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PROPERTIES OF ORTHOCENTROIDAL CIRCLES IN RELATION TO THE COSYMMEDIANS TRIANGLES

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ABSTRACT. The purpose of this work is to highlight new properties of the orthocentroidal circle via the concept of cosymmedian triangles. Given a triangle ABC, by (1.1) define the angles φ_A, φ_B, and φ_C. These angles, Proposition 2(1) and Proposition 3(2) are very useful in the elementary approach of the proposed questions. Theorem 13 establishes the connection between the Fermat and Brocard axes of the given triangle and those of the orthocentroidal triangle $A_1B_1C_1$. Theorem 14 states that $2R \cdot F^+F^- = GH \cdot J^+J^-$. The properties of sequences $(A_nB_nC_n)_{n\geq 0}$ and $(A'_nB'_nC'_n)_{n\geq 0}$ are studied, where $A_nB_nC_n$ and $A'_nB'_nC'_n$ are cosymmedian triangles in the orthocentroidal circle of rank n.

In the center of our attention are the concepts of *orthocentroidal circle* and *cosymmedian triangles*. There are many valuable results on these two concepts separately or in connection with each other (see **[3]**, **[1]**, **[5]**, **[6]**, **[9]**). For a given triangle, ABC, there is only one triangle so that they form a pair of cosymmedian triangles. This pair of triangles induces in the orthocendroidal circle of the triangle ABC, in a way *specified below*, a new pair of cosymmedian triangles. Repeating the process, a sequence of pairs of cosymmedian triangles associated with a given triangle is obtained. There is a close connection between certain centers or central lines of a given triangle and the corresponding ones in the induced triangles. The purpose of this work is to highlight new properties in this framework. We will approach all the proposed issues in an elementary way.

1. NOTATIONS AND PRELIMINARIES

Consider a triangle ABC with the sidelenghts a, b, c and area Δ. Its circumcenter, orthocenter, centroid, nine-point center, and symmedian point are denoted by O, H, G, N, K, respectively. Let h_a, m_a denote the lengths of the altitude and median from the vertice A of the triangle ABC; denote h_b, m_b and h_c, m_c analogously. The line OK is called the *Brocard axis.* The line F^+F^- (F^+- the *first Fermat (isogonic) point*, F^-- the *second Fermat (isogonic) point*) is called the *Fermat axis*. The triliniar polar of K with respect to ABC is called the *Lemoine axis.* By J^+, J^- are denoted the *isodynamic points*; it is known that J^+, J^- lie on the Brocard axis and are symmetric about the Lemoine axis. *Brocard circle* and *Fermat circle* are the circles having the line segments OK and respectively F^+F^- as

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diameter. Details of these remarkable points, lines and circles can be found in [1], [4], [5], [6], [7], [8], [9], [11].
Let A', B', C' the traces of the symmedians AK, BK, CK on (ABC), i.e. $A' = AK \cap (ABC), A' \neq A$, and similarly for B', C' ((XYZ) denotes the circle determined by the points X, Y, Z). Then, ABC and $A'B'C'$ are called *cosymmedian triangles*. For the convenience of the reader, we recall some properties regarding symmedians and cosymmedian triangles. Often, for the known results that are used in this section, alternative demonstrations to those in the places cited are given.

Proposition 1. *We have the formulas (Fig. 1):*
(i) $s_a = \dfrac{2bc}{b^2+c^2} m_a$ *($s_a = AL$ - the symmedian from A);*
(ii) $\dfrac{BL}{CL} = \dfrac{c^2}{b^2}, \quad BL = \dfrac{ac^2}{b^2+c^2}, \quad CL = \dfrac{ab^2}{b^2+c^2}$;
(iii) $AK = \dfrac{2bc}{a^2+b^2+c^2} m_a, \quad KL = \dfrac{2a^2bc}{(b^2+c^2)(a^2+b^2+c^2)} m_a$;
(iv) $s'_a = \dfrac{bc}{m_a}, \quad A'K = \dfrac{3a^2bc}{2(a^2+b^2+c^2)m_a}$ *($s'_a = AA'$ - the extension of symmedian s_a to circumcircle), and analogue formulas.*
For these and many other formulae connected with the symmedians see [**11**].

Let $\varphi_A, \varphi_B, \varphi_C$ be the angles between the altitudes and medians from the vertices A, B, C of the triangle ABC, i.e.

$$\varphi_A = \widehat{h_a, m_a}, \quad \varphi_B = \widehat{h_b, m_b}, \quad \varphi_C = \widehat{h_c, m_c}. \tag{1.1}$$

They will play an important role in the following. Because

$$DD' = \frac{|b^2-c^2|}{2a}, \quad EE' = \frac{|c^2-a^2|}{2b}, \quad FF' = \frac{|a^2-b^2|}{2c}, \tag{1.2}$$

by applying the sine and cosine laws to triangle ADD' we immediately deduce the formulas:

$$\sin\varphi_A = \frac{|b^2-c^2|}{2am_a}, \qquad \cos\varphi_A = \frac{h_a}{m_a} = \frac{2\Delta}{am_a}, \tag{1.3}$$

and then their analogues for φ_B and φ_C.

The cosymmedian triangles ABC and $A'B'C'$ have the same orientation. The order of the vertices on their circumscribed circle is $A - C' - B - A' - C - B'$. If $a > b > c$, then $m_a < m_b < m_c$ and, according to the formulae (1.6) below, we have $a' < b' < c'$.
Further, we will consider that $a > b > c$ without loss of generality.

Proposition 2. Let *ABC and $A'B'C'$ be cosymmedians triangles. Then,*
1) the corresponding angles are connected by the formulae:

$$A' = A - \varphi_B - \varphi_C, \qquad B' = B + \varphi_C - \varphi_A, \qquad C' = C + \varphi_A + \varphi_B, \tag{1.4}$$

$$A = A' + \varphi_B + \varphi_C, \qquad B = B' - \varphi_C + \varphi_A, \qquad C' = C - \varphi_A + \varphi_B; \tag{1.5}$$

2) the sides of the one are proportional to the medians of the other ([**11**; p.62], [**9**; #6.20]*); more specifically, we have:*

$$a' = \frac{3}{4}\frac{abc}{m_a m_b m_c} \cdot m_a, \qquad a = \frac{3}{4}\frac{a'b'c'}{m_{a'} m_{b'} m_{c'}} \cdot m_{a'}, \tag{1.6}$$

and analogues formulae.

Proof. 1) We show only the first equation in (1.4). With the remaining ones proceed similarly. We have $A' = \widehat{B'A'C'} = \widehat{B'A'A} + \widehat{AA'C'} = \widehat{B'BA} + \widehat{ACC'}$. Since BB' and CC' are symmedians of the triangle ABC, it follows that $\widehat{B'BA} = \widehat{GBC}$ and $\widehat{ACC'} = \widehat{GCB}$. Then, $A' = \widehat{GBC} + \widehat{GCB} = \left[B - \left(\frac{\pi}{2} - A\right) - \varphi_B\right] + \left[C - \left(\frac{\pi}{2} - A\right) - \varphi_C\right] = A - \varphi_B - \varphi_C$. Obviously, the formulas (1.5) are obtained from (1.4) by solving with respect to A, B, C.

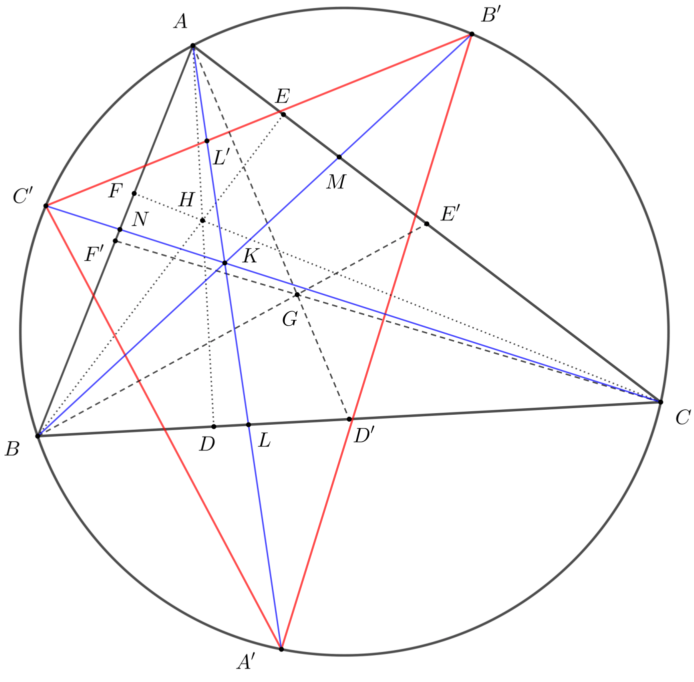

Figure 1

2) We refer only to the sides a of the triangle ABC (for a' it is enough to change the roles of the two triangles). By applying the cosine formula to the triangles KBC and $KB'C'$, we obtain:

$$\left(a'\right)^2 = B'K^2 + C'K^2 - 2B'K \cdot C'K \cdot \frac{BK^2 + CK^2 - a^2}{2BK \cdot CK}.$$

According to Proposition 1, $BK = \dfrac{2ca}{a^2+b^2+c^2} m_b, CK = \dfrac{2ab}{a^2+b^2+c^2} m_c,$

$B'K = \dfrac{3ab^2c}{2\left(a^2+b^2+c^2\right) m_b}, C'K = \dfrac{3abc^2}{2\left(a^2+b^2+c^2\right) m_c}$. By substituting these expressions and making routine calculations, we finally get the first formula from (1.6).

Second proof. By the sine formula, $a' = B'C' = 2R\sin A'$ (R - the radius of the circumcircle (ABC)). So, taking into account (1.4),
$a'{=}2R\sin(A-\varphi_B-\varphi_C) = 2R\left[\dfrac{2\Delta}{bc}\cos(\varphi_B+\varphi_C) - \dfrac{b^2+c^2-a^2}{2bc}\sin(\varphi_B+\varphi_C)\right]$. Now, by using (1.3), we get the expressions of $\sin(\varphi_B+\varphi_C)$ and $\cos(\varphi_B+\varphi_C)$ quite easily. In the end, $a' = 2R\cdot\dfrac{3\Delta}{2m_bm_c} = \dfrac{3}{4}\dfrac{abc}{m_am_bm_c}\cdot m_a$.

Proposition 3. Let ABC *and* $A'B'C'$ *be cosymmedians triangles. Then,*
1) they have the same symmedians ([**11**; pp.61], [**1**; p.265, #617]),
2) the angles between the altitudes and the medians in the corresponding vertices are equal, i.e. $\varphi_A = \varphi_{A'}, \varphi_B = \varphi_{B'}, \varphi_C = \varphi_{C'}$.
Proof. 1) Let's show that the symmedian AA' is at the same time the symmedian of the triangle $A'B'C'$. If $L' = AA'\cap B'C'$ (Fig. 1), then

$$\begin{aligned}\frac{B'L'}{L'C'} &= \frac{c'\sin\widehat{AA'B'}}{b'\sin\widehat{AA'C'}} = \frac{c'}{b'}\cdot\frac{\sin\widehat{ABB'}}{\sin\widehat{ACC'}}\\ &= \frac{c'}{b'}\cdot\frac{\sin\widehat{GBC}}{\sin\widehat{GCB}} = \frac{c'}{b'}\cdot\frac{GC}{GB} = \frac{c'}{b'}\cdot\frac{m_c}{m_b}.\end{aligned}$$

But, according to Proposition 2, m_b, m_c are proportional to b', c', respectively. Hence, $\dfrac{B'L'}{L'C'} = \left(\dfrac{c'}{b'}\right)^2$ and thus AA' is the symmedian from A' in the triangle $A'B'C'$.
2) AA' being symmedian in the triangle $A'B'C'$, we have: $\varphi_{A'} = A' - (\dfrac{\pi}{2} - C') - \widehat{AA'C'}$. Therefore, $\varphi_{A'} = \dfrac{\pi}{2} - B' - \widehat{AA'C'} = \dfrac{\pi}{2} - B' - \widehat{ACC'}$. By (1.4), it follws that $\varphi_{A'} = \dfrac{\pi}{2} - (B+\varphi_C-\varphi_A) - \widehat{ACC'}$. Further, CC' being symmedian in the triangle ABC, we have: $\widehat{ACC'} = \widehat{GCB} = C - \varphi_C - \left(\dfrac{\pi}{2} - A\right)$. Consequently, $\varphi_{A'} = \dfrac{\pi}{2} - (B+\varphi_C-\varphi_A) - \left[C - \varphi_C - \left(\dfrac{\pi}{2} - A\right)\right] = \varphi_A$, hence $\varphi_{A'} = \varphi_A$.

Remark. It is clear that the cosymmedian triangles can be defined as two triangles that had the same circumcircle and symmedians. But, two triangles that have the same circumcircle and symmedian point may not be cosymmedian (as is the case of two equilateral triangles inscribed in the same circle).

Other important properties are given by
Proposition 4. *Two cosymmedian triangles have:*
1) the same Apollonius circles, Lemoine axis and isodynamic points ([**1**; p.265, #617]);
2) the same Brocard axis, Brocard circle, Brocard points and Brocard angle ([**6**; p.283, #475], [**9**; #6.21]).

The *orthocentroidal circle* of triangle ABC is the circle on HG as diameter. Denote D, E, F the feets of the altitudes and D', E', F' the midpoints of the sides BC, CA, AB, respectively. Obviously, the orthocentroidal circle contains the orthogonal projections A_1, B_1, C_1 of G on the altitudes AD, BE, CF, and the orthogonal projections A_1', B_1', C_1' of H on the medians AD', BE', CF', respectively. Therefore, both triangles $A_1B_1C_1$ and $A'_1B'_1C'_1$ are inscribed in the orthocentroidal circle (Fig. 2). $A_1B_1C_1$ is called the *orthocetroidal triangle*. We refer to these triangles as the pair of triangles induced by ABC in its orthocentroidal circle.

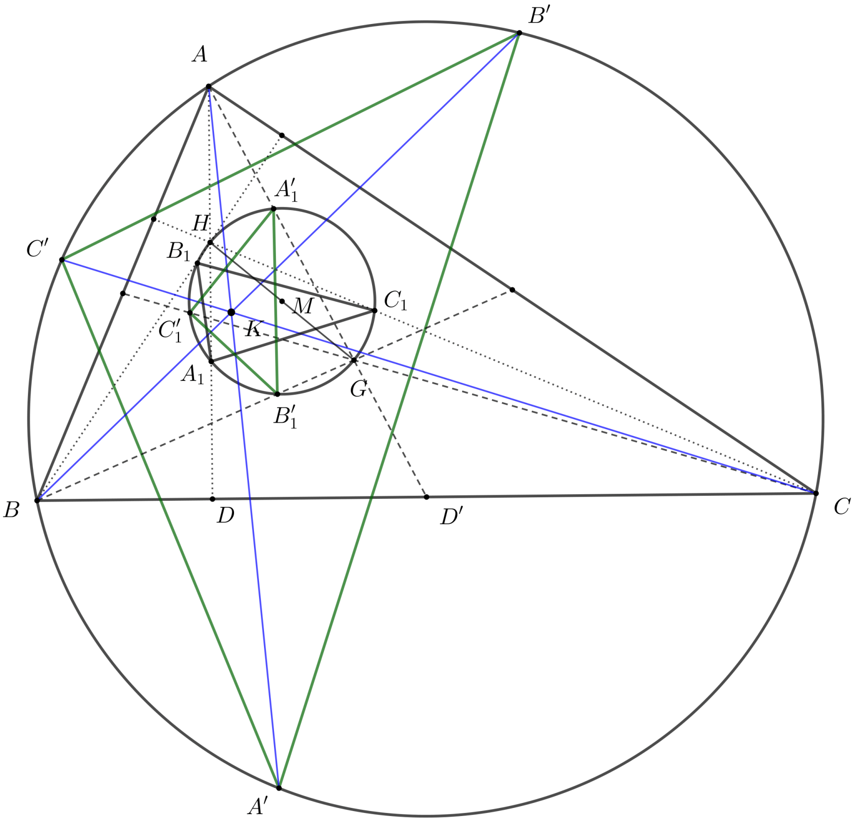

Figure 2

Proposition 5. *The following statements are true:*
1) the triangles in the pairs $(ABC, A_1B_1C_1)$ and $(A'B'C', A'_1B'_1C'_1)$ are inversely similar;
2) $A_1B_1C_1$ and $A'_1B'_1C'_1$ are cosymmedian triangles;
3) K is the symmedian point for triangles $ABC, A'B'C', A_1B_1C_1, A'_1B'_1C'_1$;
4) the triangles $ABC, A'B'C', A_1B_1C_1, A'_1B'_1C'_1$ have in the corresponding vertices equal angles between their altitudes and medians.
Proof. 1) The fact that $ABC \sim A_1B_1C_1$ is notorious and easy to prove. Indeed, because $GA_1 \perp AD$, it follows that $GA_1 \parallel CB$; similarly, $GB_1 \parallel CA$ and $GC_1 \parallel BA$. We deduce that $\widehat{A_1GB_1} = \widehat{BCA} = C$, $\widehat{B_1GC_1} = \widehat{CAB} = A$, $\widehat{C_1GA_1} = \widehat{ABC} = \pi - B$. Then, $A_1 = \widehat{B_1A_1C_1} = \widehat{B_1GC_1} = A$. Similarly, $B_1 = B$, $C_1 = C$.
Next, we have: $A'_1 = \widehat{B'_1A'_1C'_1} = \widehat{B'_1GC'_1} = \pi - \widehat{BGC} = \widehat{GBC} + \widehat{GCB} = \left[B - \varphi_B - \left(\dfrac{\pi}{2} - A\right)\right] + \left[C - \varphi_C - \left(\dfrac{\pi}{2} - A\right)\right] = A - \varphi_B - \varphi_C$. But, by (1.4), $A - \varphi_B - \varphi_C = A'$, hence $A'_1 = A'$. In the same way we find that $B'_1 = B'$ and $C'_1 = C'$. Thus, $A'B'C' \sim A'_1B'_1C'_1$.

2) We have to show that $A_1A_1', B_1B_1', C_1C_1'$ are the symmedians of the triangle $A_1B_1C_1$. We only show for A_1A_1'. So, let's show that A_1A_1' and the median m_{A_1} are equally inclined on the sides A_1B_1 and A_1C_1, i.e. $\widehat{A_1'A_1B_1} = \widehat{m_{A_1}, A_1C_1}$. On the one hand, because $A_1B_1C_1 \sim ABC$, we have:

$$\widehat{m_{A_1}, A_1C_1} = \widehat{m_A, AC} = A - \left(\frac{\pi}{2} - B\right) - \varphi_A = \frac{\pi}{2} - C - \varphi_A.$$

On the other hand, in the orthocentroidal circle we have:

$$\begin{aligned} \widehat{A_1'A_1B_1} &= \widehat{A_1'GB_1} = \widehat{A_1'GA_1} - \widehat{A_1GB_1} \\ &= \widehat{AGA_1} - \widehat{A_1C_1B_1} = \left(\frac{\pi}{2} - \varphi_A\right) - C = \frac{\pi}{2} - C - \varphi_A, \end{aligned}$$

what concludes the proof.

3) We need the following result:

Lemma ([**9**; p.59]). *If three lines, concurrent at O, are cut by a transversal at P, Q, R, and by another at P', Q', R' (Fig. 3a), then*

$$\frac{PQ}{PR} : \frac{P'Q'}{P'R'} = \frac{OQ}{OR} : \frac{OQ'}{OR'}.$$

Denote K the common symmedian point of the triangles ABC and $A'B'C'$, and K_1 that of the triangles $A_1B_1C_1$ and $A_1'B_1'C_1'$ (their existence is guaranteed by of Proposition 3(1)). Let AK_1 meets the side BC at X (Fig. 3b). By Lemma,

$$\frac{K_1A_1'}{K_1A_1} : \frac{XD'}{XD} = \frac{AA_1'}{AA_1} : \frac{AD'}{AD}.$$

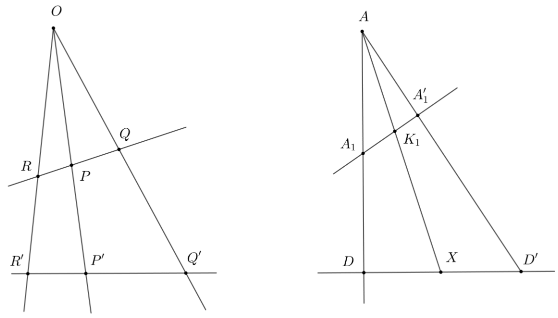

Figure 3

For $ABC \sim A_1B_1C_1$ and Proposition 1, we obtain:

$$\frac{K_1A_1'}{K_1A_1} = \frac{KA'}{KA} = \frac{s_a'}{AK} - 1 = \frac{a^2 + b^2 + c^2}{2m_a^2} - 1 = \frac{3a^2}{4m_a^2}.$$

From the equation $AH \cdot AA_1 = AG \cdot AA_1'$ (the power of A with respect to the orthocentroidal circle) it follows that

$$\frac{AA_1'}{AA_1} = \frac{AH}{AG} = \frac{3R\cos A}{m_a}.$$

Substituting and calculating, we obtain:

$$\frac{XD'}{XD} = \frac{3a^2}{4m_a^2} \cdot \frac{m_a}{3R\cos A} \cdot \frac{m_a}{h_a} = \frac{a^2}{b^2+c^2-a^2},$$

from where we get:

$$XD' = \frac{a^2}{b^2+c^2} \cdot DD' = \frac{a^2}{b^2+c^2} \cdot \frac{b^2-c^2}{2a} = \frac{a\left(b^2-c^2\right)}{2\left(b^2+c^2\right)}.$$

Then

$$\frac{BX}{XC} = \frac{BD'-XD'}{XD'+D'C} = \frac{1-\left(b^2-c^2\right)/\left(b^2+c^2\right)}{1+\left(b^2-c^2\right)/\left(b^2+c^2\right)} = \frac{c^2}{b^2},$$

i.e. AX is the symmedian through vertex A of the triangle ABC. Therefore, $K_1 \in AX$. With the same arguments it is shown that K_1 is also on the other simedianes of the triangle ABC. We conclude that K_1 coincides with K.

4) It follows directly from 1) above and from Proposition 3(2).

Remark. In [**3**; pp.215-218], as an application of the theory of figures directly similar, many properties of the point K are indicated in connection with the orthocentroidal circle. The assertion $ABC \sim A_1B_1C_1$ from 1) and property 4) are precisely Corollaire 3 and Corollaire 4, p.217.

Remark. We conclude this section by emphasizing the usefulness of angles $\varphi_A, \varphi_B, \varphi_C$ in the elementary approach to the proposed theme.

2. PROPERTIES OF INDUCED TRIANGLES

We intend to establish a connection between the Lemoine, Fermat and Brocard axes of the given triangle and those of the induced triangles $A_1B_1C_1$ and $A_1'B_1'C_1'$.

Proposition 6. *1) (AA_1A_1') is the Apollonius circle of the triangle $A_1B_1C_1$ with respect to the side B_1C_1.*
2) (AA_1A_1') is the Apollonius circle of the triangle $A_1'B_1'C_1'$ with respect to the side $B_1'C_1'$.
3) $U = AB \cap A_1C_1$ and *$V = AC \cap A_1B_1$ lie on (AA_1A_1') (Fig. 4).*
The circles $(BB_1B_1'), (CC_1C_1')$ have similar properties.
Proof. 1) We have to show that

$$\frac{AB_1}{AC_1} = \frac{A_1'B_1}{A_1'C_1} = \frac{A_1B_1}{A_1C_1}. \tag{2.1}$$

Applying Stewart's theorem to the triangle ABE and the point $B_1 \in BE$, we obtain:

$$\begin{aligned} AB_1^2 &= \frac{1}{BE}\left(AB^2 \cdot B_1E + AE^2 \cdot B_1B - BE \cdot B_1B \cdot B_1E\right) \\ &= \frac{1}{h_b}\left[c^2 \cdot \frac{h_b}{3} + \left(c^2 - h_b^2\right) \cdot \frac{2h_b}{3} - h_b \cdot \frac{2h_b}{3} \cdot \frac{h_b}{3}\right] \\ &= c^2 - \frac{8}{9}h_b^2 = c^2\left(1 - \frac{8}{9}\sin^2 A\right). \end{aligned}$$

Similarly, considering the triangle ACF and the point C_1, we get: $AC_1^2 = b^2\left(1 - \frac{8}{9}\sin^2 A\right)$.

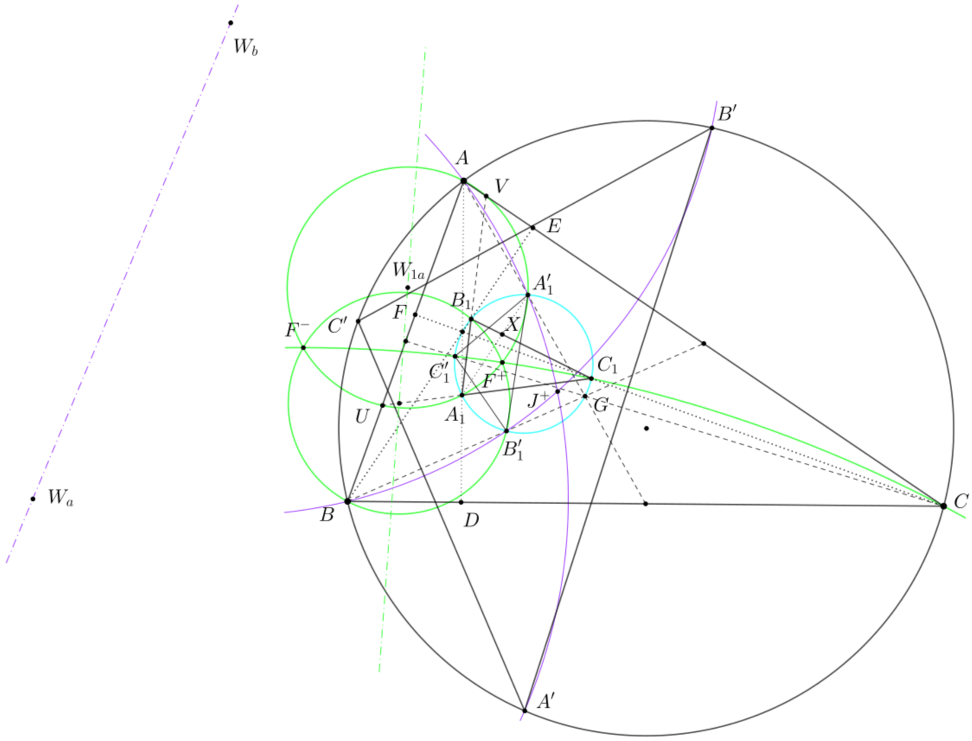

Figure 4

Then, it follow that $\dfrac{AB_1}{AC_1} = \dfrac{c}{b}$. Since the triangles ABC and $A_1B_1C_1$ are similar, $\dfrac{c}{b} = \dfrac{A_1B_1}{A_1C_1}$ and so $\dfrac{AB_1}{AC_1} = \dfrac{A_1B_1}{A_1C_1}$.
On the other hand, in the orthocentroidal circle we have:

$$\frac{A_1'B_1}{A_1'C_1} = \frac{\sin \widehat{A_1'A_1B_1}}{\sin \widehat{A_1'A_1C_1}} = \frac{\sin \widehat{XA_1B_1}}{\sin \widehat{XA_1C_1}},$$

where $X = B_1C_1 \cap A_1A_1'$. But, $\sin \widehat{XA_1B_1} = XB_1 \cdot \dfrac{\sin \widehat{A_1XB_1}}{A_1B_1}$ and $\sin \widehat{XA_1C_1} = XC_1 \cdot \dfrac{\sin \widehat{A_1XC_1}}{A_1C_1}$. Then,

$$\frac{A_1'B_1}{A_1'C_1} = \frac{XB_1}{XC_1} \cdot \frac{A_1C_1}{A_1B_1} = \frac{c^2}{b^2} \cdot \frac{b}{c} = \frac{c}{b} = \frac{A_1B_1}{A_1C_1}$$

(was used that A_1X is symmedian in the triangle $A_1B_1C_1$, according to Proposition 3). Hence (7) is established and therefore (AA_1A_1') is the Apollonius circle of the triangle $A_1B_1C_1$.
2) It follows from the fact that $A_1B_1C_1$ and $A_1'B_1'C_1'$ are cosymmedian triangles.

3) Let's show that the quadrilateral AUA_1A_1' and AUA_1V are inscribed in the circle (AA_1A_1') Indeed, $\widehat{UAA_1'} = \widehat{BAD'} = \widehat{BAD} + \widehat{DAD'} = \left(\frac{\pi}{2} - B\right) + \varphi_A$ and $\widehat{UA_1A_1'} = \pi - \widehat{A_1'A_1C_1} = \widehat{A_1A_1'C_1} + \widehat{A_1C_1A_1'} = \widehat{B_1} + \widehat{A_1GA_1'} = B + \left(\frac{\pi}{2} - \varphi_A\right)$; hence $\widehat{UAA_1'} + \widehat{UA_1A_1'} = \pi$, i.e. $U \in (AA_1A_1')$. On the other hand, we have: $\widehat{UAV} = A$ and $\widehat{UA_1V} = \pi - \widehat{B_1A_1C_1} = \pi - A$; hence $\widehat{UAV} + \widehat{UA_1V} = \pi$, i.e. $V \in (AA_1A_1')$.

The next result can be demonstrated with arguments similar to those of Proposition 6(1). We omit the details.
Proposition 7. *The circles $(AA_1'A')$, $(BB_1'B')$, $(CC_1'C')$ are the Apollonius circles of the triangle ABC with respect to the sides BC, CA, AB respectively (Fig. 4).*

Remark. Proposition 6 can also be demonstrated in another way. It is first shown that the quadrilaterals $A_1B_1A_1'C_1$, $B_1C_1B_1'A_1$, $C_1A_1C_1'B_1$ are harmonic [**5**, p.90], and then use is made of their known properties [**15**]. Same for Proposition 7.

The next result requires some preparations. Let A^+ (resp. A^-) be the point such that the triangle CBA^+ is equilateral with the same (resp. opposite) orientation as ABC; similarly for B^+, B^- and C^+, C^-. Then, $F^+ = AA^+ \cap BB^+ \cap CC^+$ and $F^+ = AA^- \cap BB^- \cap CC^-$. It is known that the segments AA^+, BB^+, CC^+ (resp. AA^-, BB^-, CC^-) have the same lengths l^+ (resp. l^-) and that

$$l^+ = \left[\frac{1}{2}\left(a^2 + b^2 + c^2 + 4\sqrt{3}\Delta\right)\right]^{\frac{1}{2}} \text{ and } l^- = \left[\frac{1}{2}\left(a^2 + b^2 + c^2 - 4\sqrt{3}\Delta\right)\right]^{\frac{1}{2}} \tag{2.2}$$

(for ex. [**6**, p.220], [**2**]). Denote α^+ (resp. α^-) the measure of the counterclockwise oriented angle BAA^+ (resp. BAA^-) (Fig. 5); β^+, β^- and γ^+, γ^- are similarly defined.

Lemma 8. *We have:*

$$\sin \alpha^+ = \frac{4\Delta + \sqrt{3}\left(c^2 + a^2 - b^2\right)}{4cl^+}, \qquad \cos \alpha^+ = \frac{b^2 + 3c^2 - a^2 + 4\sqrt{3}\Delta}{4cl^+}; \tag{2.3}$$

$$\sin \alpha^- = \frac{4\Delta - \sqrt{3}\left(c^2 + a^2 - b^2\right)}{4cl^-}, \qquad \cos \alpha^- = \frac{b^2 + 3c^2 - a^2 - 4\sqrt{3}\Delta}{4cl^-} \tag{2.4}$$

and similar formulas for β^+, β^- and γ^+, γ^-.
Proof. In both cases, see Fig. 5, it is enough to apply the sine and cosine formulas to the triangles ABA^+ and ABA^-. For example,

$$\sin \alpha^+ = \frac{BA^+ \cdot \sin\left(B + \frac{\pi}{3}\right)}{AA^+} = \frac{a\left(\sin B + \sqrt{3}\cos B\right)}{2l^+} = \frac{4\Delta + \sqrt{3}\left(c^2 + a^2 - b^2\right)}{4cl^+}$$

(I used the formulas $\sin B = \dfrac{2\Delta}{ca}$ and $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$).

Remark. If in the formulas (2.3) we put $-\sqrt{3}$ instead of $\sqrt{3}$, we get the formulas (2.4), and vice-versa. In the same way we can pass from l^+ to l^- in (2.2). Thus, the amount of calculations can be halved.

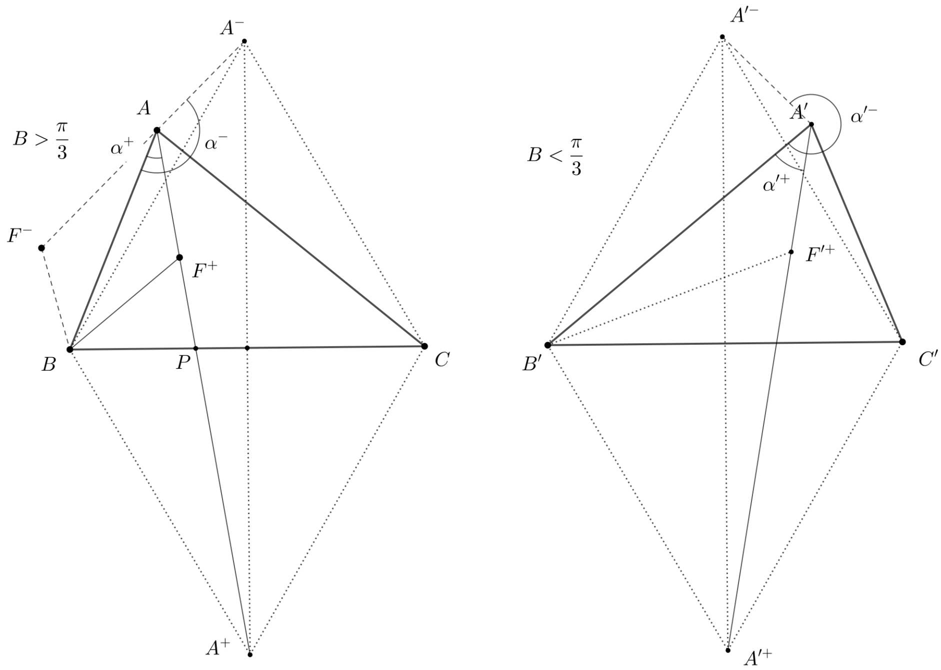

Figure 5

Lemma 9. *The distances of F^+ and F^- to the vertex A of the triangle ABC are given by*

$$F^+A = \frac{1}{2\sqrt{3}} \frac{4\Delta + \sqrt{3}\left(b^2 + c^2 - a^2\right)}{l^+},$$
$$F^-A = \frac{1}{2\sqrt{3}} \frac{4\Delta - \sqrt{3}\left(b^2 + c^2 - a^2\right)}{l^-}. \tag{2.5}$$

Proof. Consider the triangle F^+AB. Note that $\widehat{AF^+B} = \frac{2\pi}{3.}$ and $\widehat{ABF^+} = B - \beta^+$. By the sine formula,

$$F^+A = \frac{2c}{\sqrt{3}} \sin\left(B - \beta^+\right) = \frac{2c}{\sqrt{3}} \left(\frac{2\Delta}{ca} \cos\beta^+ - \frac{c^2 + a^2 - b^2}{2ca} \sin\beta^+ \right).$$

Using Lemma 8 for $\sin\beta^+$, $\cos\beta^+$ and the formula $16\Delta^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$ (Heron), we obtain by a simple calculation the first formula (2.5). For the second we can do the same in the triangle F^-AB.

Remark. With the aid of the formulas (2.3) and (2.4) we can express through the sides of the given triangle various segments associated with the points F^+ and F^- as is usually done in the case of angle-bisectors, symmedians, etc. For example, if $P = BC \cap AA^+$, we have:

$$AP = \frac{4\Delta l^{+}}{4\Delta + \sqrt{3}a^2},$$
$$BP = \frac{4\Delta + \sqrt{3}\left(c^2 + a^2 - b^2\right)}{2\left(4\Delta + \sqrt{3}a^2\right)}a,\ CP = \frac{4\Delta + \sqrt{3}\left(a^2 + b^2 - c^2\right)}{2\left(4\Delta + \sqrt{3}a^2\right)}a, \text{ etc.}$$

Previous preparations make possible an elementary demonstration of the next result.
Theorem 10. *The Fermat points of the triangle ABC are at the same time the isodynamic points of the orthocentroidal triangle $A_1B_1C_1$, i.e. $J_1^{+} = F^{+}$ and $J_1^{-} = F^{-}$.*
Proof. By definition, the isodynamic points of the triangle $A_1B_1C_1$ are the two points common to the Apollonius circles of this triangle. By Proposition 6, they are also the common points of the circles $(AA_1A_1'), (BB_1B_1'), (CC_1C_1')$ (Fig. 4). Therefore, we have to show that the points F^{+} and F^{-} lie on each of the circles $(AA_1A_1'), (BB_1B_1'), (CC_1C_1')$. Obviously, it is enough to show that $F^{+}, F^{-} \in (AA_1A_1')$. Moreover, we restrict ourselves to $F^{+} \in (AA_1A_1')$, for the remaining statement is established the same. We will do this in an elementary way, but omitting routine calculations.
According to Ptolemy's theorem, the points F^{+}, A, A_1, A_1' are concyclical if the equality

$$F^{+}A \cdot A_1A_1' = AA_1' \cdot F^{+}A_1 + AA_1 \cdot F^{+}A_1' \tag{2.6}$$

is true (Fig. 6).

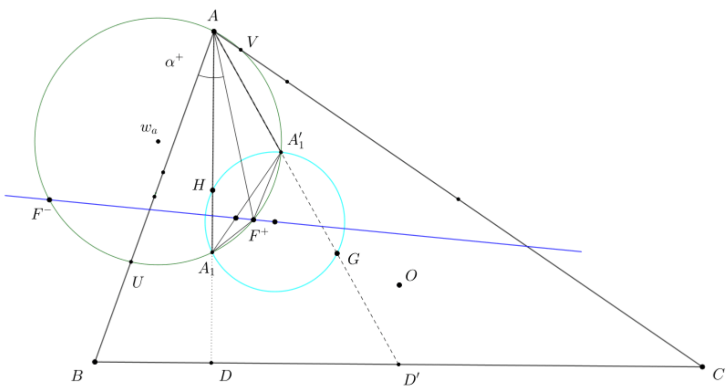

Figure 6

We have that $F^{+}A$ is given by (2.5). Also, we have: $AA_1 = \frac{2}{3}h_a$ (whereas $GA_1 \parallel CB$), hence

$$AA_1 = \frac{4\Delta}{3a}. \tag{2.7}$$

Next, $AA_1' = AH \cdot \cos\varphi_A = 2R\cos A \cdot \frac{2\Delta}{am_a}$, hence

$$AA_1' = \frac{b^2 + c^2 - a^2}{2m_a}. \tag{2.8}$$

Denote ρ_1 the radius of the orthocentroidal circle, so $\rho_1 = HG$. Then, $A_1A_1' = 2\rho_1 \sin \widehat{A_1GA_1'} = 2\rho_1 \sin\left(\frac{\pi}{2} - \varphi_A\right) = 2\rho_1 \cos\varphi_A$, and so

$$A_1A_1' = HG \cdot \frac{2\Delta}{am_a}. \tag{2.9}$$

The expressions of F^+A_1 and F^+A_1' are still to be calculated. For this, we apply the cosine formula in the triangles AA_1F^+ and AF^+A_1', respectively. We give some calculation details only on the F^+A_1; the calculation for F^+A_1' follows the same steps and we omit it. So, we start with

$$\left(F^+A_1\right)^2 = \left(F^+A\right)^2 + \left(AA_1\right)^2 - 2F^+A \cdot AA_1 \cos\left[\alpha^+ - \left(\frac{\pi}{2} - B\right)\right]. \tag{2.10}$$

But, by (2.3) and usual formulas, we obtain:

$$\begin{aligned}\cos\left[\alpha^+ - \left(\frac{\pi}{2} - B\right)\right] &= \sin\left(B + \alpha^+\right) = \sin B \cos\alpha^+ + \cos B \sin\alpha^+ \\ &= \frac{2\Delta}{ca} \cdot \frac{b^2 + 3c^2 - a^2 + 4\sqrt{3}\Delta}{4cl^+} \\ &+ \frac{c^2 + a^2 - b^2}{2ca} \cdot \frac{4\Delta + \sqrt{3}\left(c^2 + a^2 - b^2\right)}{4cl^+} \\ &= \frac{1}{8ac^2l^+}\left[16\sqrt{3}\Delta^2 + 16c^2\Delta + \sqrt{3}\left(c^2 + a^2 - b^2\right)^2\right] \\ &= \frac{4\Delta + \sqrt{3}a^2}{2al^+},\end{aligned}$$

so

$$\sin\left(B + \alpha^+\right) = \frac{4\Delta + \sqrt{3}a^2}{2al^+}. \tag{2.11}$$

If we replace the terms in the second member of (2.10) by their expressions given by (2.5), (2.7), (2.11), and effect the routine calculations, we find that

$$\left(F^+A_1\right)^2 = \frac{1}{9a^2\left(l^+\right)^2} \cdot q \qquad \text{or} \qquad F^+A_1 = \frac{\sqrt{q}}{3al^+}, \tag{2.12}$$

where

$$q = a^6 + b^6 + c^6 + 3a^2b^2c^2 - a^4b^2 - a^2b^4 - b^4c^2 - b^2c^4 - c^4a^2 - c^2a^4 \tag{2.13}$$

($q > 0$, Schur's inequality [**16**]).
For F^+A_1' we get a similar formula:

$$\left(F^+A_1'\right)^2 = \frac{1}{3\left(2b^2 + 2c^2 - a^2\right)\left(l^+\right)^2} \cdot q = \frac{1}{12m_a^2\left(l^+\right)^2} \cdot q$$

or

$$F^+A_1' = \frac{\sqrt{q}}{2\sqrt{3}m_al^+}.$$

Now, we are ready to check the equality (2.6). Indeed,

$$\frac{1}{2\sqrt{3}}\frac{4\Delta+\sqrt{3}\left(b^2+c^2-a^2\right)}{l^+}\cdot HG\cdot\frac{2\Delta}{am_a}$$
$$=\frac{b^2+c^2-a^2}{2m_a}\cdot\frac{\sqrt{q}}{3al^+}+\frac{4\Delta}{3a}\cdot\frac{\sqrt{q}}{2\sqrt{3}m_a l^+}$$
$$\Leftrightarrow\left[4\Delta+\sqrt{3}\left(b^2+c^2-a^2\right)\right]\cdot HG\cdot 2\Delta$$
$$=\frac{\sqrt{q}}{3}\left[\sqrt{3}\left(b^2+c^2-a^2\right)+4\Delta\right]\Leftrightarrow 6\Delta\cdot HG=\sqrt{q},$$

and the last equality is true, for $36\Delta^2\cdot HG^2=36\Delta^2\cdot\frac{4}{9}\left[9R^2-\left(a^2+b^2+c^2\right)\right]=9a^2b^2c^2-16\Delta^2\left(a^2+b^2+c^2\right)=9a^2b^2c^2-(2a^2b^2+2b^2c^2+2c^2a^2-a^6-b^6-c^6)\left(a^2+b^2+c^2\right)=q$. This concludes the proof.

Corollary 11. *F^+, F^- lie on the Apollonius circles of the triangles $A_1B_1C_1$ and $A_1'B_1'C_1'$, and they are mutually inverse points in orthocentroidal circle.*

Corollary 12. *1) The Fermat points of the triangle $A_1B_1C_1$ are the isogonal conjugates of F^+, F^- with respect to this triangle.*
2) The isodynamic points of the triangle $A_1'B_1'C_1'$ are F^+, F^-, and its Fermat points are the isogonal conjugates of F^+, F^- with respect to this triangle.
Proof. It is taken into account that $A_1B_1C_1$ and $A_1'B_1'C_1'$ are cosymmedian triangles and that the Fermat and isodynamic points of a triangle are isogonal conjugate.

Theorem 13. *The orthocentroidal triangle has the properties (Fig. 7):*
1) the Brocard axis of the triangle $A_1B_1C_1$ is the Fermat axis of the triangle ABC;
2) the Fermat axis of the triangle $A_1B_1C_1$ is the Brocard axis of the triangle ABC.
Proof. 1) By Theorem 10.
2) The triangles ABC and $A_1B_1C_1$ are *inversely* similar and have the same symmedian point. Let M_1 be centre of the orthocentroidal circle of the triangle $A_1B_1C_1$. The correspondence of points : $K\longrightarrow K, O\longrightarrow M, M\longrightarrow M_1$ holds. Hence, $\widehat{OKM}=\widehat{MKM_1}$. Because these angles have a common side, namely KM, and the opposite orientation, it results that the sides KO and KM_1 coincide, i.e. the assertion is true.

Remark. We consider the coaxal system consisting of all the circles that pass through the points F^+ and F^-(for more details on the coaxal systems, see **[1]**, **[6]**, and **[12]**). Obviously, its radical axis is the Fermat axis KM of the triangle ABC, and the line of centers is the perpendicular bisector of the line segments F^+F^-. It is known that the Fermat circle, Lester circle (F^+F^-ON), (F^+F^-G) and (F^+F^-H) are members of this system.
By Corollary 11, the Apollonius circles of the triangles $A_1B_1C_1$ and $A_1'B_1'C_1'$ also belong to this coaxal system. So their centers are on the radical axis of the system. On the other hand, these centers are on the (common) Lemoine axis of the triangles $A_1B_1C_1$ and $A_1'B_1'C_1'$. Therefore, the line of centers of the coaxal system is the Lemoine axis of these triangles (Fig. 7 and 8).
The orthocentroidal circle belongs to the conjugate coaxal system; indeed, because it is orthogonal to the Apollonius circles of the triangles $A_1B_1C_1$ and $A_1'B_1'C_1'$ it follows that

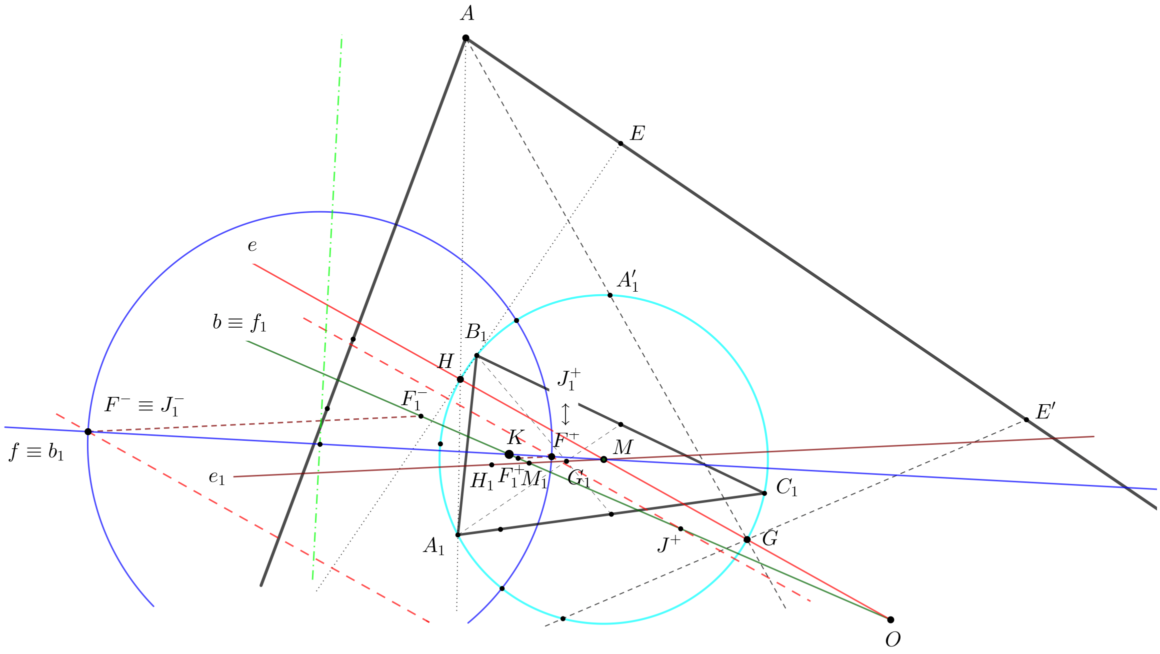

Figure 7

it is orthogonal to all circles in the system (in particular, to Fermat and Lester circles). The radical axis of this conjugate coaxal system is the Lemoine axis of the mentioned triangles. Clearly, F^+ and F^- are the limiting points of the conjugate system. Also, the (common) Brocard circle of the triangles $A_1B_1C_1$ and $A'_1B'_1C'_1$ (i.e. the circle having the line segment KM as diameter), belongs to him; in particular, it is orthogonal to Fermat and Lester circles. We can show as a simple exercise that K is the midpoint of the common chord of orthocentroidal circle and Fermat circle of the triangle ABC.

Based on Theorem 10 we can establish a simple and interesting connection between the points H, G (on the Euler line), F^+, F^- (on the Fermat axis), and J^+, J^- (on the Brocard axis):

Theorem 14. *The following formula is true:*

$$2R \cdot F^+F^- = GH \cdot J^+J^-. \tag{2.14}$$

Proof. Indeed, by similarity of the triangles ABC and $A_1B_1C_1$, we obtain:

$$\frac{J^+J^-}{J_1^+J_1^-} = \frac{2R}{GH} \quad \text{or} \quad \frac{J^+J^-}{F^+F^-} = \frac{2R}{GH},$$

from where it follows (2.14).

If the Euler line is represented by the points O and H, then (2.14) it is written in the form

$$3R \cdot F^+F^- = OH \cdot J^+J^-, \tag{2.15}$$

hence

$$3R \cdot F^+F^- = \sqrt{9R^2 - (a^2 + b^2 + c^2)} \cdot J^+J^-. \tag{2.16}$$

Denote e, e_1 the Euler lines, f, f_1 the Fermat axes, and b, b_1 the Brocard axes of the triangles ABC and, respectively, $A_1B_1C_1$. By Theorem 13, f coincides with b_1 and b with f_1.
Proposition 15. *We have (Fig. 7):*
1) $\widehat{e,e_1} = \widehat{e,b} + \widehat{b,e_1} = \widehat{e,f_1} + \widehat{f_1,e_1}$*;*
2) $\widehat{f,f_1} = \widehat{b_1,b} = \widehat{e,f} - \widehat{e,b} = \widehat{b,e_1} - \widehat{e_1,f}$*.*
Proof. 1) $\widehat{e,e_1}$ is an exterior angle of the triangle OMM_1.
2) Apply the exterior angle theorem to the triangles OKM and KMM_1.

Theorem 16. *The Fermat and isodynamic points of the triangles* ABC and $A_1B_1C_1$ *have the properties (Fig. 7):*
1) $F^+J^+ \parallel F^-J^- \parallel e$ ([**7**,Table 5.3, p.139]), $F_1^+J_1^+ \parallel F_1^-J_1^- \parallel e_1$*;*
2) $F^+F_1^+ \parallel F^-F_1^- \parallel e_1$.
Proof. 1) We demonstrate only the first statement. By Theorem 10, $J_1^+ \equiv F^+$. From $ABC \sim A_1B_1C_1$ and the correspondence $K \longrightarrow K, O \longrightarrow M, J^+ \longrightarrow J_1^+$ it results that $\dfrac{KO}{KJ^+} = \dfrac{KM}{KF^+}$. Then, $J^+F^+ \parallel OM$, i.e. $F^+J^+ \parallel e$. It is shown that $F^-J^- \parallel e$ reasoning the same.
2) It follows from 1) and Theorem 10.

Let's note that we will obtain similar results if we leave with the triangles $A'B'C'$ and $A_1'B_1'C_1'$ instead of ABC and $A_1B_1C_1$. Denote H', G', N' the orthocenter, centroid, nine-point center of the triangle $A'B'C'$, and M, M' the midpoints of the segments HG and $H'G'$, respectively. The cosymmedians triangles ABC and $A'B'C'$ have the same circumcenter O and symmedian point K, but their orthocentroidal circles (i.e. the circumcircles of the triangles $A_1B_1C_1$ and $A_1'B_1'C_1'$) having the diameters HG and $H'G'$ will not coincide. Taking into account the positions occupied by points H, G, N, M and H', G', N', M' on the Euler lines OH and OH' of the triangles ABC and $A'B'C'$, it is immediately obtained that $HH' \parallel GG' \parallel NN' \parallel MM'$. Fermat axes of the two triangles, f and f', do not coincide, but $F^+F'^+ \parallel F^-F'^- \parallel HH'$. (Fig. 8).

3. SEQUENCES OF TRIANGLES WITH THE SAME AXES

In this section, the given triangle ABC is denoted $A_0B_0C_0$, Then, we define by induction two sequences of triangle $(A_nB_nC_n)_{n\geq 0}$ and $(A_n'B_n'C_n')_{n\geq 0}$. Recall that $A_1B_1C_1$ was defined as the triangle whose vertices are the orthogonal projections of the centroid $G \equiv G_0$ of the triangle $A_0B_0C_0$ on its corresponding altitudes. By doing so, we successively obtain the triangles $A_2B_2C_2$, $A_3B_3C_3$, etc. For $n \geq 0$, let $\mathcal{C}_n$ be the circumcircle of the triangle $A_nB_nC_n$ and $O_n, H_n, G_n, M_n, F_n^+, J_n^+, ...$ be the standard notations relative to the triangle $A_nB_nC_n$. Obviously, $\mathcal{C}_{n+1}$ is the orthocentroidal circle of the triangle $A_nB_nC_n$.
On the other hand, let $A_0'B_0'C_0'$ be the triangle $A'B'C'$ and, for $n \geq 1$ fixed, $A_n'B_n'C_n'$ be the cosymmedian triangle in pair with $A_nB_nC_n$ in the circle $\mathcal{C}_n$ (or, equivalently, A_n' is the projection of the orthocenter H_{n-1} of the triangle $A_{n-1}B_{n-1}C_{n-1}$ on his median through A_{n-1}, and B_n', C_n' are difined cyclically).

Using the results from the previous sections, we immediately obtain a number of properties of the sequence $(A_nB_nC_n)_{n\geq 0}$.

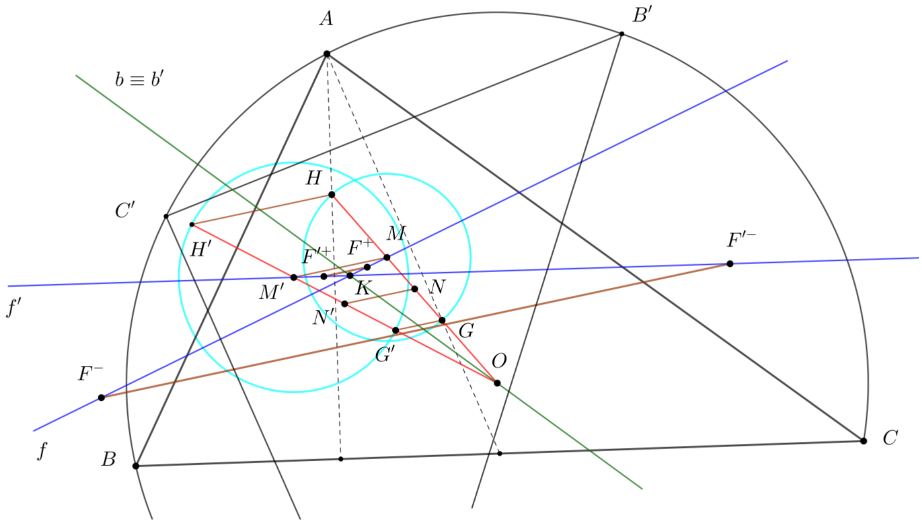

Figure 8

I. Any two triangles in the sequence $(A_nB_nC_n)_{n\geq 0}$ *are similar. The ratio of similitude of the triangles* $A_nB_nC_n$ and $A_{n+1}B_{n+1}C_{n+1}$, $n \geq 0$, is equal to $q = \dfrac{1}{3R}\sqrt{9R^2 - (a^2+b^2+c^2)}$. *The subsequences* $(A_{2n}B_{2n}C_{2n})_{n\geq 0}$ *and* $(A_{2n+1}B_{2n+1}C_{2n+1})_{n\geq 0}$ *are formed from directly similar triangles. Any triangle in subsequence* $(A_{2n}B_{2n}C_{2n})_{n\geq 0}$ *is inversely similar to any triangle in subsequence* $(A_{2n+1}B_{2n+1}C_{2n+1})_{n\geq 0}$.

According to Proposition 5(1), concerning the ratio of similitude we have: $\dfrac{a_{n+1}}{a_n} = \dfrac{H_nG_n}{H_{n-1}G_{n-1}}$ (H_nG_n being the diameter of the circumcircle of $A_{n+1}B_{n+1}C_{n+1}$) and $\dfrac{H_nG_n}{H_{n-1}G_{n-1}} = \dfrac{a_n}{a_{n-1}}$ (since $A_nB_nC_n \sim A_{n-1}B_{n-1}C_{n-1}$). Therefore, $\dfrac{a_{n+1}}{a_n} = \dfrac{a_n}{a_{n-1}} = ... = \dfrac{a_1}{a}$. But, $\dfrac{a_1}{a} = \dfrac{HG}{2R}$ ($A_1B_1C_1 \sim ABC$) and finally we get: $\dfrac{a_{n+1}}{a_n} = \dfrac{HG}{2R} = \dfrac{HO}{3R} = \dfrac{1}{3R}\sqrt{9R^2 - (a^2+b^2+c^2)}$.

II. K is the symmedian point of all triangles $A_nB_nC_n$ (by Proposition 5(3)).

III. The triangles in the subsequence $(A_{2n}B_{2n}C_{2n})_{n\geq 0}$ *have common symmedians, namely, the lines AK, BK and CK, while those in the subsequence* $(A_{2n+1}B_{2n+1}C_{2n+1})_{n\geq 0}$ *have as symmedians the lines* A_1K, B_1K *and* C_1K. In other words, for $n \geq 1$ we have: $A_{2n} \in AK$, $B_{2n} \in BK$, $C_{2n} \in CK$ and $A_{2n+1} \in A_1K$, $B_{2n+1} \in B_1K$, $C_{2n+1} \in C_1K$ (Fig. 9).

Let's just show that $A_2 \in AK$. We know that the triangles ABC and $A_2B_2C_2$ are directly similar, and have the same symmedian point, K, and Brocard axis, OK. Hence, it follows that the angles between the axis OK and the symmedians of the triangles ABC and $A_2B_2C_2$ at the vertices A and A_2 are equal. Then, $A_2 \in AK$.

IV. The corresponding sides of the triangles $A_{2n}B_{2n}C_{2n}$, $n \geq 0$ *(as well as those of the triangles* $A_{2n+1}B_{2n+1}C_{2n+1}$, $n \geq 0$*) are parallel. The homothety* $\mathcal{H}$ *with center K and ratio* $q^2 = \dfrac{1}{9R^2}\left[9R^2 - \left(a^2+b^2+c^2\right)\right]$ *transforms triangle* $A_nB_nC_n$ *into triangle* $A_{n+2}B_{n+2}C_{n+2}$, $n \geq 0$.

V. The triangles in the subsequence $(A_{2n}B_{2n}C_{2n})_{n\geq 0}$ *have as the Fermat axis the line* KM *and as the Brocard axis the line* OK*, while those in the subsequence* $(A_{2n+1}B_{2n+1}C_{2n+1})_{n\geq 0}$ *have as the Fermat axis the line* OK *and as the Brocard axis the line* KM (by Theorem 13).
Therefore, for the centers of the orthocentroidal circles $\mathcal{C}_n$ we have: $O_2, ..., O_{2n}, ... \in OK$ and $O_1, O_3, ...O_{2n+1}, ... \in KM$.

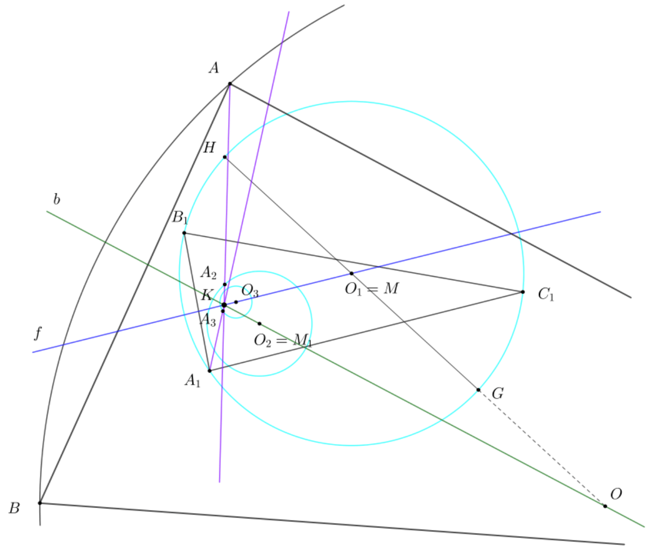

Figure 9

Since $ABC \sim A_{2n}B_{2n}C_{2n}$, it follows that the angle beetwen BC and the symmedian at A is equal to the angle beetwen $B_{2n}C_{2n}$ and the symmedian at A_{2n}. Since these symmedians coincide, it follows that $B_{2n}C_{2n} \parallel BC$. The collinearity of the points B_{2n} as well as those of the points C_{2n} and the property I ensure that the homothety $\mathcal{H}$ verifies the desired statement if n is even. Analogously, if n is odd.
VI. The Lemoine axes of the triangles $A_{2n}B_{2n}C_{2n}$, $n \geq 0$, *are parallel to each other and perpendicular to* OK*, while those of the triangles* $A_{2n+1}B_{2n+1}C_{2n+1}$, $n \geq 0$, *are parallel to each other and perpendicular to* KM*.*
From V and the fact that in any triangle the Lemoine axis is perpendicular to the Brocard axis.
VII. The Euler lines of the triangles $A_{2n}B_{2n}C_{2n}$, $n \geq 0$, *are parallel to the Euler line of* ABC *and those of the triangles* $A_{2n+1}B_{2n+1}C_{2n+1}$, $n \geq 0$, *are parallel to the Euler line of* $A_1B_1C_1$*, i.e.* $e_{2n} \parallel e$ and $e_{2n+1} \parallel e_1$.
From the similarity $ABC \sim A_{2n}B_{2n}C_{2n}$ and the fact that KM is the common Fermat axis of $A_{2n}B_{2n}C_{2n}, n \geq 0$, we infer that the lines e_{2n} are equally inclined on KM, hence they are parallel. Hence, $e_{2n} \parallel e$. Similarly, for the remaining part.
VIII. For any $n \geq 0$ *we have: 1)* $F^+_{2n}, F^-_{2n} \in KM$ *and* $F^+_{2n+1}, F^-_{2n+1} \in OK$*, 2)* $J^+_{2n}, J^-_{2n} \in OK$ *and* $J^+_{2n+1}, J^-_{2n+1} \in KM$*, 3)* $J^+_{2n+1} = F^+_{2n}$ *and* $J^-_{2n+1} = F^-_{2n}$*.*
Consequence of Theorem 10 and the property V.
IX. $F^+_{2n}J^+_{2n} \parallel F^-_{2n}J^-_{2n} \parallel e$*, and* $F^+_{2n+1}J^+_{2n+1} \parallel F^-_{2n+1}J^-_{2n+1} \parallel e_1$ *(or* $F^+_{2n-1}F^+_{2n} \parallel F^-_{2n-1}F^-_{2n} \parallel e$ *and* $F^+_{2n}F^+_{2n+1} \parallel F^-_{2n}F^-_{2n+1} \parallel e_1$*).*
Consequence of the property VII, Theorem 16 and Theorem 10.

The sequence $(A'_n B'_n C'_n)_{n\geq 0}$ retains unchanged some of the preceding properties, while others require adaptations or modifications to remain valid. The properties I-IV and VI are also valid for the sequence $(A'_n B'_n C'_n)_{n\geq 0}$ we just have to change $A_n \to A'_n$, $B_n \to B'_n$, and $C_n \to C'_n$. K is the symmedian point of all triangles $A_n B_n C_n$ and $A'_n B'_n C'_n$, $n \geq 0$.
According to Proposition 4, the triangles $A_n B_n C_n$ and $A'_n B'_n C'_n$, for any $n \geq 0$, have the same axis Brocard. Because neither centroid nor orthocenter of these triangles do not coincide, the midpoints of the line segments $G_n H_n$ and $G'_n H'_n$, say M_n and M'_n, are different. Moreover, $\mathcal{C}_n$ is not the orthocentroidal circle of the triangle $A'_{n-1} B'_{n-1} C'_{n-1}$. But, it is easy to see that $K, M'_0, M'_2, ..., M'_{2n}, ...$ and $K, M'_1, M'_3, ...M'_{2n+1}, ...$ are sequence of collinear points. Then, corresponding to the properties V, and VII-IX we must take the following statements:
V'. The triangles in the subsequence $(A'_{2n} B'_{2n} C'_{2n})_{n\geq 0}$ have as the Fermat axis the line KM'_0 and as the Brocard axis the line OK, while those in the subsequence $\left(A'_{2n+1} B'_{2n+1} C'_{2n+1}\right)_{n\geq 0}$ have as the Fermat axis the line KM'_1 and as the Brocard axis the line KM (M'_0 and *M'_1 are the midpoints of $G'H'$ and $G'_1 H'_1$, respectively).*
VII'. The Euler lines of the triangles $A'_{2n} B'_{2n} C'_{2n}$, $n \geq 0$, are parallel to the Euler line of $A'B'C'$ (i.e. $e'_{2n} \parallel e' \equiv G'H'$) and those of the triangles $A'_{2n+1} B'_{2n+1} C'_{2n+1}$, $n \geq 0$, are parallel to the Euler line of $A'_1 B'_1 C'_1$, (i.e. $e'_{2n+1} \parallel e'_1 \equiv G'_1 H'_1$).
$VIII'$. For any $n \geq 0$ we have: 1) $F'^{+}_{2n}, F'^{-}_{2n} \in KM'$ and $F'^{+}_{2n+1}, F'^{-}_{2n+1} \in KM'_1$, 2) $J'^{+}_{2n}, J'^{-}_{2n} \in OK$ and $J^{+}_{2n+1}, J^{-}_{2n+1} \in KM$.
IX'. $F'^{+}_{2n} J'^{+}_{2n} \parallel F'^{-}_{2n} J'^{-}_{2n} \parallel e'$, and $F'^{+}_{2n+1} J'^{+}_{2n+1} \parallel F'^{-}_{2n+1} J'^{-}_{2n+1} \parallel e'_1$.

4. CENTERS AND CENTRAL LINES

Below, we will use the following statements relative to the centers and central lines of the given triangle ABC ([**7**]): 1) the line joining two centers is a central line, 2) the point of intersection of two distinct central lines is a center, 3) if P and Q are centers, then the reflection of P in Q is also a center, 4) the line through a center and parallel to a central line is a central line.

Proposition 17. *For any $n \geq 1$, O_n, G_n, H_n are centers and e_n are central lines of the triangle ABC.*
Proof. The vertices of the triangle $A_1 B_1 C_1$ have the following trilinear coordinates ([**8**, #X(5476)], Peter Moses, 2014):
$A_1 = a : \dfrac{1}{b}\left(a^2 + b^2 - c^2\right) : \dfrac{1}{c}\left(a^2 - b^2 + c^2\right),$
$B_1 = \dfrac{1}{a}\left(b^2 - c^2 + a^2\right) : b : \dfrac{1}{c}\left(b^2 + c^2 - a^2\right),$
$C_1 = \dfrac{1}{a}\left(c^2 + a^2 - b^2\right) : \dfrac{1}{b}\left(c^2 - a^2 + b^2\right) : c$. Then, for the centroid G_1 of this triangle we obtain:

$$G_1 = a\left[a^2 b^2 + b^2 c^2 + c^2 a^2 - \left(b^2 - c^2\right)^2\right] ::$$

Hence G_1 is a center of the triangle ABC. Now, it is known that $O_1 \equiv M \equiv X_{381}$ ([**7**],[**8**]) is a center. Then, $M_1 \equiv O_2$ as reflection of O_1 in G_1 and H_1 as reflection of G_1 in M_1 are centers of the triangle ABC. On the other hand, the Euler line e_1 joining the centers $O_1 \equiv M$ and G_1 is a central line. Thus, the property is verified for $n = 1$.

The Euler line e_2 of the triangle $A_2B_2C_2$ passes through $O_2 \equiv M_1$ and is parallel to e; so, e_2 is a central line of the triangle ABC. On the other hand, $H_2 \in KH$, $G_2 \in KG$ and $M_2 \in KM$ according to property IV. Hence $H_2 \in KH \cap e_2$ and $G_2 \in KG \cap e_2$, from which it follows that the points H_2, G_2, and $M_2 \equiv O_3$ are centers of the triangle ABC. The remaining claims to be proven are deduced by induction.

Proposition 18. *For any $n \geq 0$, F_n^+, F_n^- are centers and $F_n^+F_{n+1}^+, F_n^-F_{n+1}^-$ are central lines of the triangle ABC.*

Proof. The lines $F^+F_1^+$ and $F^-F_1^-$ are parallel to e_1 (Theorem 16 or property IX) and pass through the centers F^+ and F^-, respectively. It follows, therefore, that $F^+F_1^+$ and $F^-F_1^-$ are central lines of the triangle ABC.

Now, taking into account the formulae $F_1^+ = F^+F_1^+ \cap OK$ and $F_1^- = F^-F_1^- \cap OK$ it follows that F_1^+ and F_1^- are centers of the triangle ABC. The lines $F_1^+F_2^+$ and $F_1^-F_2^-$ are parallel to e (property IX) and pass through the centers F_1^+ and F_1^-, respectively. Hence, $F_1^+F_2^+$ and $F_1^-F_2^-$ are central lines of the triangle ABC. Obviously, we can proceed by induction to conclude the proof.

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