigma)\eta(X) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] \eta(X) + 2a\sigma^2\eta(X) = 0. \tag{2.51}$$

Putting $X = \xi$ in (2.51),

$$\xi\sigma = a\sigma^2 + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]. \tag{2.52}$$

Using (2.52) in (2.51),

$$X\sigma = \left[a\sigma^2 + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right]\eta(X).$$

$$Or,\ d\sigma = \left[a\sigma^2 + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right]\eta.$$

Applying d on the above equation, we get

$$d\left[a\sigma^2 + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right]\eta + \left[a\sigma^2 + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right]d\eta = 0.$$

Since $d\eta = 0$, we have

$$2a\sigma(d\sigma)\eta = 0.$$

This shows that, either $a = 0$ or σ is constant.

□

2.16. **Generalized $*$-conformal η-Ricci soliton with respect to generalized Tanaka connection.**

Theorem 2.28. *If a Lorentzian Knemotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to generalized Tanaka connection inherits generalized $*$-conformal η-Ricci soliton, then the soliton is expanding or steady or shrinking according as $p < 2a + 2\mu - \frac{2}{2n+1}$ or $p = 2a + 2\mu - \frac{2}{2n+1}$ or $p > 2a + 2\mu - \frac{2}{2n+1}$.*

Proof. In equation (1.12), Taking $V^\#(X) = g(X, V)$ and replacing S^* by $\mathring{S}^*$. Then equation (1.12) becomes

$$\mathcal{L}_V g(X,Y) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y)$$
$$+2ag(X,V)g(Y,V) + 2b\mathring{S}^*(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{2.53}$$

As the given space form $(M, \eta, \varphi, \xi, g)$ is a generalized $*$-conformal η-Ricci soliton, equation (2.53) implies

$$g(\mathring{\nabla}_X\xi, Y) + g(X, \mathring{\nabla}_Y\xi) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y)$$
$$+2a\eta(X)\eta(Y) + 2b\mathring{S}^*(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

By (2.14),

$$-\left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) + 2a\eta(X)\eta(Y) + 2b\mathring{S}^*(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{2.54}$$

Putting $Y = \xi$ and using (2.39), we get

$$\lambda = \frac{1}{2}\left(p + \frac{2}{2n+1}\right) - a - \mu. \tag{2.55}$$

Thus, the soliton is expanding or steady or shrinking according as $p < 2a + 2\mu - \frac{2}{2n+1}$ or $p = 2a + 2\mu - \frac{2}{2n+1}$ or $p > 2a + 2\mu - \frac{2}{2n+1}$. □

Theorem 2.29. *If a Lorentzian Knemotsu space form $(M, \eta, \varphi, \xi, g)$ with respect to generalized Tanaka connection inherits generalized ∗-conformal η-Ricci soliton and V is is pointwise colinear vector field with ξ. Then V is constant multiple of ξ.*

Proof. In equation (2.53), putting $V = \sigma\xi$, where σ is a function on M. Then we have

$$g(\mathring{\nabla}_X(\sigma\xi), Y) + g(X, \mathring{\nabla}_Y(\sigma\xi)) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X, Y)$$
$$+2a\sigma^2\eta(X)\eta(Y) + 2b\mathring{S}^*(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Using (2.14),

$$-(X\sigma)\eta(Y) - (Y\sigma)\eta(X) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X, Y)$$
$$+2a\sigma^2\eta(X)\eta(Y) + 2b\mathring{S}^*(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Replacing Y by ξ and using equation (2.39), we get

$$-(X\sigma) - (\xi\sigma)\eta(X) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] \eta(X) + 2a\sigma^2\eta(X) + 2\mu\eta(X) = 0. \quad (2.56)$$

Putting $X = \xi$ in (2.56),

$$\xi\sigma = a\sigma^2 + \mu + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]. \quad (2.57)$$

Using (2.57) in (2.56),

$$X\sigma = \left[a\sigma^2 + \mu + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right] \eta(X).$$
$$Or,\ d\sigma = \left[a\sigma^2 + \mu + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right] \eta.$$

Applying d on the above equation we get,

$$d\left[a\sigma^2 + \mu + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right] \eta + \left[a\sigma^2 + \mu + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\right] d\eta = 0.$$

Since $d\eta = 0$, we have

$$2a\sigma(d\sigma)\eta = 0.$$

This shows that, either $a = 0$ or σ is constant. □

References

[1] Arindam Bhattacharyya and Nirabhra Basu, *Conformal Ricci soliton in Kenmotsu manifold*,Global Journal of Advanced Research on Classical and Modern Geometries, 4(1) (2015):15-21.

[2] Arthur E. Fischer, *An introduction to conformal Ricci flow*, Classical and Quantum Gravity, 21(3)(2004): pages-S171.

[3] David E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, volume 203, (2002).

[4] George Kaimakamis and Konstantina Panagiotidou, **-Ricci solitons of real hypersurfaces in non-flat complex space forms*, Journal of Geometry and Physics, 86 (2014): 408-413.

[5] J. T. Cho and M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J. 61(2) (2009): 205-212.

[6] Katsuei Kenmotsu, *A class of almost contact Riemannian manifolds*, The Tohoku Mathematical Journal, 24 (1972): 93-103.
[7] Levy Harry, *Symmetric tensors of the second order whose covariant derivatives vanish*, Annals of Mathematics, (1925):91-98.
[8] Mohd Danish Siddiqi, *Conformal η-Ricci solitons in δ-Lorentzian Trans Sasakian manifolds*, International Journal of Maps in Mathematics, 1(1) (2018): 15-34.
[9] Mohd Danish Siddiqi, *Generalized Ricci solitons on trans-Sasakian manifolds*, Khayyam Journal of Mathematics, 4(2) (2018): 178-186.
[10] N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japanese journal of mathematics, New series, 2(1) (1976):131-190.
[11] Nurowski Paweł and Randall, Matthew, *Generalized Ricci solitons*, The Journal of Geometric Analysis, 26(2) (2016):1280-1345.
[12] R. Rosca, *On Lorentzian Kenmotsu manifolds*, Atti Accad. Peloritana Pericolanti Cl. Aci. Fis. Mat. Natur, 69 (1991): 15-30.
[13] Richard S. Hamilton and others, *Three-manifolds with positive Ricci curvature*, J. Differential geom, 17(2) (1982): 255-306.
[14] Shun-ichi Tachibana, *On almost-analytic vectors in almost-Kählerian manifolds*, Tohoku Mathematical Journal, Second Series,11(2) (1959): 247-265.
[15] S. Tanno, *Variational problems on contact Riemannian manifolds*, Transactions of the American Mathematical society, 314(1) (1989):349-379.
[16] Soumendu Roy, Santu Dey, Arindam Bhattacharyya and Shyamal Kumar Hui, **-Conformal η-Ricci soliton on Sasakian manifold*, Asian-European Journal of Mathematics, 15(2) (2022): 17 pages.
[17] S.M. Webster, *Pseudo-Hermitian structures on a real hypersurface*, Journal of Differential Geometry, 13(1)(1978):25-41.
[18] Tatsuyoshi Hamada, *Real Hypersurfaces of Complex Space Forms in Terms of Ricci $*$-Tensor*, Tokyo Journal of Mathematics,25(2)(2002): 473-483.

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