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ON CEVA'S AND SEEBACH'S THEOREMS

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ABSTRACT. In this article we discuss the relation of Ceva's condition for three secants of a triangle to the cross ratio formed by these secants. Subsequently we relate the obtained result to Seebach's theorem on the inscription of a triangle of given similarity type, as a Cevian one of another given triangle. We supply also a method to construct the triangle center $X(370)$ whose Cevian triangle is equilateral.

1. INTRODUCTION

This work grew out of an attempt to find a simple construction of an equilateral $A'B'C'$ inscribed as a Cevian triangle inside a given triangle ABC. The intersection point of the Cevians of such a triangle is the *"triangle center"* $X(370)$ of Kimberling's list [1]. It is however noticeable the small amount of information recorded there for this point. It is one of the triangle centers, whose barycentric coordinates are not given explicitly. What is given instead is a system of three quadratic equations satisfied by its barycentrics and the fact that it lies on the Neuberg cubic of the triangle ABC.

More general, the existence of points D inside a triangle ABC, whose Cevian tri-

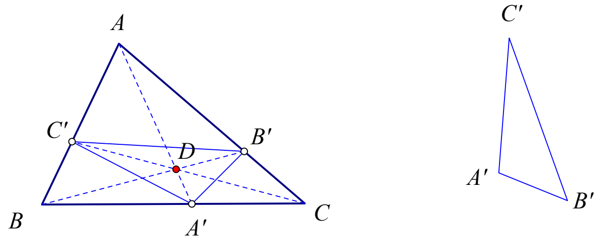

Figure 1. Inscribing as a Cevian $\triangle\ A'B'C'$ in $\triangle\ ABC$

angle has a given similarity type is guaranteed by Seebach's theorem ([3]), according to which (see Figure 1):

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Theorem 1.1. *Given two triangles there is precisely one triangle $A'B'C'$ similar to the second inscribed in the first triangle ABC as a Cevian triangle of a point D lying inside ABC and with a prescribed correspondence of vertices of $A'B'C'$ lying on sides of ABC :*

$$A' \in BC,\ B' \in CA,\ C' \in AB\ .$$

The original proof by Seebach combined geometric arguments and a considerable amount of computation. A subsequent elegant proof by Hajja ([2]) succeeded with less computations. The present work gives two alternative proofs based on well known theorems, without resort to any substantial computation. In addition, supplies a third method especially adapted to the equilateral, to solve geometrically the corresponding problem of inscription.

A central idea of our method is to search for the appropriate Cevian point D by reversing the problem, starting with $\triangle A'B'C'$ and trying to construct an *"anticevian"* triangle ABC of $\triangle A'B'C'$ w.r.t. D.

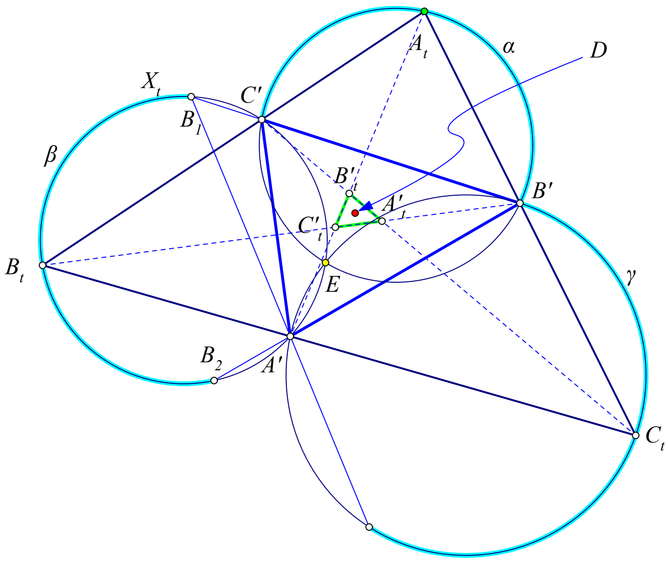

Figure 2. Similar triangles $\{A_tB_tC_t\}$ circumscribing $\triangle A'B'C'$

In fact, triangles $\{A_tB_tC_t\}$ similar to ABC circumscribing the triangle $A'B'C'$ and having a prescribed correspondence for {angles of ABC} $\leftrightarrow$ {sides of $A'B'C'$}, have their vertices respectively on three circles $\{\alpha, \beta, \gamma\}$. However, joining the corresponding vertices, the lines $\{A_tA', B_tB', C_tC'\}$ do not pass through the same point, but form instead a triangle $\tau_t = A'_tB'_tC'_t$ (see Figure 2). It will be seen below, that varying this triangle, equivalently, varying one of its vertices, A_t say, the corresponding $\triangle\ \tau_t$ degenerates for an appropriate t to a point D. The existence and uniqueness of the inscribed triangle $A'B'C'$ is proved under the assumption that it lies inside the triangle ABC. If we allow for it a position also outside the triangle, then, as has been shown by Hvala ([4]), there are in general one to three triangles of a given similarity type inscribed in $\triangle\ ABC$.

Regarding the organization of the article, section 2 discusses an aspect of Ceva's theorem used in the first proof of Seebach's theorem. Section 3 deals mainly with the question of *"strict"* circumscription of $\triangle\ A'B'C'$ by $\triangle\ ABC$ in the sense that we want the vertices of ABC to view the sides of $A'B'C'$ by the proper angles of triangle ABC and not their complements. This restricts the vertices of $A_tB_tC_t$ in certain arcs of the circles, shown in figure 2 . Section 4 gives a proof of Seebach's theorem. Section 5 gives a more general proof of existence of Cevian inscribed triangles including the case $\triangle A'B'C'$ is *"escribed"* in $\triangle ABC$. Section 6 deals with a property related to Maclaurin's theorem used in the last section 7, which gives a method to construct the triangle center $X(370)$.

2. CEVA'S THEOREM

Ceva's theorem, formulated below for convenience of reference, and whose proof can be found in every book of geometry ([5, p.145], [6, p.158]), guarantees that (see figure 3-(I)):

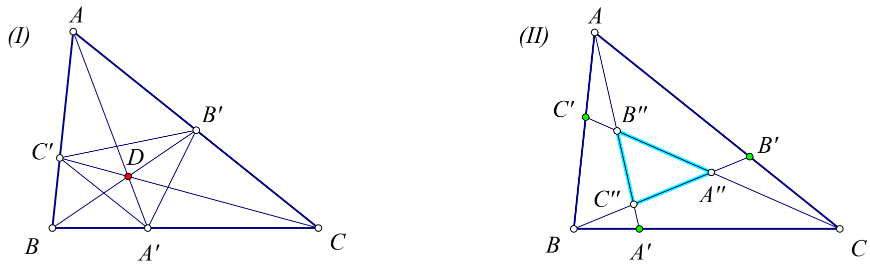

Figure 3. Ceva's theorem

Theorem 2.1. *A necessary and sufficient condition for three lines $\{AA', BB', CC'\}$ through the vertices of the triangle ABC to meet at a point D, not lying on any side-line of the triangle is the "Ceva's condition" for the ratios of oriented segments:*

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = -1. \tag{2.1}$$

Points $\{A' \in BC, B' \in CA, C' \in AB\}$ are the *"traces"* of the point D on respective sides of the triangle ABC. If we choose the three points $\{A', B', C'\}$ at random on the sides of the triangle, then the corresponding lines $\{AA', BB', CC'\}$ do not pass in general through the same point but form instead a triangle $A''B''C''$ (see figure 3-(II)). In this case we can still form the expression of the left side of Ceva's condition and following theorem, which, despite its simplicity seems to have been unnoticed, shows that this expression is related to the cross ratio formed by the quadruples of points on the lines $\{AA', BB', CC'\}$.

Theorem 2.2. *For three points $\{A', B', C'\}$ on the side-lines $\{BC, CA, AB\}$, different from the vertices of the triangle ABC, the lines $\{AA', BB', CC'\}$ form a triangle $A''B''C''$ and the cross ratio $(A'A; C''B'') = \frac{C''A'}{C''A} : \frac{B''A'}{B''A}$ is equal to the other corresponding cross ratios $(B'B; A''C'')$ and $(C'C; B''A'')$. Further this cross ratio is equal to the negative of Ceva's expression for the points $\{A', B', C'\}$:*

$$(A'A; C''B'') = (B'B; A''C'') = (C'C; B''A'') = -\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B}.$$

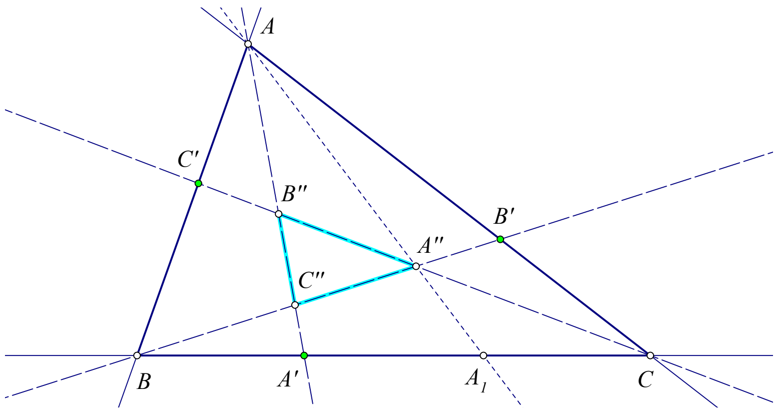

Figure 4. $(A'A; C''B'') = -\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B}$

Proof. Consider the intersection $A_1 = AA'' \cap BC$ (see Figure 4). By the well known invariance of cross ratio on a secant of a pencil of four lines ([7, p.89]), the pencil of four lines from A defines the same cross ratio on the secant lines

$$(CB; A_1A') = (B'B; A''C'') = (CC'; A''B'') = (C'C; B''A''),$$

latter equality following from the symmetry properties of cross ratios ([7, p.88]). This proves the first part of the theorem on the independence of the cross ratio from the particular line $\{AA', BB', CC'\}$.

To prove the other claim, we notice that

$$(C'C; B''A'') = (CB; A_1A') = (BC; A'A_1) = \frac{A'B}{A'C} : \frac{A_1B}{A_1C}. \tag{2.2}$$

Equating this with the negative of the Ceva expression we have:

$$-\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = \frac{A'B}{A'C} : \frac{A_1B}{A_1C} \quad \Leftrightarrow \quad \frac{A_1B}{A_1C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = -1,$$

which is Ceva's condition for the point A'' to which concur lines $\{AA_1, BB', CC'\}$. □

3. Miquel's configuration and the arcs of restriction

The *"Miquel configuration"* results by selecting three points $\{A' \in BC, B' \in CA, C' \in AB\}$ on respective sides of the triangle ABC (see Figure 5). It is then proved that the three circles $\{\alpha = (AB'C'), \beta = (BC'A'), \gamma = (CA'B')\}$ pass through the same point E, called *"Miquel point "* of the *"Miquel triangle"* $A'B'C'$ w.r.t. triangle ABC ([8, p.79], [5, p.131]).

Changing viewpoint, we fix the triangle $A'B'C'$, the point E and the three circles $\{\alpha, \beta, \gamma\}$ and consider point A varying on the circle α. Drawing then $\{AC', AB'\}$ and extending them until to meet a second time the circles $\{\beta, \gamma\}$ we see by a simple angle chasing argument, that they define the variable triangle ABC whose vertices lie on the respective circles $\{\alpha, \beta, \gamma\}$ and consequently has constant angles, or, as we say, it is of *"constant similarity type"*. A simple angle chasing argument shows, that we can create such a configuration for arbitrary similarity types of triangles $A'B'C'$ and ABC, the second circumscribing the first and having prescribed angle $\widehat{A}$ viewing $B'C'$, angle $\widehat{B}$ viewing $C'A'$ and angle $\widehat{C}$ viewing $A'B'$. This shows that for any two given triangles

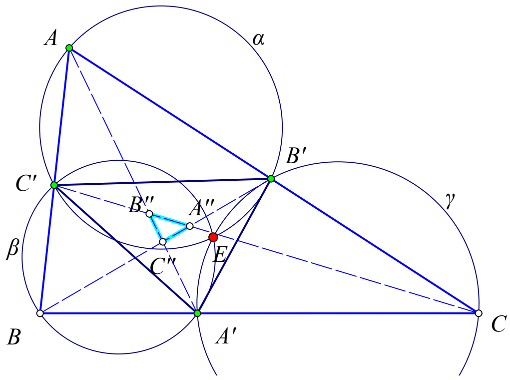

Figure 5. A Miquel configuration

$\{ABC, A'B'C'\}$ we can find similar to them triangles, such that the second is inscribed in the first in a prescribed correspondence of vertices of the first viewing sides of the second.

Regarding the angles, under which these vertices view the corresponding sides, they can be equal to those of the circumscribing triangle or equal to their supplements. If we insist in having the proper angles of ABC opposite to the prescribed sides of $A'B'C'$ and not their complements, then the vertices of ABC must be restricted into certain arcs, which we call *"arcs of restriction"*, subtending equal central angles, equivalently, subtending equal inscribed angles at E (see Figure 6). The relative location of the three

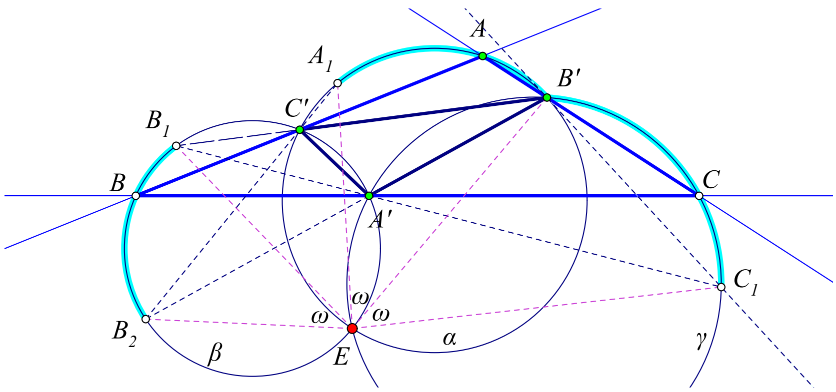

Figure 6. Restrictions of vertices of ABC on certain arcs (I)

arcs of restriction for the vertices of ABC vary in dependence of the angles of the two triangles $\{ABC, A'B'C'\}$ and the location of the point E of the three circles carrying the vertices of ABC.

The existence of these arcs follows from a standard exercise ([6, p.44]), according to which, given two triangles $\{ABC, A'B'C'\}$ we can inscribe the second into the first, so that the sides of the inscribed are parallel to those of the second (see Figure 7). For this we start with an arbitrary segment A_1C_1 parallel to $A'C'$ with endpoints on the sides $\{BA, BC\}$ of the triangle and construct on it the similar to $A'B'C'$ triangle $A_1B_1C_1$. Then, define $B' = BB_1 \cap AC$ and project it parallel to the sides of $\{A_1B_1, C_1B_1\}$ at points

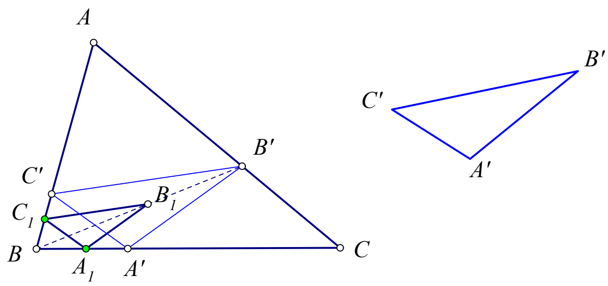

Figure 7. Inscribing $\triangle\ A'B'C'$ into $\triangle\ ABC$

$\{A' \in BC\ ,\ C' \in BA\}$. Obviously selecting the appropriate orientation of $\{A'B'C'\}$ we can realize every combination of an angle of ABC viewing a given side of $A'B'C'$.

Having at least a point on each circle $\{\alpha, \beta, \gamma\}$, we can pass to the maximal connected arc on that circle, whose points see the corresponding side under the same angle and from the three maximal arcs take the one with minimal in measure central angle. The arcs on the other circles result by the vertical inscribed angles at appropriate vertices of $\triangle\ A'B'C'$. For example in figure 6, taking the minimal arc (B_1BB_2) of circle β, the other arcs result by the vertical angles at the vertices $\{A', C'\}$.

The endpoints of these arcs can be found by varying a vertex, A say, of ABC and taking the limiting positions of the other vertices, when a side of ABC tends to coincide with a side of $A'B'C'$. Thus, these endpoints coincide either with vertices of the triangle $A'B'C'$ or with intersections of the circles with the sides of $A'B'C'$ or with the second intersection of a circle with a tangent at its intersection with one of the other circles. One example of the last case is point C_1 of figure 6, which is the second intersection of circle γ with the tangent to α at B'. This results from the limiting position of ABC coinciding with that of the triangle $B'B_1C_1$.

Figure 8 shows two other configurations with the corresponding arcs of restriction. Some other characteristics of these arcs, proved by simple angle chasing arguments, are: (i) each side-line of ABC divides the corresponding two arcs in proportional parts, and

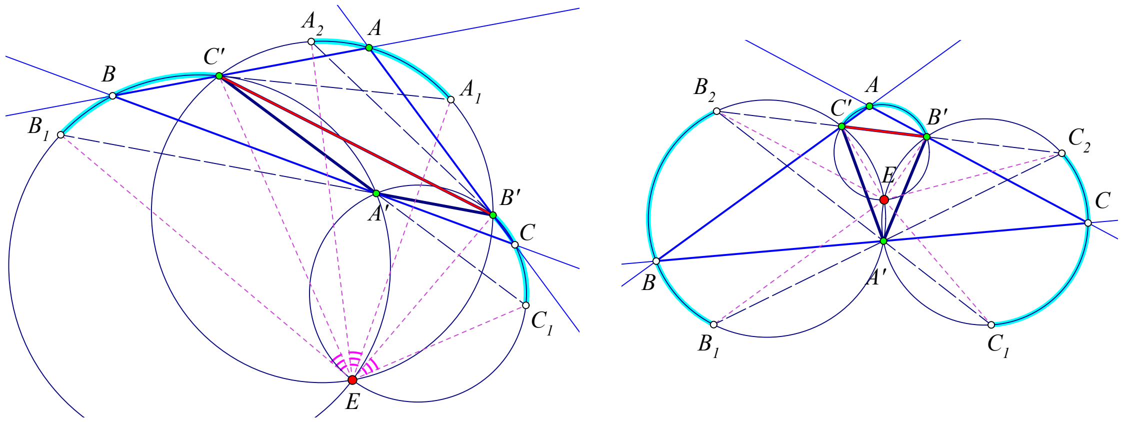

Figure 8. Restrictions of vertices of ABC on certain arcs (II)

(ii) point E, together with the endpoints of the arcs and the corresponding vertex of

ABC define quadrangles, like $\{EB_1BC', EA_2AA_1, EB'CC_1\}$ of the first figure 8, which are similar, their similarity center being point E.

We formulate these remarks in the form of a theorem.

Theorem 3.1. *For two triangles of given similarity type, the vertices of a triangle ABC of the first similarity type, circumscribing a triangle $A'B'C'$ of the second type, with $\{A, B, C\}$ viewing respectively the sides $\{B'C', C'A', A'B'\}$, vary on three circles $\{\alpha, \beta, \gamma\}$ intersecting at a point E. The angles under which $\{A, B, C\}$ view the sides $\{B'C', C'A', A'B'\}$ are equal to those of the triangle ABC or their supplements. The vertices of the triangles ABC which view the corresponding sides under angles equal to those of the triangle ABC and not equal to their supplements, lie on three similar w.r.t. E arcs of the circles $\{\alpha, \beta, \gamma\}$, which are divided by corresponding sides of $\triangle ABC$ in proportional parts.*

4. Seebach's theorem

Seebach's theorem ([3]), guarantees the existence of points D inside a triangle ABC, whose Cevian triangle has a given similarity type (see Figure 1):

Theorem 4.1. *Given two similarity types of triangles there is precisely one triangle $A'B'C'$ of the second type inscribed in a triangle ABC of the first type as a Cevian triangle of a point D with $\{A' \in BC, B' \in CA, C' \in AB\}$.*

In this section we give a proof based on the theorems of the two preceding sections. For this, we locate the appropriate Cevian point D by reversing the problem and starting with $\triangle A'B'C'$, try to construct an *"anticevian"* triangle ABC of $\triangle A'B'C'$ w.r.t. D. In fact, we saw that triangles $\{A_tB_tC_t\}$ similar to ABC circumscribing the triangle $A'B'C'$ and having a prescribed correspondence for {angles of ABC} $\leftrightarrow$ {sides of $A'B'C'$}, have their vertices respectively on three arcs of circles $\{\alpha, \beta, \gamma\}$ (see figure 9-(I)). However,

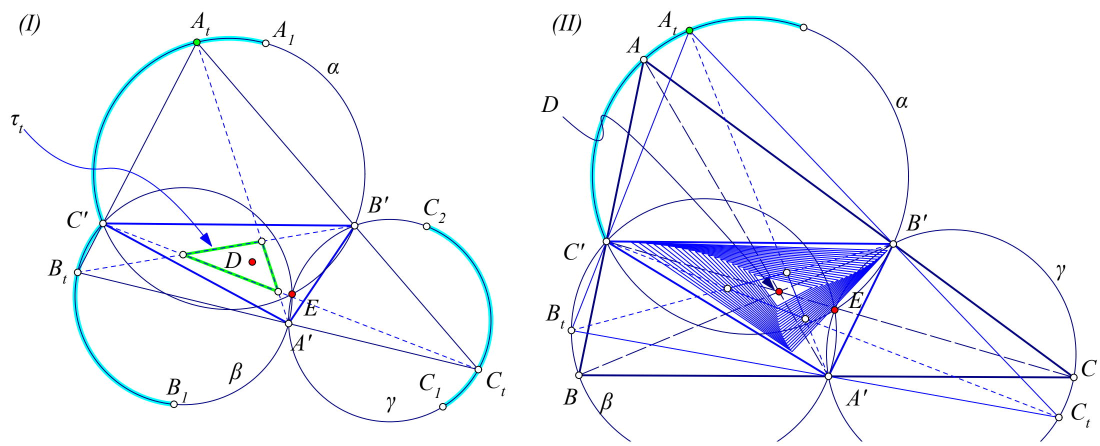

Figure 9. Similar triangles $\{A_tB_tC_t\}$ circumscribing $\triangle A''B''C''$

joining the corresponding vertices, the lines $\{A_tA', B_tB', C_tC'\}$ do not pass through the same point, but form instead a triangle $\tau_t = A'_tB'_tC'_t$. Figure 9-(II) shows some of the triangles $\{\tau_t\}$ corresponding to various positions of the triangle $A_tB_tC_t$.

It will be seen below, that varying this triangle, equivalently, varying one of its vertices, A_t say on the corresponding arc of restriction $(C'A_tB')$ (see Figure 10), the corresponding triangle τ_t degenerates to a point D precisely one time and defines the required point. For this, denote by $\tau_t = \triangle A'_tB'_tC'_t$ the triangle formed by the intersections of lines $\{A'A_t, B'B_t, C'C_t\}$. We show, that as A_t moves on the arc of restriction from

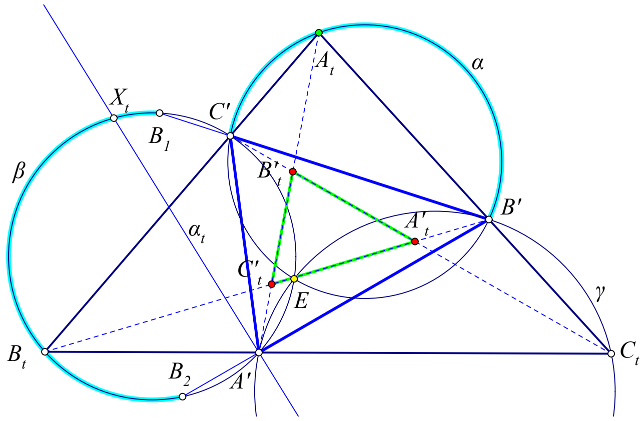

Figure 10. Studying the cross product $(A''A_t; C_1B_1)$

C' to B', the cross ratio $c_t = (A'A_t; C'_tB'_t)$ varies continuously from 0 to $+\infty$. In fact, for A_t tending to C', we see that B'_t tends also to C', implying that $c_t \to 0$. For A_t tending to B', we see that C'_t tends also to B', implying that $c_t \to +\infty$. Hence, by continuity, there is a $t = t_0$ for which $c_0 = 1$. This translates to $\frac{C'_tA'}{C'_tA_t} = \frac{B'_tA'}{B'_tA_t}$, i.e. the points $\{B'_t, C'_t\}$ are identical and by theorem 2.2 also points $\{C'_t, A'_t\}$ are identical. Consequently $\tau_0 = A'_tB'_tC'_t$ degenerates to a point D and $A'B'C'$ becomes the Cevian triangle of D w.r.t. $\triangle A_{t_0}B_{t_0}C_{t_0}$ as required.

Regarding the uniqueness of the point D, we use a known property of the anticevian triangle ([9, proposition 4, p.147]), which, for convenience of reference, I formulate as a theorem.

Theorem 4.2. *Given the triangle $A'B'C'$ and the point D, the harmonic conjugate line of D w.r.t. side-lines $\{A'B', A'C'\}$ coincides with the side-line through A' of the anticevian triangle ABC of $A'B'C'$ w.r.t. D.*

Having that, we consider the relative positions of the lines $\{B_tC_t, \alpha_t\}$, latter being the harmonic conjugate of A_t w.r.t. $\{A'B', A'C'\}$ (see Figure 10). As A_t moves clockwise on the arc of restriction from C' to B', point B_t moves also clockwise on the corresponding arc of restriction of B_t from B_2 to B_1 and the second intersection point X_t of α_t with β moves counterclockwise on β from C' to B_2. Triangle $A_tB_tC_t$ becomes anticevian of $A'B'C'$ precisely when $\alpha_t \equiv B_tC_t$, equivalently, when $X_t \equiv B_t$. This, because of the opposite monotonicity of the variations of $\{X_t, B_t\}$ on the arc $(B_1B_2,)$ can happen there only once. It follows that there is a unique position on the arc (B_1B_2) for which $X_t \equiv B_t$ and consequently the point D for which $\triangle A_tB_tC_t$ becomes anticevian w.r.t. $\triangle A''B''C''$ is unique.

5. A more general proof of existence

Next proof of the existence part of Seebach's theorem relies on theorem 4.2 and reveals a connection to the problem at hand of a structure from the realm of projective conics. Next lemma formulates a key property for this approach.

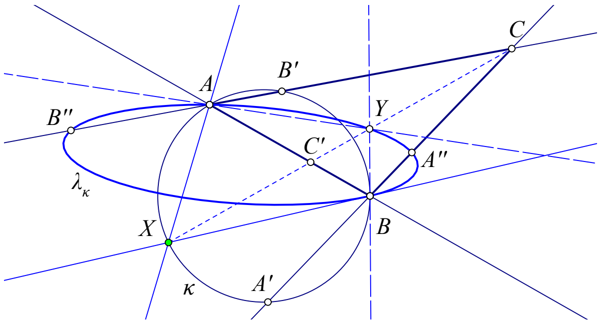

Figure 11. Harmonic perspectivity transformation and the conic λ_κ

Lemma 5.1. *Let κ be a circle through the vertices $\{A,B\}$ of the triangle ABC. For each point $X \in \kappa$ consider the harmonic conjugate lines $\{AY,BY\}$ of X w.r.t. the pairs of side-lines (AB,AC) and (BA,BC) intersecting at Y. The correspondence $Y = f_C(X)$ is a "harmonic perspectivity" with axis the line AB and center C, and point Y lies on the image λ_k via f of the circle κ.*

Proof. Recall that the *"harmonic perspectivity"* or *"harmonic homology"* ([10, p.55]) with axis AB and center C is the projectivity mapping a point of the plane X to a point $Y \in XC$, having cross ratio $(XY;CC') = -1$, where $C' = AB \cap CX$. The proof of the lemma follows directly from theorem 4.2, since X is the vertex of the anticevian triangle of ABC w.r.t. Y. □

Remark 5.1. *A triangle ABC defines three harmonic perspectivities $\{f_A, f_B, f_C\}$ with corresponding centers the vertices of the triangle and axes the opposite to them side-lines. It is easy to see (*[11]*) that the three perspectivities, together with the identity transformation e of the plane, build a group of four elements satisfying*

$$f_A^2 = e \quad \Leftrightarrow \quad f_A^{-1} = f_A \quad \text{and} \quad f_A \circ f_B = f_B \circ f_A = f_C,$$

and the analogous relations for cyclic permutations of $\{A,B,C\}$.

Returning to the existence part of Seebach's theorem, we consider the circles $\{\alpha,\beta,\gamma\}$ carrying the vertices of the circumscribing triangles ABC of $A'B'C'$ (see Figure 12). Applying to each circle the corresponding harmonic perspectivity $\{f_A, f_B, f_C\}$ of the preceding lemma, we obtain three conics $\{\lambda_\alpha = f_A(\alpha)\ ,\ \lambda_\beta = f_B(\beta)\ ,\ \lambda_\gamma = f_C(\gamma)\}$. If a certain point on a circle, A say on α, is a vertex of an anticevian triangle of $A'B'C'$, w.r.t. some point D, then, by theorem 4.2 and lemma 5.1, point D will be on the conic λ_α.

The existence of D results from the fact that two of the conics, $\{\lambda_\alpha, \lambda_\gamma\}$ say, intersecting at a vertex (B') intersect also at least at one more point. For, suppose that they had no other intersection point and were tangent there, having a common tangent line t trhough B'. This would imply that the circles $\{\alpha,\gamma\}$ have also a common tangent at B' and since all three circles are supposed to pass through the same point E, this would be

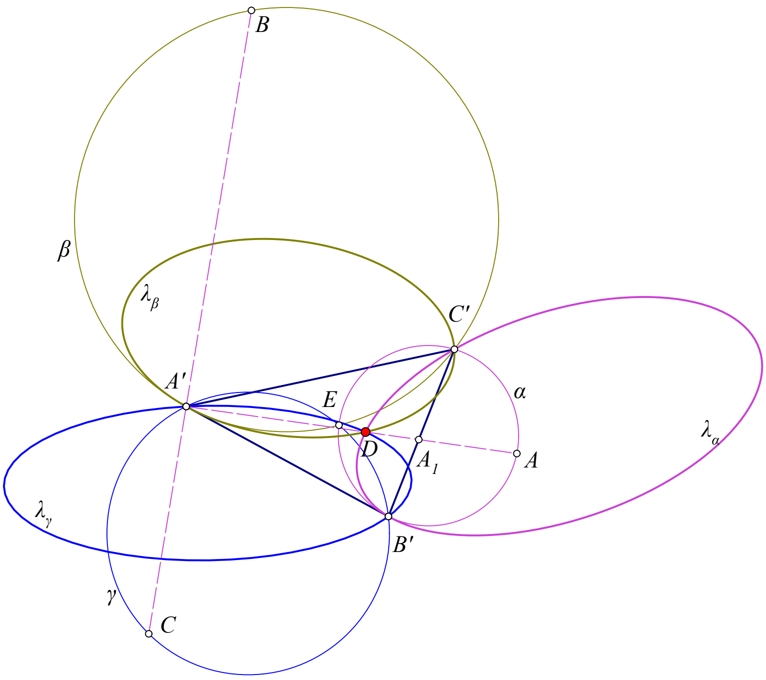

Figure 12. Construction of the point D using three conics

coincident with B', which, by assumption, is not allowed. Thus, there is a point D common to $\{\lambda_\alpha, \lambda_\gamma\}$ different from the vertex B'. This, using the preceding remark, implies that the third conic λ_β passes also through D. In fact,

$$\begin{aligned} D &= f_A(A) = f_C(C) && \Rightarrow \\ C &= f_C(f_A(A)) = f_B(A) && \Rightarrow \quad A = f_B(C)\,. \end{aligned}$$

By the definition of f_B this implies that $\{A, C\}$ and B' are collinear and

$$\begin{aligned} D &= AA' \cap CC' \quad \Rightarrow \\ f_B(D) &= (f_B(A)f_B(A')) \cap (f_B(C)f_B(C')) \\ f_B(D) &= CA' \cap AC'\,. \end{aligned}$$

Since point $B' \in AC$, this implies that $f_B(D) = CA' \cap AC'$ is a point of the circle β, hence $D = f_B(CA' \cap AC')$ lies also on the conic λ_β.

Remark 5.2. *Since the conics $\{\alpha, \beta, \gamma\}$ intersect pairwise at a point, the three conics cannot share more than three common points. This conforms to the result by Hvala who gives a generalization of Seebach's theorem ([4]), its paper including the history of the subject and further references to it. Seebach's theorem guarantees one common point D. Hence they can exist at most two additional points $\{D', D''\}$ whose anticevian triangles w.r.t. $A'B'C'$ are similar to ABC and circumscribe $A'B'C'$. Necessarily then, the two other triangles must have at least one of their vertices on the complement of the arc of restriction on the corresponding circle. Hence they are "circumscribing" $A'B'C'$ in a wider sense, since that vertex will view the corresponding side of $A'B'C'$ under the supplement of the respective angle of the triangle ABC and not under the angle itself.*

Figure 13 shows a case of an equilateral $A'B'C'$ circumscribed by three similar triangles $\{A_1B_1C_1,\ A_2B_2C_2,\ A_3B_3C_3\}$. The first one $A_1B_1C_1$ is cevian w.r.t. D and circumscribes the equilateral in the strict sense. The other two are cevian correspondingly w.r.t. the points $\{D', D''\}$ and circumscribe the equilateral in the wider sense, having, each, two vertices viewing the sides of the equilateral by a supplemental angle of the triangle $A_1B_1C_1$.

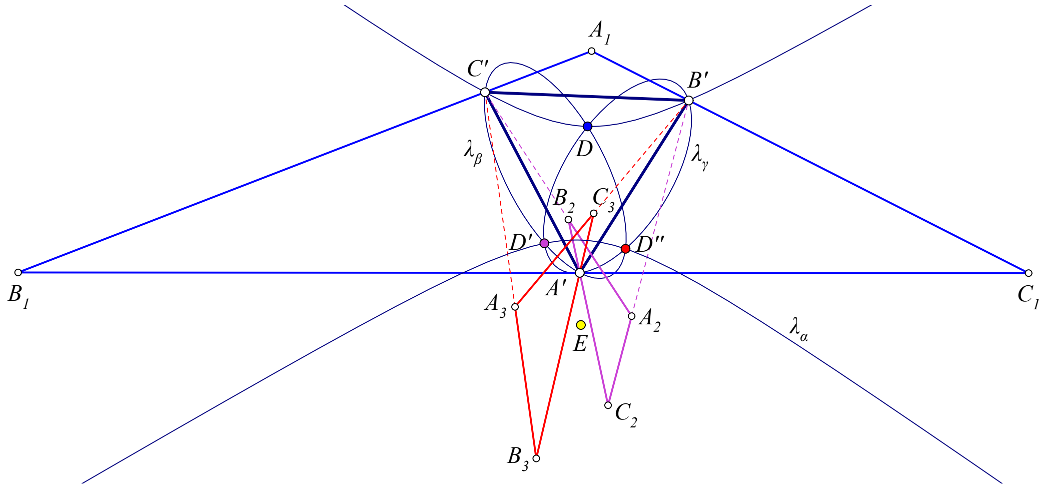

Figure 13. Three similar triangles circumscribing an equilateral

6. Two conics and a line

The two preceding sections lead to two different approaches to the problem of construction of an equilateral $A'B'C'$ inscribed as a Cevian triangle inside a given triangle ABC. The intersection of the Cevians of such a triangle, to which this construction amounts, is the *"triangle center"* $D = X(370)$ of ABC in Kimberling's list [1].

In this section we examine a configuration, different from the preceding two, leading to a method of construction of this point, which is completed in the next section. The present configuration is based on Maclaurin's theorem ([12, p.77], [13, p.247]). This considers four lines $\{\alpha, \beta, \gamma, \delta\}$ in general position, their six intersection points and two arbitrary points $\{A \in \alpha, C \in \gamma\}$. The theorem guarantees, that a triangle XYI, whose side-lines pass through the fixed points $\{O = \alpha \cap \gamma, A, C\}$ with $\{A \in IX, C \in IY, O \in XY\}$ and two of its vertices $\{X, Y\}$ move on the fixed lines $\{X \in \delta, Y \in \beta\}$, have their third vertex I describing a conic λ passing through points $\{A, C\}$. Further, the conic passes also through the intersection points $\{B = \beta \cap \delta, D = \gamma \cap \delta, E = \alpha \cap \beta\}$ of the quadruple of fixed lines.

In figure 14 illustrating this configuration, appears a second conic μ enveloping the lines UV created by the intersections $\{U = OC \cap AX\ ,\ V = OA \cap CY\}$. Its existence is guaranteed through the Chasles-Steiner theorem ([10, p.77]), by which two points $\{U, V\}$ varying on two lines $\{\gamma, \alpha\}$ and corresponding by a homographic relation $V = f(U)$ define lines UV enveloping a conic. The homography relating the two points is the composition $f = r \circ q \circ p$ of perspectivities between lines $\gamma \ni U \overset{p}{\mapsto} X \in \delta$ with lines through A, $\delta \ni X \overset{q}{\mapsto} Y \in \beta$ with lines through O, and $\beta \ni Y \overset{r}{\mapsto} V \in \alpha$ with lines through C.

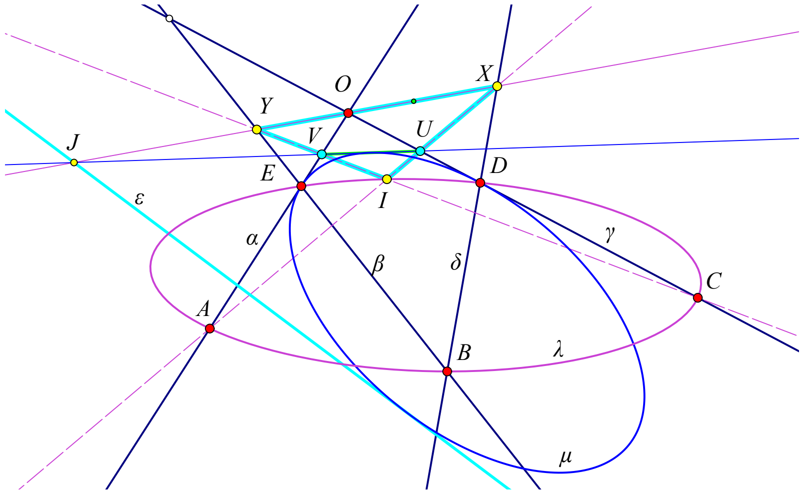

Figure 14. Maclaurin's theorem for the variable triangle

Line ε, appearing in figure 14 and containing the intersections $J = XY \cap UV$, results from the following lemma.

Lemma 6.1. *With the notation and conventions of this section, the lines* $\{XY, UV\}$ *intersect at a point* J *describing a line as* X *varies on line* δ *.*

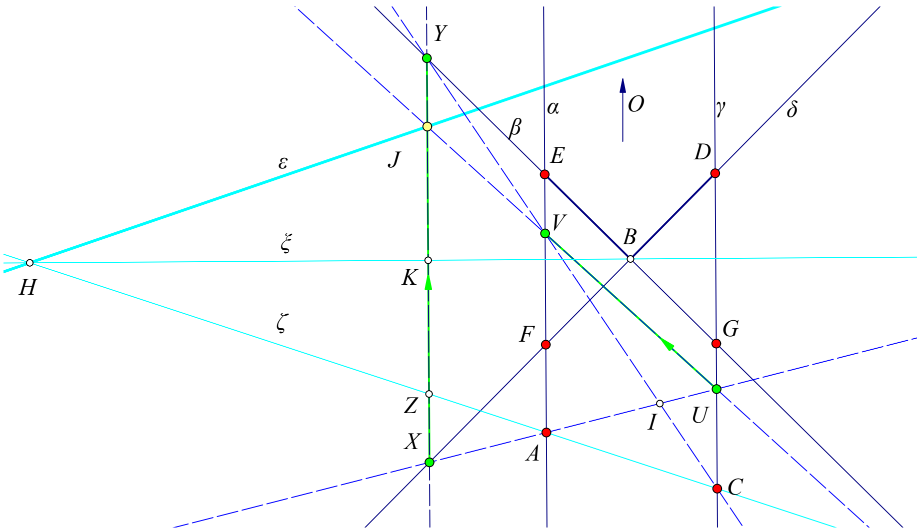

Figure 15. A reduction by a projectivity of Maclaurin's configuration

Proof. Since all the ingredients of the configuration at hand are invariant under projective transformations, we simplify our reasoning by transforming the configuration of figure 14 to a simpler one. For this we consider the projective transformation sending BDE to an isosceles right triangle and also sending point O to infinity in the direction of the bisector of the angle $\widehat{DBE}$ (see Figure 15). Then, lines $\{\alpha = AO\,, \gamma = CO\}$ become parallel and also the line XY passing through O becomes parallel to these two lines for any $X \in \delta = BD$ and the corresponding $Y \in \beta = BE$.

Consider now the line $\xi = BK$ through B orthogonal to the direction of these parallels. Points $J = XY \cap UV$ vary on line ε which is the reflection on ξ of the line $\zeta = AC$. To see this it suffices to verify that $XZ = JY$, where $Z = XY \cap \zeta$. In fact, from their definition triangles $\{XAZ\ ,\ UAC\}$ are similar and also triangles $\{JYV\ ,\ UCV\}$ are similar and the required equality $XZ = JY$ results immediately. $\square$

7. The point $X(370)$

Given the triangle ABC, We start with an angle $\widehat{B'AC'}$ of measure $\pi/3$ revolving about the vertex A and intersecting the line BC at $\{B', C'\}$ (see Figure 16). On the sides $\{AB', AC'\}$ we construct equilaterals $\{AB'B'', AC'C''\}$ and consider the intersections of the variable lines $\{C_1 = AB \cap CB''\ ,\ B_1 = AC \cap BC''\}$. From these points we draw parallels respectively to $\{B'B'', C'C''\}$, intersect at a point A_1 with the following properties.

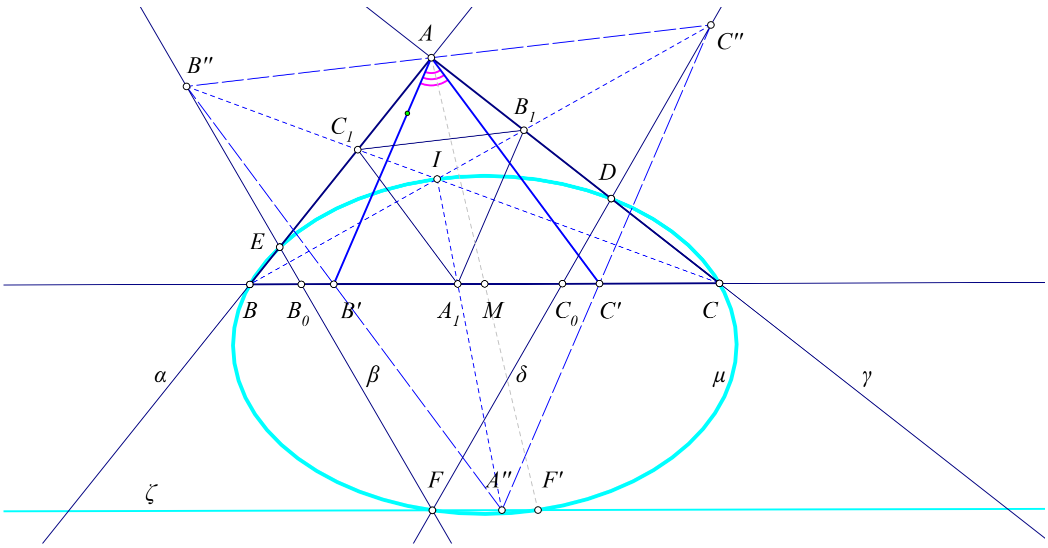

Figure 16. Searching for a point whose cevian is equilateral

Theorem 7.1. *With the notation and conventions of this and the preceding section, the following are valid properties:*

(1) *The intersection $A'' = B'B'' \cap C'C''$ moves on the parallel ζ to BC containing the reflection of A in BC.*
(2) *Line B_1C_1 is parallel to line $B''C''$.*
(3) *The triangle $A_1B_1C_1$ is equilateral and its vertex A_1 lies on BC.*
(4) *The conic μ carrying the intersection points $\{I = CB'' \cap BC''\}$ passes through the reflected point F of the vertex A w.r.t. BC and also through the symmetric F' of A w.r.t to the middle M of BC.*
(5) *The conic μ passes also through the vertices $\{B, C\}$ and the intersection points $\{E, D\}$ of the sides $\{AB, AC\}$ with the sides of the equilateral B_0FC_0 having a vertex at F and opposite side contained in line BC.*
(6) *One axis of the conic μ is parallel to BC.*

Proof. Nr-1. We notice first, that the equilateral triangle $AC'C''$ varies having the vertex A fixed and the vertex C' moving on line BC. Hence, by the well known property of similar triangles ([6, p.49]) varying that way, the third vertex varies on a line $\delta = FD$ intersecting line BC at 60° degrees. The same happens with vertex B'', seen to vary on a line $\beta = EF$ intersecting BC also at 60° degrees. The triangle $A''B''C''$ created by the extensions of $\{B''B', C''C'\}$ is equilateral, its vertex A'' defines the parallelogram $AB'A''C''$ and varies on a line through F, parallel to BC, point F being the reflection of A on BC.

Nr-2. The resulting configuration is that of lemma 6.1, guaranteeing that the intersection $E_1 = BC'' \cap CB''$ varies on a conic μ passing through the five points $\{E, B, F, C, D\}$ and the intersection point J of lines $\{B''C'', C_1B_1\}$ varies on a line ε. When A'' takes the position F, we see easily that $B''C''$ becomes parallel to BC and C_1B_1 is parallel to BC. Hence line ε is either the line at infinity or is a parallel to BC. The parallelism of lines $\{B''C'', B_1C_1\}$ is equivalent with the coincidence of ε with the line at infinity. This follows by one more point at infinity contained in ε and obtained through an appropriate position of triangle $\widehat{B'A'C'}$.

This position for $B'AC'$ is the one for which $B' = C_0$ and C' is the point at infinity of BC, i.e. $AC'C''$ becomes an infinite triangle with $B''C''$ parallel to δ. Then, it is easily seen that $C_1 = E_1 = B$, $B'' = B_0$ and $B_1 = C''B \cap \gamma$ defines B_1C_1 parallel to $B''C''$. This means that ε passes through the point at infinity of AB_0, hence, having two different points at infinity coincides with the line at infinity.

Nr-3. This follows from the preceding *nrs*. In fact, the triangle $A_1B_1C_1$ is by definition equilateral and has its sides parallel to corresponding sides of the equilateral $B'AB''$. Besides, the two lines $\{AB_1 \ , \ B''C_1\}$ joining corresponding vertices pass through the same point C. Hence the $B'A_1$ joining the third couple of corresponding vertices will pass through C too.

Nr-4. Follows by observing the position of I when triangle $B'AC'$ becomes isosceles with apical angle of measure 120° degrees. Then all four points $\{A'', B'', C'', I\}$ coincide with F. An analogous argument involving a special position of line $B''C$ shows that μ passes through F'. This is the position of I when $B''C$ becomes parallel to AB.

Nr-5. Follows from the Maclaurin configuration of figure 14, of which the present configuration is a special case.

Nr-6. This follows from the preceding *nrs*, since $\{BC, FF'\}$ are parallel chords of the conic μ. It is then easily seen, that they are both symmetric w.r.t. the bisector line of the side BC, hence the directions of this bisector line and of BC are two orthogonal conjugate lines w.r.t. the conic μ. □

The theorem implies, that point $X(370)$, whose Cevian triangle is equilateral, is an intersection point of μ and the other two analogous conics $\{\mu', \mu''\}$ constructed w.r.t. to the other sides of the triangle ABC (see Figure 17). The interesting fact is that we have, for each of these conics, six easily constructible points, such as $\{B, C, D, E, F, F'\}$ in the case of μ, through which passes the conic.

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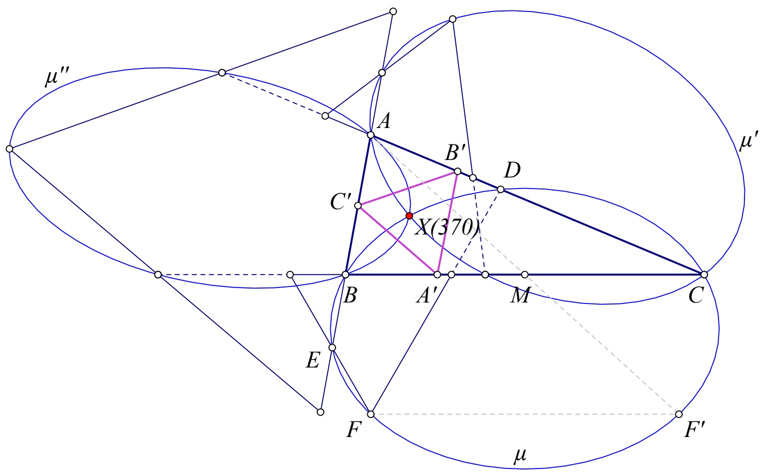

Figure 17. Construction of the Cevian equilateral $A'B'C'$ of $\triangle ABC$

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