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OSSERMAN CONDITION ON SASAKIAN MANIFOLDS

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ABSTRACT. In this paper, we introduce a new notion of Osserman Condition Type in the Sasakian manifolds setting, saidly the contact structure Osserman type. As results, we obtain the Theorem 4.1. We also construct examples of $3-$dimensional and $5-$dimensional Sasakian contact structure manifolds.

1. INTRODUCTION AND MOTIVATIONS

The study of the behaviour of the Jacobi operators is an important topic in Riemannian geometry. More precisely, let (M, g) be a Riemannian manifold with curvature tensor R and consider a point p in M. For any unit vector $X \in T_pM$, the symmetric endomorphism $R_X = R_p(\cdot, X)X : X^{\perp} \to X^{\perp}$ is called the Jacobi operator with respect to X. If the eigenvalues of R_X are independent of the choices of X and p, one says that (M, g) is an Osserman manifold. The Osserman conjecture [19] states that an Osserman manifold is either flat or locally a rank-one symmetric space, and some progress towards this conjecture was made in [6, 7, 8, 16, 17, 18]. Osserman manifolds were also studied in the Lorentzian context [12], where a complete solution is available, also in Pseudo-Riemannian setting [13] and in affine geometry [9, 10, 11].

For a Lorentzian manifold, in [12] a new Osserman-type condition is introduced and studied: the null Osserman condition. Lorentzian almost contact manifolds are studied in this context. It is shown that any Lorentzian Sasakian manifolds with constant ϕ-sectional curvature is null Osserman with respect to the Reeb vector field [12]. Lorentzian S-structures constitute a generalization of the Lorentzian Sasakian ones, and so it becomes natural to study the null Osserman condition on Lorentzian S-manifolds. In [2], the author proved that the null Osserman condition does not hold for Lorentzian S-manifolds with constant ϕ-sectional curvature. Therefore, a specialized version of the null Osserman condition, the ϕ-null Osserman condition, has been introduced in [2], where some of the first properties have been also considered. In [4], the authors obtain some results concerning the relationships among the classical Osserman condition, the null Osserman one and the ϕ-null Osserman condition.

In this paper, we study the Osserman condition on a Sasakian manifold. The paper is organized as follows: In section 2, we recall some basic definitions and necessary geometric concepts that are used throughout this paper. In section3, we review some

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recent results on null Osserman condition and the ϕ-null Osserman condition. In section 4, we introduce the notion of Osserman type Sasakian manifolds and we prove that any Sasakian manifolds with constant ϕ-sectional couvature of dimension ≥ 5 is Osserman Type.

2. Preliminaries

Let M be an odd-dimensional differentiable manifold, ϕ is a tensor field of type $(1,1)$, ξ the Reeb vector field of type $(1,0)$ and η be a 1-form of type $(0,1)$. If the system (ϕ, ξ, η) satisfies the conditions such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \text{and} \quad \eta \circ \phi = 0; \tag{2.1}$$

then (ϕ, ξ, η) is said to be an almost contact structure on M, and such a manifold is called an almost contact manifold. Let g be a compatible Riemannian metric with the almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any $X, Y \in \Gamma(TM)$. If g is such a metric, then the quadruplet (ϕ, ξ, η, g) is called an almost contact metric structure on M, and (M, ϕ, ξ, η, g) is an almost contact metric manifold. From (2.2) we have immediately

$$g(X, \xi) = \eta(X), \quad g(X, \phi Y) = -g(\phi X, Y) \tag{2.3}$$

for any $X, Y \in \Gamma(TM)$. In this case, the Reeb vector field ξ is orthonormal with respect to g $[g(\xi, \xi) = \eta(\xi) = 1]$. An almost contact metric structure becomes a contact metric if

$$g(X, \phi Y) = d\eta(X, Y) \tag{2.4}$$

where $X, Y \in \Gamma(TM)$. For an almost contact metric manifold, the 2-form Φ defined by

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.5}$$

is called second fondamental form.
Next, let ∇ denote the Levi Civita connection on (M, g). Then (M, g, ϕ, ξ, η) is said to be a Sasakian manifold if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.6}$$

for $X, Y \in \mathfrak{X}(M)$, which be the Lie algebra of smooth vector fields on M. This condition implies

$$\nabla_X \xi = -\phi X \text{ and } \nabla_\xi \varphi X = 0 \tag{2.7}$$

from which it follows that ξ is a Killing vector field. Further, the Riemannian curvature tensor

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

of a Sasakian manifold satisfies

$$R(X, \xi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.8}$$

and

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.9}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Let M be a $(2n+1)$-dimensional Sasakian manifold with Sasakian structure (ϕ, ξ, η, g). From (2.6) we easily see

$$\begin{aligned}\mathcal{R}(X,Y)\phi Z &= \phi\mathcal{R}(X,Y)Z + g(X,Z)\phi Y - g(Y,Z)\phi X \\ &\quad + g(\phi X, Z)Y - g(\phi Y, Z)X. \qquad (2.10)\end{aligned}$$

From (2.10) we also have the following equations:

$$\begin{aligned}\mathcal{R}(X,Y)Z &= -\phi\mathcal{R}(X,Y)\phi Z + g(Y,Z)X - g(X,Z)Y \\ &\quad - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y \qquad (2.11)\\ g(R(\phi X,\phi Y)\phi Z, \phi W) &= g(R(X,Y)Z,W) - \eta(Y)\eta(Z)g(X,W) \\ &\quad - \eta(X)\eta(W)g(Y,Z) + \eta(Y)\eta(W)g(X,Z) \\ &\quad + \eta(X)\eta(Z)g(Y,W). \qquad (2.12)\end{aligned}$$

A plane section in T_pM is called a ϕ-section if there exists a unit vector $X \in T_pM$ orthogonal to ξ such that $\{X, \phi X\}$ is an orthonormal basis of the plane section. Then, the sectional curvature $K(X, \phi X) = g(R(X, \phi X)\phi X, X)$ is called a ϕ-sectional curvature. On a Sasakian manifold, the ϕ-sectional curvatures determine completely the curvature .

Theorem 2.1. [1, Theorem 7.19, p139] *If the ϕ-sectional curvature at any point of a Sasakian manifold of dimension ≥ 5 is independent of the choice of ϕ-section at the point, then it is constant on the manifold and the curvature tensor is given by*

$$\begin{aligned}\mathcal{R}(X,Y)Z &= \frac{c+3}{4}(g(Y,Z)X - g(X,Z)Y) \\ &\quad + \frac{c-1}{4}\Big(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &\quad - g(Y,Z)\eta(X)\xi + d\eta(Z,Y)\varphi X - d\eta(Z,X)\varphi Y \\ &\quad + 2d\eta(X,Y)\varphi Z\Big) \qquad (2.13)\end{aligned}$$

where c is the φ -sectional curvature.

A Sasakian manifold M is called a Sasakian space form if M has constant ϕ-sectional curvature c, and will be denoted by $M(c)$.

Theorem 2.2. *Let M be a $(2n+1)$-dimensional complete simply connected Sasakian manifold with constant ϕ-sectional curvature c.*

(1) If $c > -3$, then M is isomorphic to $S^{2n+1}(c)$ or M is D-harothetic to S^{2n+1};
(2) If $c = -3$, then M is isomorphic to $R^{2n+1}(-3)$;
(3) If $c < -3$, then M is isomorphic to $(\mathbb{R}, CD^n)(c)$, where CD^n is a simply connected bounded complex domain in $\mathbb{C}^n$ with constant holomorphic sectional curvature $c < 0$.

3. NULL OSSERMAN CONDITION AND LORENTZ S-MANIFOLDS

Following [12], let us recall the notion of null Osserman condition. First, we need the notion of Jacobi operator with respect to a lightlike vector. Let (M, g) be a Lorentzian manifold, $p \in M$ and $u \in T_pM$ a null vector, that is $u \neq 0$ and $g_p(u,u) = 0$. Then the orthogonal complement $u^\perp$ of u is a degenerate vector space since $\mathrm{span}(u) \subset u^\perp$ and we put $\bar{u}^\perp = u^\perp/\mathrm{span}(u)$ the quotient space. The canonical projection is $\pi : u^\perp \to \bar{u}^\perp$. A positive definite inner product $\bar{g}$ can be defined on $\bar{u}^\perp$ by putting $\bar{g}(\bar{x}, \bar{y}) = g_p(x,y)$, where $\pi(x) = \bar{x}$ and $\pi(y) = \bar{y}$, so that $(\bar{u}^{\perp}, \bar{g})$ becomes an Euclidean vector space. The Jacobi operator with respect to u is the endomorphism $\bar{R}_u : \bar{u}^{\perp} \to \bar{u}^{\perp}$ defined by $\bar{R}_u(\bar{x}) = \pi(R_p(u,x)x)$, for all $\bar{x} = \pi(x) \in \bar{u}^{\perp}$. One sees that $\bar{R}_u$ is self-adjoint, hence diagonalizable.

In Lorentzian geometry it is well-known that a null vector u and a timelike vector z are never orthogonal. Hence, in a Lorentz manifold (M, g) the null congruence set determined by a timelike vector $z \in T_pM$ at $p \in M$ is defined by $N(z) = \{u \in T_pM, g_p(u,u) = 0, g_p(u,z) = -1\}$. Now, we can define the null Osserman condition.

Definition 3.1. [12] *A Lorentzian manifold (M, g) is said to be null Osserman with respect to $z, z \in T_pM$ being a unit timelike vector, if the eigenvalues of $\bar{R}_u$, counted with multiplicities, are independent of $u \in N(z)$.*

It is known that Lorentz Sasakian manifolds with constant ϕ- sectional curvature are null Osserman with respect to the characteristic vector field. Lorentzian S-structures constitute a generalization of the Lorentzian Sasakian ones.L. Brunetti and V. Caldarella have proved the following result:

Proposition 3.1. [3] *Let $(M, \phi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian S-manifold, with $\dim(M) = 2n + s, s > 2$. Then M does not verify the null Osserman condition with respect to $(\xi_1)_p$, for all $p \in M$.*

Motivated by the previous result, in [2] the author considered the subset

$$N_\phi\big((\xi_1)_p\big) = \{u = \xi_1 + x \in T_pM, \quad \text{with unit} \quad x \in \mathrm{Im}(\phi)\},$$

called the ϕ-null congruence set of ξ_1, and the following definition has been given.

Definition 3.2. [2] *A Lorentzian S-manifold $(M, \phi, \xi_\alpha, \eta^\alpha, g)$, with $\alpha \in \{1, \ldots, s\}$, is said to be ϕ-null Osserman at a point $p \in M$ if and only if the eigenvalues of $\bar{R}_u$, counted with multiplicities, are independent of $u \in N_\phi\big((\xi_1)_p\big)$.*

The notion of ϕ-null Osserman condition seems to be suitable for Lorentzian S-manifold. Indeed, it generalizes the null Osserman condition, since in a Lorentz Sasakian manifold one has clearly $N_\phi(\xi) = N(\xi)$ and the ϕ-null Osserman condition reduces to the null Osserman one. Moreover, in [2] it is proved that any Lorentzian S-manifold with constant ϕ-sectional curvature is ϕ-null Osserman with respect to $(\xi_1)_p$, at any point $p \in M$, thus recovering the similar known result for Lorentz Sasakian space forms.

Let $(M, \phi, \xi_\alpha, \eta^\alpha, g), \alpha \in \{1, \ldots, s\}$, be a $(2n+s)$-dimensional Lorentzian S-manifold, $s > 1$. Fix $p \in M$ and consider $u \in N_\phi\big((\xi_1)_p\big)$. Then $u = (\xi_1)_p + x$, with unit $x \in \mathrm{Im}(\phi_p)$, and we can consider the Jacobi operator $R_x : x^{\perp} \to x^{\perp}$ corresponding to $\bar{R}_u : \bar{u}^{\perp} \to \bar{u}^{\perp}$, and vice-versa. Studying the links between these two operators, the authors get the following.

Proposition 3.2. [3] *Let $(M, \phi, \xi_\alpha, \eta^\alpha, g), \alpha \in \{1, \ldots, s\}$ be a Lorentzian S-manifold, with $\dim(M) = 2n + s, s > 1$. For any $p \in M$, M is ϕ-null Osserman at p if and only if the eigenvalues of R_x, counted with multiplicities, are independent of any unit $x \in \mathrm{Im}(\phi_p)$.*

4. Osserman Condition on Sasakian manifolds

Let M be a Sasakian manifold, for any unit vector $X \in T_pM$, the endomorphism $R_X = R_p(\cdot, X)X : X^{\perp} \to X^{\perp}$ is called the Jacobi-Type operator with respect to X. From (2.12), we have:

$$\mathcal{R}_X Y = -\varphi\mathcal{R}(Y,X)\varphi X + Y + g(\varphi Y, X)\varphi X \tag{4.1}$$

where $Y \in X^{\perp}$. We have the following: The Jacobi-Type operator with respect to Reeb vector field ξ satisfies

$$\mathcal{R}_{\xi} X = X \tag{4.2}$$

for all unit vector X orthogonal to ξ and for a unit vector X orthogonal to ξ, we have

$$\mathcal{R}_X X = 0 \tag{4.3}$$

Proposition 4.1. *The Jacobi-Type operator is self-adjoint.*

Proof. On one hand, we have:

$$\begin{aligned} g(\mathcal{R}_X Y, Z) &= g(-\varphi\mathcal{R}(Y,X)\varphi X, Z) + g(Y,Z) \\ &\quad -g(Y,X)g(X,Z) + g(\varphi Y, X)g(\varphi X, Z). \end{aligned}$$

Since $-\varphi\mathcal{R}(Y,X)\varphi X = Y - g(X,Y)X \; g(X,\varphi Y) = -g(\varphi X, Y)$ we obain:

$$\begin{aligned} g(\mathcal{R}_X Y, Z) &= g(Y,Z) - g(X,Y)g(X,Z) + g(Y,Z) \\ &\quad -g(Y,X)g(X,Z) + g(\varphi X, Y)g(\varphi Z, X). \end{aligned} \tag{4.4}$$

On the other hand,

$$\begin{aligned} g(\mathcal{R}_X Z, Y) &= g(-\varphi\mathcal{R}(Z,X)\varphi X, Z) + g(Z,Y) \\ &\quad -g(Z,X)g(X,Y) + g(\varphi Z, X)g(\varphi X, Y) \\ &= g(Z,Y) - g(X,Z)g(X,Y) + g(Z,Y) \qquad (4.5) \\ &\quad -g(Z,X)g(X,Y) + g(\varphi Z, X)g(\varphi X, Y) \qquad (4.6) \end{aligned}$$

Hence, we get

$$g(\mathcal{R}_X Y, Z) = g(\mathcal{R}_X Z, Y) \tag{4.7}$$

□

Definition 4.1. *We say that a Sasakian manifold $(M^{2n+1}, \phi, \eta, \xi, g)$ is Osserman-Type if the characteristic polynomial of the Jacobi-Type operator is independent of the unit vector X orthogonal to ξ.*

It is easy to show that:

Proposition 4.2. *Any Sasakian manifolds with constant ϕ-sectional couvature of dimension ≥ 5 is Osserman Type.*

Definition 4.2. *Let (M, g) be a Riemannian manifold and H be a distribution of TM. We say that a Riemannian manifold is H-Osserman Type if $\forall X \in H$, the characteristic polynomial of the Jacobi-Type operator $\mathcal{R}_X$ is independent of the unit vector X. We say that a contact manifold $(M^{2n+1}, \phi, \eta, \xi, g)$ is contact stucture Osserman Type if $H = \{\xi\}^{\perp}$.*

Proposition 4.3. *Let $\mathcal{B} = \{e_1, \ldots, e_n, \phi e_1, \ldots, \phi e_n, \xi\}$ be a ϕ-basis and X a unit vector orthogonal to ξ such that*

$$X = \sum_{i=1}^{n} \alpha_i e_i + \sum_{i=1}^{n} \alpha_{n+i} \varphi e_i \tag{4.8}$$

with respect to the basis $\mathcal{B}$; then

$$\begin{cases} \mathcal{R}_X\xi = \xi \\ \mathcal{R}_X e_i = \alpha_{i+n}\varphi X \\ \mathcal{R}_X\varphi e_i = -\alpha_i\varphi X \end{cases} \tag{4.9}$$

Proof. Proof of the first equation of the system (4.9). From the equation (2.8), we get

$$\mathcal{R}(\xi, X)X = g(X, X)\xi - \eta(X)X = \xi$$

Proof of the second equation of the system (4.9), we obtain:

$$\begin{aligned} \mathcal{R}_X e_i &= -\varphi\mathcal{R}(e_i, X)\varphi X + e_i - g(e_i, X)X + g(\varphi e_i, X)\varphi X \\ &= -\varphi\mathcal{R}(e_i, X)\varphi X + e_i - \alpha_i X + \alpha_{i+n}\varphi X \end{aligned} \tag{4.10}$$

$$\mathcal{R}(e_i, X)\varphi X = \nabla_{e_i}\nabla_X\varphi X - \nabla_X\nabla_{e_i}\varphi X - \nabla_{[e_i,X]}\varphi X \tag{4.11}$$

From the equations (2.6) and (2.7), we get

$$\nabla_X\varphi X = g(X, X)\xi - \eta(X)X = \xi$$

then

$$\nabla_{e_i}\nabla_X\varphi X = -\varphi e_i \tag{4.12}$$

$$\nabla_{e_i}\varphi X = g(e_i, X)\xi - \eta(X)X = g(e_i, X)\xi = \alpha_i\xi$$

hence

$$\nabla_X\nabla_{e_i}\varphi X = \alpha_i\nabla_X\xi = -\alpha_i\varphi X \tag{4.13}$$

$$\nabla_{[e_i,X]}\varphi X = g([e_i, X], X)\xi \tag{4.14}$$

Consequently

$$\mathcal{R}(e_i, X)\varphi X = -\varphi e_i + \alpha_i\varphi X - g([e_i, X], X)\xi \tag{4.15}$$

and

$$-\varphi\mathcal{R}(e_i, X)\varphi X = -e_i + \alpha_i X \tag{4.16}$$

$$\begin{aligned} \mathcal{R}_X e_i &= -\varphi\mathcal{R}(e_i, X)\varphi X + e_i - \alpha_i X + \alpha_{i+n}\varphi X \\ &= -e_i + \alpha_i X + e_i - \alpha_i X + \alpha_{i+n}\varphi X \\ &= \alpha_{i+n}\varphi X \end{aligned} \tag{4.17}$$

We get the proof of the third equation of the system (4.9), by the same way as above. □

Theorem 4.1. *Sasakian manifolds $(M^{2n+1}, \phi, \eta, \xi, g)$, are contact stucture Osserman Type.*

Proof. Let consider the proposition 4.3 and the system (4.9). Then, we have:

$$\begin{cases} \mathcal{R}_X\xi = \xi \\ \mathcal{R}_X e_i = -\sum_{k=1}^{n}\alpha_{i+n}\alpha_{k+n}e_k + \sum_{k=1}^{n}\alpha_{i+n}\alpha_k\varphi e_k \\ \mathcal{R}_X\varphi e_i = \sum_{k=1}^{n}\alpha_i\alpha_{k+n}e_k - \sum_{k=1}^{n}\alpha_i\alpha_k\varphi e_k \end{cases}$$

Thus, the matrix of the Jacobi-Type operator is given by:

$$\mathcal{J}_{\mathcal{R}_X} = \begin{pmatrix} -\alpha_{n+1}^2 & \dots & -\alpha_{n+1}\alpha_{n+j} & \dots & -\alpha_{n^2}\alpha_{n+1} & \alpha_{n+1}\alpha_1 & \dots & \alpha_{n+1}\alpha_k & \dots & \alpha_{n+1}\alpha_n & 0 \\ -\alpha_{n+1}\alpha_{n+2} & \dots & -\alpha_{n+2}\alpha_{n+j} & \dots & -\alpha_{n^2}\alpha_{n+2} & \alpha_{n+2}\alpha_1 & \dots & \alpha_{n+2}\alpha_k & \dots & \alpha_{n+2}\alpha_n & 0 \\ \dots & \ddots & & & & & \dots & \dots & \dots & \dots & 0 \\ -\alpha_{n+1}\alpha_{n+i} & \dots & -\alpha_{n+i}\alpha_{n+j} & \dots & -\alpha_{n^2}\alpha_{n+i} & \alpha_{n+i}\alpha_1 & \dots & \alpha_{n+i}\alpha_k & \dots & \alpha_{n+i}\alpha_n & 0 \\ \vdots & \ddots & & & & & \dots & \dots & \dots & \dots & 0 \\ -\alpha_{n+1}\alpha_{n^2} & \dots & -\alpha_{n^2}\alpha_{n+j} & \dots & -\alpha_{n^2}^2 & \alpha_{n^2}\alpha_1 & \dots & \alpha_{n^2}\alpha_k & \dots & \alpha_{n^2}\alpha_n & 0 \\ \alpha_{n+1}\alpha_1 & \dots & \alpha_1\alpha_{n+j} & \dots & \alpha_{n^2}\alpha_1 & -\alpha_1^2 & \dots & \alpha_1\alpha_k & \dots & \alpha_1\alpha_n & 0 \\ \vdots & \ddots & & & & & \dots & \dots & \dots & \dots & 0 \\ \alpha_{n+1}\alpha_l & \dots & \alpha_l\alpha_{n+j} & \dots & \alpha_{n^2}\alpha_l & -\alpha_l\alpha_1 & \dots & -\alpha_l\alpha_k & \dots & -\alpha_l\alpha_n & 0 \\ \vdots & \ddots & & & & & \dots & \dots & \dots & \dots & 0 \\ \alpha_{n+1}\alpha_n & \dots & \alpha_n\alpha_{n+j} & \dots & \alpha_{n^2}\alpha_n & -\alpha_n\alpha_1 & \dots & -\alpha_n\alpha_k & \dots & -\alpha_n^2 & 0 \\ 0 & \dots & & & & \dots & & \dots & 0 & 0 & 1 \end{pmatrix}$$

Hence, the characteristic polynomial $\mathcal{P}_{\mathcal{R}_X}(\lambda)$ of the Jacobi-Type operator is given by

$$\mathcal{P}_{\mathcal{R}_X}(\lambda) = \lambda^{2n-1}(1-\lambda^2) \tag{4.18}$$

□

5. Examples

5.1. **Example: 3-dimensionnal Manifolds.** Let $(M^{2n+1}, \eta, \xi, \varphi, g)$ be a 3 dimensionnal Sasakian manifold. Then, according to equation (4.9), we have:

$$\begin{cases} \mathcal{R}_X e_1 = -\alpha_2^2 e_1 + \alpha_1\alpha_2\varphi e_1 \\ \mathcal{R}_X \varphi e_1 = \alpha_1\alpha_2 e_1 - \alpha_1^2\varphi e_1 \\ \mathcal{R}_X\xi = \xi \end{cases} \tag{5.1}$$

where $X = \alpha_1 e_1 + \alpha_2\varphi e_1$ is a unit vector fields orthogonal to ξ. In additionnal, we find

$$\mathcal{J}_{\mathcal{R}_X} = \begin{pmatrix} -\alpha_2^2 & \alpha_1\alpha_2 & 0 \\ \alpha_1\alpha_2 & -\alpha_1^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.2}$$

and

$$\mathcal{P}_{\mathcal{J}_{\mathcal{R}_X}}(\lambda) = \lambda(1-\lambda)(1+\lambda) \tag{5.3}$$

5.2. **Example: 5-dimensionnal Manifolds.** Let $(M^{2n+1}, \eta, \xi, \varphi, g)$ be a 5 dimensionnal Sasakian manifold. Then according to equation (4.9), we have

$$\begin{cases} \mathcal{R}_X e_1 = -\alpha_3^2 e_1 - \alpha_3\alpha_4 e_2 + \alpha_1\alpha_3\varphi e_1 + \alpha_3\alpha_2\varphi e_2 \\ \mathcal{R}_X e_2 = -\alpha_3\alpha_4 e_1 - \alpha_4^2 e_2 + \alpha_1\alpha_4\varphi e_1 + \alpha_2\alpha_4\varphi e_2 \\ \mathcal{R}_X\varphi e_1 = \alpha_1\alpha_3 e_1 + \alpha_1\alpha_4 e_2 - \alpha_1^2\varphi e_1 - \alpha_1\alpha_2\varphi e_2 \\ \mathcal{R}_X\varphi e_2 = \alpha_2\alpha_3 e_1 + \alpha_2\alpha_4 e_2 - \alpha_1\alpha_2\varphi e_1 - \alpha_2^2\varphi e_2 \\ \mathcal{R}_X\xi = \xi \end{cases} \tag{5.4}$$

where $X = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 \varphi e_1 + \alpha_4 \varphi e_2$ is a unit vector fields orthogonal to ξ. In addition, we have

$$\mathcal{J}_{\mathcal{R}_X} = \begin{pmatrix} -\alpha_3^2 & -\alpha_3\alpha_4 & \alpha_1\alpha_3 & \alpha_2\alpha_3 & 0 \\ -\alpha_3\alpha_4 & -\alpha_4^2 & \alpha_1\alpha_4 & \alpha_2\alpha_4 & 0 \\ \alpha_1\alpha_3 & \alpha_1\alpha_4 & -\alpha_1^2 & -\alpha_1\alpha_2 & 0 \\ \alpha_3\alpha_2 & \alpha_2\alpha_4 & -\alpha_1\alpha_2 & -\alpha_2^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.5}$$

and

$$\mathcal{P}_{\mathcal{R}_X}(\lambda) = -\lambda^3(1 - \lambda^2) \tag{5.6}$$

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