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RICCI CONSTANT METRIC ON WARPED PRODUCT FINSLER SPACE

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ABSTRACT. In this paper, we consider the Ricci tensor on the warped product Finsler space(cf. [14]) and so provide a necessary and sufficient condition for the warped Finsler space to have Ricci-constant. Close relationships between the geometry of the warped product Finsler spaces of Ricci-constant metric and the spectral theory of the Laplacian operator of the well-known Sasaki-Finsler metrics of the base space F_1 is established by detailed investigation of the above mentioned PDEs.

1. INTRODUCTION

The warped product extended for Finslerian metrics by the work of Kozma et al.(cf. [15]). In [1, 17, 4, 5], some objects of Finsler manifolds are expanded to the warped product Finsler manifolds. One of the important problems in Finsler geometry is to characterize and construct Finsler metrics with constant flag curvature and Ricci curvature. In 2004, D. Bao, C. Robles and Shen classified Randers metrics with constant flag curvature (cf. [8]). Moreover, with the help of the navigation problem, D. Bao and C. Robles give a characterization for Einstein metrics of Randers type (cf. [7]).

In this paper by applying the concept of Ricci tensor to the warped product Finslerian spaces, we introduce Ricci-constant metrics on the warped product Finsler spaces and present the necessary and sufficient conditions for the warped product Finslerian space to be Ricci-constant. So, relationships between the geometry of the warped product Finsler spaces of Ricci-constant and the spectral theory of the Laplacian of the Sasaki-Finsler metrics of the base space is established by detailed investigation of the above conditions. Moreover, several results are obtained in special cases for example the case of Riemannian, Locally Minkowski and Berwalds spaces are considered. Also, it is shown that Ricci-constant warped product Finsler manifold must be Ricci flat where the base space is Locally Minkowski space.

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2. Preliminaries

Let M be a real smooth m-dimensional manifold. Denote by (TM, π, M) the tangent bundle of M and $\overset{\circ}{T}M = TM \setminus \{0\}$. As usual, we denote by $(x, y) = (x^i, y^i)$, $1 \leq i \leq n$ the local coordinates of a point $u \in TM$, where $\pi(u) = x$. The transformation of local coordinates on TM are given by (cf.[9])

$$\tilde{x}^i = \tilde{x}^i(x^1, \cdots, x^m), \quad \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j.$$

Consequently the local frame fields $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i})$ and $(\frac{\partial}{\partial \tilde{x}^i}, \frac{\partial}{\partial \tilde{y}^i})$ on $\overset{\circ}{T}M$ satisfies

$$\frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^i}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} + \frac{\partial^2 \tilde{x}^j}{\partial x^i \partial x^k} y^k \frac{\partial}{\partial \tilde{y}^j}, \quad \frac{\partial}{\partial y^i} = \frac{\partial \tilde{x}^i}{\partial x^i} \frac{\partial}{\partial \tilde{y}^j}. \tag{2.1}$$

Now, consider a smooth function $F : TM \to (0, +\infty)$ where, M' is an open subset of TM°. The triple $F^m = (M, M', F)$ is called a Finsler manifold if for any coordinate system $(x^i, y^i) \in M'$, the following conditions are fulfilled:

1) F is positively homogeneous of degree one with respect to $(y^1, \cdots, y^m)$.
2) The $m \times m$ Hessian matrix

$$g_{ij}(x, y) = \frac{1}{2} \Big(\frac{\partial^2 F^2}{\partial y^i \partial y^j} \Big)(x, y),$$

is positive-definite at every point of $\overset{\circ}{T}M$.

Next, we denote by F_V the foliation on M' determined by fibers of the canonical projection $\pi : M' \to M$, and call it the *vertical foliation* on M' . Then the tangent distribution to F_V is the vertical distribution VM' on M' (cf. [10]). According to (2.1), the vertical distribution VM' is locally spanned by $\frac{\partial}{\partial y^i}$, $i \in \{1, \cdots, m\}$. There also exists a complementary distribution HM' to VM' in TM' which is called *horizontal distribution*, and is locally spanned by

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^i}, \tag{2.2}$$

where

$$G_i^j = \frac{\partial G^j}{\partial y^i}, \qquad G^j = \frac{1}{4} g^{jk} \Big(\frac{\partial^2 F^2}{\partial y^k \partial x^h} y^h - \frac{\partial F^2}{\partial x^k} \Big).$$

According to the decomposition $TM' = HM' \oplus VM'$, we may consider $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ as a local frame field on TM'. We show the dual basis to $\{\frac{\delta}{\delta x^i}\}$ by $\{dx^i\}$, and the dual basis to $\{\frac{\partial}{\partial y^i}\}$ by $\{\delta y^i\}$ (see [9]).

Let us consider the Levi-Civita connection ∇ on the vertical distribution VM'. We have

$$\nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} = F_{ij}^k \frac{\partial}{\partial y^k} \quad , \quad \nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} = C_{ij}^k \frac{\partial}{\partial y^k},$$

where

$$F_{ij}^k = \frac{1}{2} g^{kh} \Big(\frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h} \Big) \quad , \quad C_{ij}^k = \frac{1}{2} g^{kh} \frac{\partial g_{ij}}{\partial y^h}.$$

The linear connection ∇ together with the canonical non-linear connection (G_i^j) defines the Cartan connection (cf. [9]) which, we denote by

$$FC = (G_k^i, F_{jk}^i, C_{jk}^i).$$

The Ricci tensor is defined, as the sum of $m-1$ flag curvatures $K(x, y, e_\nu)$

$$\Re ic := \sum_{\nu=1}^{m-1} K(x, y, e_\nu) = R_i^i := \frac{1}{F^2}(y^j R_{jil}^i y^l),$$

where $\{e_\nu; \nu = 1, \cdots, m-1\}$ is a collection of orthonormal transverse edges perpendicular to the flagpole (cf. [7]). The Berwald's spray curvature is defined entirely in terms of the geodesic spray coefficients (cf. [11]),

$$K_k^i := 2(G^i)_{x^k} - (G^i)_{y^j}(G^j)_{y^k} - y^j(G^i)_{x^j y^k} + 2G^j(G^i)_{y^j y^k}.$$

This spray curvature is related to the predecessor of the flag curvature, in the following manner

$$F^2 R_i^i = K_k^i.$$

The Ricci tensor has the same geometrical content as the Ricci scalar and defines as follows:

$$Ric_{ij} := (\frac{1}{2} F^2 \Re ic)_{y^i y^j}. \tag{2.3}$$

The definition 2.3, due to Akbar-Zadeh (see [2]), is motivated by the fact that, when F arises from any Riemannian metric ρ, the curvature tensor depends on x alone and the y-Hessian in question reduces to the familiar expression $R_{jil}^i|_\rho$, which is $Ric_{ij}|_\rho$.
In Finsler geometry, *Einstein metrics* on an m-dimensional manifold are Finsler metrics $F = F(x, y)$ whose Ricci scalar depends only on the position x i.e.

$$\Re ic = K(x),$$

where $K(x)$ is a scalar function on the manifold. Finsler metric F is said to be *Ricci-constant* if Ricci scalar does not depend on the location x either, in this case, the function K is constant.

3. Warped product of Finsler spaces

Let $F_1 = (M_1, M_1', F_1)$ and $F_2 = (M_2, M_2', F_2)$ be Finsler manifolds, and let $f : M_1 \to R^+$ be a smooth function. Consider the triple $F^m = (M, M', F)$ where, M is the product manifold $M_1 \times M_2$, M' is the product $M_1' \times M_2'$ which is an open subset of $\overset{\circ}{T}(M_1 \times M_2)$ and the function F is defined as

$$F^2(x_1, x_2, y_1, y_2) = F_1^2(x_1, y_1) + f(x_1)^2 F_2^2(x_2, y_2), \tag{3.1}$$

where F is a smooth and positively homogeneous of degree 1 with respect to (y_1, y_2). For the function F,

$$(g_{ab}(\mathrm{x}, \mathrm{y})) = (\frac{1}{2}\frac{\partial^2 F^2}{\partial \mathrm{y}^a \partial \mathrm{y}^b}) = \begin{pmatrix} \hat{g}_{ij}(x_1, y_1) & 0 \\ 0 & f^2(x_1)\check{g}_{\alpha\beta}(x_2, y_2) \end{pmatrix}, \tag{3.2}$$

are the components of a positive definite quadratic form at every point (x,y). Therefore $F^m = (M, M', F)$ defines a Finsler manifold and is called the warped product of F_1 and F_2 and is denoted by $F = F_1 \times_f F_2$ ([1, 15]).
Lower case Latin letters like $\{i, j, k, l, \cdots\}$, $\{\alpha, \beta, \gamma, \cdots\}$ and $\{a, b, c, d, \cdots\}$ are used in the upper position for variable indexes. They belong to the set $\{1, \cdots, m_1\}$, $\{1, \cdots, m_2\}$ and $\{1, \cdots, m_1 + m_2\}$ respectively. According to the spaces F_1, F_2 or $F = F_1 \times_f F_2$ they represent. Variables of F_1 and F_2 have lower indexes 1 and 2 respectively, like x_1^i, y_1^j and x_2^α, y_2^β. When there is no appropriate position to place indexes 1 and 2, objects of F_1 and F_2 will be hat and check respectively, like $\hat{g}_{ij}$ and $\check{g}_{\alpha\beta}$, to indicate their relevant spaces.
The inverse g^{ab} of g_{ab} is given by

$$(g^{ab}(\mathrm{x},\mathrm{y})) = \begin{pmatrix} \hat{g}^{ij}(x_1, y_1) & 0 \\ 0 & f^{-2}(x_1)\check{g}^{\alpha\beta}(x_2, y_2) \end{pmatrix}.$$

The local coordinates (x_1, x_2, y_1, y_2) on M' are transformed by the rules

$$\begin{aligned} \tilde{x}_1^i &= \tilde{x}_1^i(x_1^1, \cdots, x_1^{m_1}), \quad \tilde{y}_1^i = \tfrac{\partial \tilde{x}_1^i}{\partial x_1^j} y_1^j \\ \tilde{x}_2^\alpha &= \tilde{x}_2^\alpha(x_2^1, \cdots, x_2^{m_2}), \quad \tilde{y}_2^\alpha = \tfrac{\partial \tilde{x}_2^\alpha}{\partial x_2^\beta} y_2^\beta. \end{aligned}$$

and

$$\frac{\partial}{\partial y_1^i} = \frac{\partial \tilde{x}_1^j}{\partial x_1^i}\frac{\partial}{\partial \tilde{y}_1^j} \quad , \quad \frac{\partial}{\partial y_2^\alpha} = \frac{\partial \tilde{x}_2^j}{\partial x_2^i}\frac{\partial}{\partial \tilde{y}_2^j}.$$

Now, we put

$$\hat{G}^i(x_1, y_1) = \frac{1}{4}\hat{g}^{ih}(x_1, y_1)\Big(\frac{\partial^2 F_1^2}{\partial y_1^h \partial x_1^j} y_1^j - \frac{\partial F_1^2}{\partial x_1^h}\Big)(x_1, y_1), \tag{3.3}$$

$$\check{G}^\alpha(x_2, y_2) = \frac{1}{4}\check{g}^{\alpha\gamma}(x_2, y_2)\Big(\frac{\partial^2 F_2^2}{\partial y_2^\gamma \partial x_2^\beta} y_2^\beta - \frac{\partial F_2^2}{\partial x_2^\gamma}\Big)(x_2, y_2), \tag{3.4}$$

$$G^a(\mathrm{x},\mathrm{y}) = \frac{1}{4}g^{ac}(\mathrm{x},\mathrm{y})\Big(\frac{\partial^2 F^2}{\partial \mathrm{y}^c \partial \mathrm{x}^b}\mathrm{y}^b - \frac{\partial F^2}{\partial \mathrm{x}^c}\Big)(\mathrm{x},\mathrm{y}). \tag{3.5}$$

Using (3.1),(3.3), (3.4) and (3.5), it can be deduced by straightforward calculation as follows $G^a(\mathrm{x},\mathrm{y}) = (G^i(\mathrm{x},\mathrm{y}), G^\alpha(\mathrm{x},\mathrm{y}))$ where

$$\begin{cases} G^i(\mathrm{x},\mathrm{y}) = \hat{G}^i(x_1, y_1) - \frac{1}{4}\hat{g}^{ih}\frac{\partial f^2(x_1)}{\partial x_1^h} F_2^2(x_2, y_2) \\ G^\alpha(\mathrm{x},\mathrm{y}) = \check{G}^\alpha(x_2, y_2) + \frac{1}{2}\frac{1}{f^2(x_1)} y_2^\alpha \frac{\partial f^2(x_1)}{\partial x_1^h} y_1^h. \end{cases}$$

The coefficients $G^a_b = (G^i_j, G^i_\beta, G^\alpha_j, G^\alpha_\beta)$ of the *warped non-linear connection* are introduced as follows (cf. [1])

$$\begin{aligned}
G^i_j &= \hat{G}^i_j - \frac{1}{4}\frac{\partial \hat{g}^{ih}}{\partial y^j_1}\frac{\partial f^2}{\partial x^h_1}F^2_2,\\
G^i_\beta &= -\frac{1}{4}\hat{g}^{ih}\frac{\partial f^2}{\partial x^h_1}\frac{\partial F^2_2}{\partial y^\beta_2},\\
G^\alpha_j &= \frac{1}{2}\frac{1}{f^2}y^\alpha_2\frac{\partial f^2}{\partial x^j_1},\\
G^\alpha_\beta &= \check{G}^\alpha_\beta + \frac{1}{2}\frac{1}{f^2}y^h_1\frac{\partial f^2}{\partial x^h_1}\delta^\alpha_\beta.
\end{aligned}$$

Corollary 3.1. *([1]) The coefficients*

$$G^a_{bc} = (G^i_{jk}, G^i_{\beta k}, G^i_{\beta\gamma}, G^\alpha_{jk}, G^\alpha_{j\gamma}, G^\alpha_{\beta\gamma}),$$

of the warped non-linear connection on the $F = F_1 \times_f F_2$ are gotten as

$$\begin{aligned}
G^i_{jk} &= \hat{G}^i_{jk} - \frac{1}{4}\frac{\partial^2 \hat{g}^{ih}}{\partial y^j_1 \partial y^k_1}\frac{\partial f^2}{\partial x^h_1}F^2_2 = G^i_{kj}\ ,\\
G^i_{\beta k} &= -\frac{1}{4}\frac{\partial \hat{g}^{ih}}{\partial y^k_1}\frac{\partial f^2}{\partial x^h_1}\frac{\partial F^2_2}{\partial y^\beta_2} = G^i_{k\beta}\ ,\\
G^i_{\beta\gamma} &= -\frac{1}{2}\hat{g}^{ih}\frac{\partial f^2}{\partial x^h_1}\check{g}_{\beta\gamma} = G^i_{\gamma\beta}\ ,\\
G^\alpha_{jk} &= 0\ ,\\
G^\alpha_{j\gamma} &= \frac{1}{2}\frac{1}{f^2}\frac{\partial f^2}{\partial x^j_1}\delta^\alpha_\gamma = G^\alpha_{\gamma j}\ ,\\
G^\alpha_{\beta\gamma} &= \check{G}^\alpha_{\beta\gamma} = G^\alpha_{\gamma\beta}.
\end{aligned}$$

Next, $V(M')$ kernel of the differential of the projection map

$$\pi = (\pi_1, \pi_2) : TM'_1 \oplus TM'_2 \to M_1 \times M_2$$

which is known as *vertical bundle* on TM' is considered. Locally, $\Gamma(V(M'))$ is spanned by the natural vector fields $\left\{\frac{\partial}{\partial y^i_1}, \frac{\partial}{\partial y^\alpha_2}\right\}$. So, using the functions G^i_j, G^i_β, G^α_j and G^α_β, nonholonomic vector fields are defined as follows

$$\begin{aligned}
\frac{\delta^*}{\delta x^i_1} &= \frac{\partial}{\partial x^i_1} - G^j_i\frac{\partial}{\partial y^j_1} - G^\beta_i\frac{\partial}{\partial y^\beta_2},\\
\frac{\delta^*}{\delta x^\alpha_2} &= \frac{\partial}{\partial x^\alpha_2} - G^j_\alpha\frac{\partial}{\partial y^j_1} - G^\beta_\alpha\frac{\partial}{\partial y^\beta_2},
\end{aligned} \tag{3.6}$$

which make it possible to construct a complementary vector subbundle $H(M')$ to $V(M')$ in $T(M')$ which is locally $H(M') = span\left\{\frac{\delta^*}{\delta x^i_1}, \frac{\delta^*}{x^\alpha_2}\right\}$. The subbundle $H(M')$ is called the *warped horizontal distribution* on $TM' = TM'_1 \oplus TM'_2$. That is, $\mathbf{N} = (G^a_b) = (G^i_j, G^i_\beta, G^\alpha_j, G^\alpha_\beta)$ is a warped non-linear connection on $TM = TM_1 \oplus TM_2$.

4. The Laplacian of the Sasaki-Finsler metric

It is well known that various kinds of Laplace operators play a very important role in differential geometry and physics, especially in the theory of harmonic integral and Bochner technique. In [13], Chunping and Tongde generalized the Laplace operator in Riemannian manifolds to Finsler vector bundles as such bundles arise naturally in

Finsler geometry (cf. [6]). They defined the h-harmonic function and h-harmonic horizontal Finsler vector fields and using the h-Laplace operator, and they proved some integral formulas of the scalar fields and horizontal Finsler vector fields on E.
Suppose, (M, F) is an m-dimensional Finsler manifold then, the tangent bundle TM endowed with the Sasaki-type metric constructed from the given Finsler metric F, is a Riemannian vector bundle. Therefor,

$$dV = \det(g_{ij})dx^1 \wedge \cdots \wedge dx^m \wedge \delta y^1 \wedge \cdots \wedge \delta y^m,$$

is the volume form associated to the Riemannian structure,

$$G = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j,$$

on TM. The divergence of $X = X^i \frac{\delta}{\delta x^i} + \bar{X}^i \frac{\partial}{\partial y^i}$ is defined by

$$L_X dV = (divX)dV.$$

Also, we put $div_h X = div(X^i \frac{\delta}{\delta x^i})$ and $div_v X = div(\bar{X}^i \frac{\partial}{\partial y^i})$.

Lemma 4.1. ([13]) *Let* $X = X^i \frac{\delta}{\delta x^i} + \bar{X}^i \frac{\partial}{\partial y^i} \in \chi(TM)$ *then*

$$divX = div_h X + div_v X,$$

where

$$div_h X = \nabla_{\frac{\delta}{\delta x^i}} X^i - (G^k_{ik} - F^k_{ik})X^i \quad , \quad div_v X = \nabla_{\frac{\partial}{\partial y^i}} \bar{X}^i + C^k_{ik}\bar{X}^i.$$

For a Riemannian manifold (TM, G), the gradient vector field of a function $f \in C^\infty(TM)$ is given by

$$G(\nabla f, X) = df(X) \ , \ \forall X \in \chi(TM).$$

Hence, in the adapted frame $\left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right\}$ we have

$$\nabla f = \nabla_h f + \nabla_v f,$$

where

$$\nabla_h f = g^{ij} \frac{\delta f}{\delta x^j} \frac{\delta}{\delta x^i} \quad , \qquad \nabla_v f = g^{ij} \frac{\partial f}{\partial y^j} \frac{\partial}{\partial y^i}.$$

A simple calculation shows that

$$G(\nabla_h f^2, \nabla_h f^2) = g^{ij} \frac{\partial f^2}{\partial x^i} \frac{\partial f^2}{\partial x^j}. \tag{4.1}$$

Now the $h(v)$-Laplace operator on TM is defined by

$$\Delta_h = div_h \circ \nabla_h \quad , \quad \Delta_v = div_v \circ \nabla_v.$$

Applying Proposition 3.2 of [13] in the case of Finsler manifold, we can state.

Lemma 4.2. *Let* (M, F) *be Finsler space and* $f \in C^\infty(TM)$. *Then*

$$\Delta f = \Delta_h f + \Delta_v f,$$

where

$$\Delta_h f = \tfrac{\delta g^{ij}}{\delta x^i}\tfrac{\delta f}{\delta x^j} + g^{ij}\tfrac{\delta}{\delta x^i}(\tfrac{\delta f}{\delta x^j}) - (G^k_{ik} - F^k_{ik})g^{ij}\tfrac{\delta f}{\delta x^j}$$

$$\Delta_v f = \tfrac{\partial g^{ij}}{\partial y^i}\tfrac{\partial f}{\partial y^j} + g^{ij}\tfrac{\partial}{\partial y^i}(\tfrac{\partial f}{\partial y^j}) + g^{ij}\tfrac{\partial f}{\partial y^j}C^k_{ik}.$$

Consider the Finsler manifold to be Riemannian space and $f : M \to R$, so the horizontal Laplacian of f^2 is given by

$$\Delta_h f^2 = \frac{\partial g^{ij}}{\partial x^i}\frac{\partial f^2}{\partial x^j} + g^{ij}\frac{\partial^2 f^2}{\partial x^i \partial x^j} + G^k_{ik}g^{ij}\frac{\partial f^2}{\partial x^j}. \tag{4.2}$$

When the Finsler space is locally Minkowski space, the horizontal Laplacian of $f \in C^\infty(M)$ is gotten by

$$\Delta_h f^2 = g^{ij}\frac{\partial^2 f^2}{\partial x^i \partial x^j}. \tag{4.3}$$

5. Ricci-constant metric on WPFM

The importance of the Ricci tensor can be seen from the Bonnet-Myers theorem. The Riemannian version of this result is one of the most useful comparison theorems in differential geometry, (cf. [12]). It was first extended to Finsler manifolds in [3]. Akbar-Zadeh generalized the concept of Ricci tensor to Finsler geometry in [2]. In this section, we extend the concept of Ricci tensor on warped product Finsler manifolds and give some conditions for warped product Finsler metric to be Ricci-constant.
Let us begin by introducing *Ricci scalar* for the warped product Finsler space as follows

$$\Re ic := R^a_a = R^i_i + R^\alpha_\alpha,$$

where

$$\begin{aligned} R^i_i &:= \tfrac{1}{F^2}(R^i_{jik}y^j_1 y^k_1 + 2R^i_{ji\gamma}y^j_1 y^\gamma_2 + R^i_{\beta i\gamma}y^\beta_2 y^\gamma_2)\ , \\ R^\alpha_\alpha &:= \tfrac{1}{F^2}(R^\alpha_{j\alpha k}y^j_1 y^k_1 + +2R^\alpha_{\beta\alpha k}y^\beta_2 y^k_1 + R^\alpha_{\beta\alpha\gamma}y^\beta_2 y^\gamma_2), \end{aligned} \tag{5.1}$$

and R^a_{bcd} are the h-curvature tensor field of the Cartan connection of warped product Finsler manifold that are given by

$$
\begin{aligned}
R^i_{jik} &:= \tfrac{\delta^* F^i_{ji}}{\delta x^k_1} - \tfrac{\delta^* F^i_{jk}}{\delta x^i_1} + F^h_{ji}F^i_{hk} + F^\gamma_{ji}F^i_{\gamma k} \\
&\quad - F^h_{jk}F^i_{hi} - F^\gamma_{jk}F^i_{\gamma i} + R^h_{ik}C^i_{jh} + R^\gamma_{ik}C^i_{j\gamma}, \\
R^i_{\beta i\gamma} &:= \tfrac{\delta^* F^i_{\beta i}}{\delta x^\gamma_2} - \tfrac{\delta^* F^i_{\beta\gamma}}{\delta x^i_1} + F^\zeta_{\beta i}F^i_{\zeta\gamma} + F^k_{\beta i}F^i_{k\gamma} \\
&\quad - F^\zeta_{\beta\gamma}F^i_{\zeta i} - F^k_{\beta\gamma}F^i_{ki} + R^\zeta_{i\gamma}C^i_{\beta\zeta} + R^k_{i\gamma}C^i_{\beta k}, \\
R^\alpha_{j\alpha k} &:= \tfrac{\delta^* F^\alpha_{j\alpha}}{\delta x^k_1} - \tfrac{\delta^* F^\alpha_{jk}}{\delta x^\alpha_2} + F^h_{j\alpha}F^\alpha_{hk} + F^\gamma_{j\alpha}F^\alpha_{\gamma k} \\
&\quad - F^h_{jk}F^\alpha_{h\alpha} - F^\gamma_{jk}F^\alpha_{\gamma\alpha} + R^h_{\alpha k}C^\alpha_{jh} + R^\gamma_{\alpha k}C^\alpha_{j\gamma}, \\
R^\alpha_{\beta\alpha\gamma} &:= \tfrac{\delta^* F^\alpha_{\beta\alpha}}{\delta x^\gamma_2} - \tfrac{\delta F^\alpha_{\beta\gamma}}{\delta x^\alpha_2} + F^\zeta_{\beta\alpha}F^\alpha_{\zeta\gamma} + F^k_{\beta\alpha}F^\alpha_{k\gamma} \\
&\quad - F^\zeta_{\beta\gamma}F^\alpha_{\zeta\alpha} - F^k_{\beta\gamma}F^\alpha_{k\alpha} + R^\zeta_{\alpha\gamma}C^\alpha_{\beta\zeta} + + R^k_{\alpha\gamma}C^\alpha_{\beta k}.
\end{aligned} \tag{5.2}
$$

The Ricci tensor has the same geometrical content as the Ricci scalar. We define *Ricci tensor* of warped product Finsler manifold as follows:

$$Ric_{bc} := (\frac{1}{2}F^2\Re ic)_{\mathrm{y}^b\mathrm{y}^c}. \tag{5.3}$$

The definition of the Ricci tensor is not practical if one wants to compute it, so we use the generalized Berwarld's formula on warped product Finsler manifold that is defined as

$$K^a_a := 2\tfrac{\partial G^a}{\partial \mathrm{x}^a} - \tfrac{\partial G^a}{\partial \mathrm{y}^b}\tfrac{\partial G^b}{\partial \mathrm{y}^a} - y^b\tfrac{\partial G^a}{\partial \mathrm{x}^b \partial \mathrm{y}^a} + 2G^b\tfrac{\partial G^a}{\partial \mathrm{y}^b \partial \mathrm{y}^a}. \tag{5.4}$$

The Ricci scalar is related to the generalized Berwald's Formula in the following manner:

$$F^2 R^a_a = K^a_a. \tag{5.5}$$

By using this relation, we obtain the Ricci tensor of warped product Finsler space as follows

$$
\begin{aligned}
Ric_{kl} = \hat{Ric}_{kl} &+ \tfrac{1}{2}F_2^2[-\tfrac{1}{2}\frac{\partial^3\hat{g}^{ih}}{\partial x_1^i\partial y_1^k\partial y_1^l}\frac{\partial f^2}{\partial x_1^h} - \tfrac{1}{2}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^k\partial y_1^l}\frac{\partial^2 f^2}{\partial x_1^h\partial x_1^i} + \tfrac{1}{4}y_1^j\frac{\partial^4\hat{g}^{ih}}{\partial x_1^j\partial y_1^i\partial y_1^k\partial y_1^l}\frac{\partial f^2}{\partial x_1^h}\\
&+\tfrac{1}{4}y_1^j\frac{\partial^3\hat{g}^{ih}}{\partial y_1^i\partial y_1^k\partial y_1^l}\frac{\partial^2 f^2}{\partial x_1^j\partial x_1^h} + \tfrac{1}{4}\frac{1}{f^2}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^k\partial y_1^l}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^i} - \tfrac{1}{2}\frac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial^3\hat{g}^{is}}{\partial y_1^i\partial y_1^k\partial y_1^l}]\\
&+\tfrac{1}{8}\frac{\partial f^2}{\partial x_1^s}F_2^2[\hat{G}^i_{klj}\frac{\partial\hat{g}^{js}}{\partial y_1^i} + \hat{G}^i_{kj}\frac{\partial^2\hat{g}^{js}}{\partial y_1^i\partial y_1^l} + \hat{G}^i_{lj}\frac{\partial^2\hat{g}^{js}}{\partial y_1^i\partial y_1^k}\hat{G}^i_j\frac{\partial^3\hat{g}^{js}}{\partial y_1^i\partial y_1^k\partial y_1^l}]\\
&+\tfrac{1}{8}\frac{\partial f^2}{\partial x_1^h}F_2^2[\hat{G}^j_{kli}\frac{\partial\hat{g}^{ih}}{\partial y_1^j} + \hat{G}^j_{ki}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^j\partial y_1^l} + \hat{G}^j_{li}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^j\partial y_1^k}\hat{G}^j_i\frac{\partial^3\hat{g}^{ih}}{\partial y_1^j\partial y_1^k\partial y_1^l}]\\
&-\tfrac{1}{4}\frac{\partial f^2}{\partial x_1^s}F_2^2[\hat{G}^j_{kl}\frac{\partial^2\hat{g}^{is}}{\partial y_1^j\partial y_1^i} + \hat{G}^j_k\frac{\partial^3\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^l} + \hat{G}^j_l\frac{\partial^3\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^k}\hat{G}^j\frac{\partial^4\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^k\partial y_1^l}]\\
&-\tfrac{1}{4}\frac{\partial f^2}{\partial x_1^h}F_2^2[\hat{G}^i_{ji}\frac{\partial^2\hat{g}^{jh}}{\partial y_1^k\partial y_1^l} + \hat{G}^i_{lji}\frac{\partial\hat{g}^{jh}}{\partial y_1^k} + \hat{G}^i_{kji}\frac{\partial\hat{g}^{jh}}{\partial y_1^l}\hat{G}^i_{klji}\hat{g}^{jh}]\\
&+\tfrac{1}{2}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}F_2^4[-\tfrac{1}{16}\frac{\partial^3\hat{g}^{ih}}{\partial y_1^j\partial y_1^k\partial y_1^l}\frac{\partial\hat{g}^{js}}{\partial y_1^i} - \tfrac{1}{16}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^j\partial y_1^k}\frac{\partial^2\hat{g}^{js}}{\partial y_1^i\partial y_1^l} - \tfrac{1}{16}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^j\partial y_1^l}\frac{\partial^2\hat{g}^{js}}{\partial y_1^i\partial y_1^k}\\
&-\tfrac{1}{16}\frac{\partial\hat{g}^{ih}}{\partial y_1^j}\frac{\partial^3\hat{g}^{js}}{\partial y_1^i\partial y_1^k\partial y_1^l} + \tfrac{1}{8}\frac{\partial^2\hat{g}^{jh}}{\partial y_1^k\partial y_1^l}\frac{\partial^2\hat{g}^{is}}{\partial y_1^j\partial y_1^i} + \tfrac{1}{8}\frac{\partial\hat{g}^{jh}}{\partial y_1^k}\frac{\partial^3\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^l} + \tfrac{1}{8}\frac{\partial\hat{g}^{jh}}{\partial y_1^l}\frac{\partial^3\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^k}\\
&+\tfrac{1}{8}\hat{g}^{jh}\frac{\partial^4\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^k\partial y_1^l}] + \tfrac{1}{8}y_2^\beta\frac{\partial^3\hat{g}^{ih}}{\partial y_1^i\partial y_1^k\partial y_1^l}\frac{\partial f^2}{\partial x_1^h}\frac{\partial F_2^2}{\partial x_2^\beta} - \tfrac{1}{4}\check{G}^\beta\frac{\partial^3\hat{g}^{is}}{\partial y_1^i\partial y_1^k\partial y_1^l}\frac{\partial f^2}{\partial x_1^s}\frac{\partial F_2^2}{\partial y_2^\beta} + \tfrac{1}{2}\frac{1}{f^2}\hat{G}^j_{kl}\frac{\partial f^2}{\partial x_1^j},
\end{aligned}
\tag{5.6}
$$

$$
\begin{aligned}
Ric_{k\nu} = \tfrac{1}{2}\frac{\partial F_2^2}{\partial y_2^\nu}&[-\tfrac{1}{2}\frac{\partial^2\hat{g}^{ih}}{\partial x_1^i\partial y_1^k}\frac{\partial f^2}{\partial x_1^h} - \tfrac{1}{2}\frac{\partial\hat{g}^{ih}}{\partial y_1^k}\frac{\partial^2 f^2}{\partial x_1^h\partial x_1^i} + \tfrac{1}{4}\hat{G}^i_{kj}\frac{\partial\hat{g}^{js}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s} + \tfrac{1}{4}\hat{G}^i_j\frac{\partial^2\hat{g}^{js}}{\partial y_1^i\partial y_1^k}\frac{\partial f^2}{\partial x_1^s}\\
&+\tfrac{1}{4}\hat{G}^j_{ki}\frac{\partial\hat{g}^{ih}}{\partial y_1^j}\frac{\partial f^2}{\partial x_1^h} + \tfrac{1}{4}\hat{G}^j_i\frac{\partial^2\hat{g}^{ih}}{\partial y_1^j\partial y_1^k}\frac{\partial f^2}{\partial x_1^h} + \tfrac{1}{4}y_1^j\frac{\partial^3\hat{g}^{ih}}{\partial x_1^j\partial y_1^i\partial y_1^k}\frac{\partial f^2}{\partial x_1^h} + \tfrac{1}{4}y_1^j\frac{\partial^2\hat{g}^{ih}}{\partial y_1^i\partial y_1^k}\frac{\partial^2 f^2}{\partial x_1^j\partial x_1^h}\\
&-\tfrac{1}{2}\hat{G}^j_k\frac{\partial^2\hat{g}^{is}}{\partial y_1^j\partial y_1^i}\frac{\partial f^2}{\partial x_1^s} - \tfrac{1}{2}\hat{G}^j\frac{\partial^3\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^k}\frac{\partial f^2}{\partial x_2^s} - \tfrac{1}{2}\frac{\partial\hat{g}^{jh}}{\partial y_1^k}\frac{\partial f^2}{\partial x_1^h}\hat{G}^i_{ji} - \tfrac{1}{2}\hat{g}^{jh}\frac{\partial f^2}{\partial x_1^h}\hat{G}^i_{kji}\\
&+\tfrac{1}{4}\frac{1}{f^2}\frac{\partial\hat{g}^{ih}}{\partial y_1^k}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^i} - \tfrac{1}{2}\frac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial^2\hat{g}^{is}}{\partial y_1^i\partial y_1^k}] + \tfrac{1}{2}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial F_2^4}{\partial y_2^\nu}[-\tfrac{1}{16}\frac{\partial^2\hat{g}^{ih}}{\partial y_1^j\partial y_1^k}\frac{\partial\hat{g}^{js}}{\partial y_1^i}\\
&-\tfrac{1}{16}\frac{\partial\hat{g}^{ih}}{\partial y_1^j}\frac{\partial^2\hat{g}^{js}}{\partial y_1^i\partial y_1^k} + \tfrac{1}{8}\frac{\partial\hat{g}^{jh}}{\partial y_1^k}\frac{\partial^2\hat{g}^{is}}{\partial y_1^j\partial y_1^i} + \tfrac{1}{8}\hat{g}^{jh}\frac{\partial^3\hat{g}^{is}}{\partial y_1^j\partial y_1^i\partial y_1^k}] + \tfrac{1}{8}y_1^\beta\frac{\partial^2\hat{g}^{ih}}{\partial y_1^i\partial y_1^k}\frac{\partial f^2}{\partial x_1^h}\frac{\partial^2 F_2^2}{\partial x_2^\beta\partial y_2^\nu}\\
&-\tfrac{1}{4}\frac{\partial^2\hat{g}^{is}}{\partial y_1^i\partial y_1^k}\frac{\partial f^2}{\partial x_1^s}[\check{G}^\beta_\nu\frac{\partial F_2^2}{\partial y_2^\beta} + \check{G}^\beta\frac{\partial^2 F_2^2}{\partial y_2^\beta\partial y_2^\nu}],
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
Ric_{\mu\nu} = \check{Ric}_{\mu\nu} &+ \tfrac{1}{2}\frac{\partial^2 F_2^2}{\partial y_2^\mu \partial y_2^\nu}[-\tfrac{1}{2}\frac{\partial \hat{g}^{ih}}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^h} - \tfrac{1}{2}\hat{g}^{ih}\frac{\partial^2 f^2}{\partial x_1^h \partial x_1^i} + \tfrac{1}{4}y_1^j\frac{\partial^2 \hat{g}^{ih}}{\partial x_1^j \partial y_1^i}\frac{\partial f^2}{\partial x_1^h} \\
&+ \tfrac{1}{4}y_1^j\frac{\partial \hat{g}^{ih}}{\partial y_1^i}\frac{\partial^2 f^2}{\partial x_1^j \partial x_1^h} - \tfrac{1}{2}\hat{G}^j\frac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i}\frac{\partial f^2}{\partial x_1^s} - \tfrac{1}{2}\hat{g}^{jh}\frac{\partial f^2}{\partial x_1^h}\hat{G}_{ji}^i + \tfrac{1}{4}\tfrac{1}{f^2}\hat{g}^{ih}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^i} \\
&- \tfrac{1}{2}\tfrac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial \hat{g}^{is}}{\partial y_1^i} + \tfrac{1}{4}\hat{G}_j^i\frac{\partial \hat{g}^{js}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s} + \tfrac{1}{4}\hat{G}_i^j\frac{\partial \hat{g}^{ih}}{\partial y_1^j}\frac{\partial f^2}{\partial x_1^h}] \\
&- \tfrac{1}{4}\frac{\partial \hat{g}^{is}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s}[\check{G}_{\mu\nu}^\beta\frac{\partial F_2^2}{\partial y_2^\beta} + \check{G}_\mu^\beta\frac{\partial F_2^2}{\partial y_2^\beta \partial y_2^\nu} + \check{G}^\beta\frac{\partial^3 F_2^2}{\partial y_2^\beta \partial y_2^\mu \partial y_2^\nu} + \check{G}_\nu^\beta\frac{\partial F_2^2}{\partial y_2^\beta \partial y_2^\mu}] \\
&+ \tfrac{1}{2}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial^2 F_2^4}{\partial y_2^\mu \partial y_2^\nu}[-\tfrac{1}{16}\frac{\partial \hat{g}^{ih}}{\partial y_1^j}\frac{\partial \hat{g}^{js}}{\partial y_1^i} + \tfrac{1}{8}\hat{g}^{jh}\frac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i}] \\
&+ \tfrac{1}{8}y_2^\beta\frac{\partial \hat{g}^{ih}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^h}\frac{\partial^3 F_2^2}{\partial x_2^\beta \partial y_2^\mu \partial y_2^\nu} + \tfrac{1}{2}\tfrac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\check{G}_{\mu\nu\alpha}^\alpha.
\end{aligned}
\tag{5.8}
$$

Definition 5.1. *Warped product Finsler metric F is called Ricci-constant metric if there exists a constant $K \in R$ such that*

$$Ric_{ab} = Kg_{ab}. \tag{5.9}$$

In Riemannian geometry, the warped product manifold $(M, g) = M_1 \times_f M_2$ is Einstein with $Ric = kg$ if and only if $(M_2, \check{g})$ is Einstein, i.e., $\check{Ric} = k_2\check{g}$ for a constant k_2 and the followings hold [16]:

$$
\begin{aligned}
k\hat{g} &= \hat{Ric} - \tfrac{d}{f}H^f\ , \\
k &= \tfrac{k_2}{f^2} - \tfrac{\Delta f}{f} - (d-1)|\tfrac{\nabla f}{f}|_{\hat{g}}^2.
\end{aligned}
\tag{5.10}
$$

Now, we want to generalize this result on warped product Finsler manifolds. In fact, we answer Chern's question on warped product Finsler space. He asked "whenever a smooth manifold admits a Ricci-constant Finsler metric?"

Theorem 5.1. *Let $F = F_1 \times_f F_2$ be a warped product Finsler space. Consider the warped product Finsler metric F is a Ricci-constant of constant K then the following conditions hold:*

$$\hat{Ric}_{kl} = K\hat{g}_{kl} - \frac{1}{2}\frac{1}{f^2}\frac{\partial f^2}{\partial x_1^j}\hat{G}_{kl}^j,$$

$$
\begin{aligned}
K\check{g}_{\mu\nu} = \tfrac{1}{f^2}\check{Ric}_{\mu\nu} &+ \tfrac{1}{f^2}\check{g}_{\mu\nu}[-\tfrac{1}{2}\frac{\partial \hat{g}^{ih}}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^h} - \tfrac{1}{2}\hat{g}^{ih}\frac{\partial^2 f^2}{\partial x_1^i \partial x_1^h} - \tfrac{1}{2}\hat{G}^j\frac{\partial^2 g^{is}}{\partial y_1^j \partial y_1^i}\frac{\partial f^2}{\partial x_1^s} \\
&- \tfrac{1}{2}\hat{g}^{jh}\frac{\partial f^2}{\partial x_1^h}\hat{G}_{ji}^i + \tfrac{1}{4}\tfrac{1}{f^2}\hat{g}^{ih}\frac{\partial f^2}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^h} + \tfrac{1}{4}\hat{G}_j^i\frac{\partial g^{js}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s} + \tfrac{1}{4}\hat{G}_i^j\frac{\partial g^{ih}}{\partial y_1^j}\frac{\partial f^2}{\partial x_1^h}] + \tfrac{1}{2}\tfrac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\check{G}_{\mu\nu\alpha}^\alpha.
\end{aligned}
$$

Proof. By definition of Ricci-constant, $Ric_{k\nu} = 0$. So the follow equation holds

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{\partial F_2^2}{\partial y_2^\nu}[-\tfrac{1}{2}\tfrac{\partial^2 \hat{g}^{ih}}{\partial x_1^i \partial y_1^k}\tfrac{\partial f^2}{\partial x_1^h} - \tfrac{1}{2}\tfrac{\partial \hat{g}^{ih}}{\partial y_1^k}\tfrac{\partial^2 f^2}{\partial x_1^h \partial x_1^i} + \tfrac{1}{4}\hat{G}^i_{kj}\tfrac{\partial \hat{g}^{js}}{\partial y_1^i}\tfrac{\partial f^2}{\partial x_1^s} + \tfrac{1}{4}\hat{G}^i_j \tfrac{\partial^2 \hat{g}^{js}}{\partial y_1^i \partial y_1^k}\tfrac{\partial f^2}{\partial x_1^s}\\
&+\tfrac{1}{4}\hat{G}^j_{ki}\tfrac{\partial \hat{g}^{ih}}{\partial y_1^j}\tfrac{\partial f^2}{\partial x_1^h} + \tfrac{1}{4}\hat{G}^j_i \tfrac{\partial^2 \hat{g}^{ih}}{\partial y_1^j \partial y_1^k}\tfrac{\partial f^2}{\partial x_1^h} + \tfrac{1}{4}y_1^j \tfrac{\partial^3 \hat{g}^{ih}}{\partial x_1^j \partial y_1^i \partial y_1^k}\tfrac{\partial f^2}{\partial x_1^h} + \tfrac{1}{4}y_1^j \tfrac{\partial^2 \hat{g}^{ih}}{\partial y_1^i \partial y_1^k}\tfrac{\partial^2 f^2}{\partial x_1^j \partial x_1^h}\\
&-\tfrac{1}{2}\hat{G}^j_k \tfrac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i}\tfrac{\partial f^2}{\partial x_1^s} - \tfrac{1}{2}\hat{G}^j \tfrac{\partial^3 \hat{g}^{is}}{\partial y_1^j \partial y_1^i \partial y_1^k}\tfrac{\partial f^2}{\partial x_1^s} - \tfrac{1}{2}\tfrac{\partial \hat{g}^{jh}}{\partial y_1^k}\tfrac{\partial f^2}{\partial x_1^h}\hat{G}^i_{ji} - \tfrac{1}{2}\hat{g}^{jh}\tfrac{\partial f^2}{\partial x_1^h}\hat{G}^i_{kji}\\
&+\tfrac{1}{4}\tfrac{1}{f^2}\tfrac{\partial \hat{g}^{ih}}{\partial y_1^k}\tfrac{\partial f^2}{\partial x_1^h}\tfrac{\partial f^2}{\partial x_1^i} - \tfrac{1}{2}\tfrac{1}{f^2}y_1^h \tfrac{\partial f^2}{\partial x_1^h}\tfrac{\partial f^2}{\partial x_1^s}\tfrac{\partial^2 \hat{g}^{is}}{\partial y_1^i \partial y_1^k}] + \tfrac{1}{2}\tfrac{\partial f^2}{\partial x_1^h}\tfrac{\partial f^2}{\partial x_1^s}\tfrac{\partial F_2^4}{\partial y_2^\nu}[-\tfrac{1}{16}\tfrac{\partial^2 \hat{g}^{ih}}{\partial y_1^j \partial y_1^k}\tfrac{\partial \hat{g}^{js}}{\partial y_1^i}\\
&-\tfrac{1}{16}\tfrac{\partial \hat{g}^{ih}}{\partial y_1^j}\tfrac{\partial^2 \hat{g}^{js}}{\partial y_1^i \partial y_1^k} + \tfrac{1}{8}\tfrac{\partial \hat{g}^{jh}}{\partial y_1^k}\tfrac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i} + \tfrac{1}{8}\hat{g}^{jh}\tfrac{\partial^3 \hat{g}^{is}}{\partial y_1^j \partial y_1^i \partial y_1^k}] + \tfrac{1}{8}y_2^\beta \tfrac{\partial^2 \hat{g}^{ih}}{\partial y_1^i \partial y_1^k}\tfrac{\partial f^2}{\partial x_1^h}\tfrac{\partial^2 F_2^2}{\partial x_2^\beta \partial y_2^\nu}\\
&-\tfrac{1}{4}\tfrac{\partial^2 \hat{g}^{is}}{\partial y_1^i \partial y_1^k}\tfrac{\partial f^2}{\partial x_1^s}[\check{G}^\beta_\nu \tfrac{\partial F_2^2}{\partial y_2^\beta} + \check{G}^\beta \tfrac{\partial^2 F_2^2}{\partial y_2^\beta \partial y_2^\nu}] = 0.
\end{aligned} \tag{5.11}$$

Differentiating (5.11) with respect to y_1^l and then contracting it with $\frac{1}{2}y_2^\nu$ and replacing this result to (5.6), thus by applying (5.9) we get

$$\begin{aligned}
K\hat{g}_{kl} = \hat{Ric}_{kl} &- \tfrac{1}{2}\tfrac{\partial f^2}{\partial x_1^h}\tfrac{\partial f^2}{\partial x_1^s}F_2^4[-\tfrac{1}{16}\tfrac{\partial^3 \hat{g}^{ih}}{\partial y_1^j \partial y_1^k \partial y_1^l}\tfrac{\partial \hat{g}^{js}}{\partial y_1^i} - \tfrac{1}{16}\tfrac{\partial^2 \hat{g}^{ih}}{\partial y_1^j \partial y_1^k}\tfrac{\partial^2 \hat{g}^{js}}{\partial y_1^i \partial y_1^l} - \tfrac{1}{16}\tfrac{\partial^2 \hat{g}^{ih}}{\partial y_1^j \partial y_1^l}\tfrac{\partial^2 \hat{g}^{js}}{\partial y_1^i \partial y_1^k}\\
&-\tfrac{1}{16}\tfrac{\partial \hat{g}^{ih}}{\partial y_1^j}\tfrac{\partial^3 \hat{g}^{js}}{\partial y_1^i \partial y_1^k \partial y_1^l} + \tfrac{1}{8}\tfrac{\partial^2 \hat{g}^{jh}}{\partial y_1^k \partial y_1^l}\tfrac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i} + \tfrac{1}{8}\tfrac{\partial \hat{g}^{jh}}{\partial y_1^k}\tfrac{\partial^3 \hat{g}^{is}}{\partial y_1^j \partial y_1^i \partial y_1^l} + \tfrac{1}{8}\tfrac{\partial \hat{g}^{jh}}{\partial y_1^l}\tfrac{\partial^3 \hat{g}^{is}}{\partial y_1^j \partial y_1^i \partial y_1^k}\\
&+\tfrac{1}{8}\hat{g}^{jh}\tfrac{\partial^4 \hat{g}^{is}}{\partial y_1^j \partial y_1^i \partial y_1^k \partial y_1^l}] + \tfrac{1}{4}\check{G}^\beta \tfrac{\partial^3 \hat{g}^{is}}{\partial y_1^i \partial y_1^k \partial y_1^l}\tfrac{\partial f^2}{\partial x_1^s}\tfrac{\partial F_2^2}{\partial y_2^\beta} + \tfrac{1}{2}\tfrac{1}{f^2}\hat{G}^j_{kl}\tfrac{\partial f^2}{\partial x_1^j}.
\end{aligned} \tag{5.12}$$

If we differentiate (5.12) with respect to y_2^ν and then contract it with y_2^ν and apply the result to (5.12), we obtain

$$K\hat{g}_{kl} = \hat{Ric}_{kl} + \tfrac{1}{4}\check{G}^\beta \tfrac{\partial^3 \hat{g}^{is}}{\partial y_1^i \partial y_1^k \partial y_1^l}\tfrac{\partial f^2}{\partial x_1^s}\tfrac{\partial F_2^2}{\partial y_2^\beta} + \tfrac{1}{2}\tfrac{1}{f^2}\hat{G}^j_{kl}\tfrac{\partial f^2}{\partial x_1^j}. \tag{5.13}$$

Differentiating (5.13) again with respect to y_2^μ, then contracting it with y_2^μ and using this result to (5.13), we get

$$\hat{Ric}_{kl} = K\hat{g}_{kl} - \frac{1}{2}\frac{1}{f^2}\frac{\partial f^2}{\partial x_1^j}\hat{G}^j_{kl}. \tag{5.14}$$

Using (5.8) and (5.9), we deduce that

$$\begin{aligned} Kf^2\check{g}_{\mu\nu} &= \check{Ric}_{\mu\nu} + \frac{1}{2}\frac{\partial^2 F_2^2}{\partial y_2^\mu \partial y_2^\nu}[-\frac{1}{2}\frac{\partial \hat{g}^{ih}}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^h} - \frac{1}{2}\hat{g}^{ih}\frac{\partial^2 f^2}{\partial x_1^h \partial x_1^i} + \frac{1}{4}y_1^j\frac{\partial^2 \hat{g}^{ih}}{\partial x_1^j \partial y_1^i}\frac{\partial f^2}{\partial x_1^h} \\ &+\frac{1}{4}y_1^j\frac{\partial \hat{g}^{ih}}{\partial y_1^i}\frac{\partial^2 f^2}{\partial x_1^j \partial x_1^h} - \frac{1}{2}\hat{G}^j\frac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i}\frac{\partial f^2}{\partial x^s} - \frac{1}{2}\hat{g}^{jh}\frac{\partial f^2}{\partial x_1^h}\hat{G}^i_{ji} + \frac{1}{4}\frac{1}{f^2}\hat{g}^{ih}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^i} \\ &-\frac{1}{2}\frac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial \hat{g}^{is}}{\partial y_1^i} + \frac{1}{4}\hat{G}^i_j\frac{\partial \hat{g}^{js}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s} + \frac{1}{4}\hat{G}^j_i\frac{\partial \hat{g}^{ih}}{\partial y_1^j}\frac{\partial f^2}{\partial x_1^h}] + \frac{1}{2}\frac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\check{G}^\alpha_{\mu\nu\alpha} \\ &-\frac{1}{4}\frac{\partial \hat{g}^{is}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s}[\check{G}^\beta_{\mu\nu}\frac{\partial F_2^2}{\partial y^\beta} + \check{G}^\beta_\mu\frac{\partial F_2^2}{\partial y_2^\beta \partial y_2^\nu} + \check{G}^\beta\frac{\partial^3 F_2^2}{\partial y_2^\beta \partial y_2^\mu \partial y_2^\nu} + \check{G}^\beta_\nu\frac{\partial F_2^2}{\partial y_2^\beta \partial y_2^\mu}] \\ &+\frac{1}{2}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^s}\frac{\partial^2 F_2^4}{\partial y_2^\mu \partial y_2^\nu}[-\frac{1}{16}\frac{\partial \hat{g}^{ih}}{\partial y_1^j}\frac{\partial \hat{g}^{js}}{\partial y_1^i} + \frac{1}{8}\hat{g}^{jh}\frac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i}] + \frac{1}{8}y_2^\beta\frac{\partial \hat{g}^{ih}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^h}\frac{\partial^3 F_2^2}{\partial x_2^\beta \partial y_2^\mu \partial y_2^\nu}. \end{aligned} \tag{5.15}$$

Let us different of the above equation with respect to y_1^k and contract it by y_1^k, then replace this result to (5.15), we obtain

$$\begin{aligned} Kf^2\check{g}_{\mu\nu} &= \check{Ric}_{\mu\nu} + \check{g}_{\mu\nu}[-\frac{1}{2}\frac{\partial \hat{g}^{ih}}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^h} - \frac{1}{2}\hat{g}^{ih}\frac{\partial^2 f^2}{\partial x_1^h \partial x_1^i} - \frac{1}{2}\hat{G}^j\frac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i}\frac{\partial f^2}{\partial x_1^s} \\ &-\frac{1}{2}\hat{g}^{jh}\frac{\partial f^2}{\partial x_1^h}\hat{G}^i_{ji} + \frac{1}{4}\frac{1}{f^2}\hat{g}^{ih}\frac{\partial f^2}{\partial x_1^h}\frac{\partial f^2}{\partial x_1^i} + \frac{1}{4}\hat{G}^i_j\frac{\partial \hat{g}^{js}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s} + \frac{1}{4}\hat{G}^j_i\frac{\partial \hat{g}^{ih}}{\partial y_1^j}\frac{\partial f^2}{\partial x_1^h}] \\ &+3\check{g}_{\mu\nu}F_2^2[\frac{1}{8}\frac{\partial \hat{g}^{ih}}{\partial y_1^j}\frac{\partial \hat{g}^{js}}{\partial y_1^i} - \frac{1}{4}\hat{g}^{jh}\frac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i}] + \frac{1}{2}\frac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\check{G}^\alpha_{\mu\nu\alpha}. \end{aligned} \tag{5.16}$$

Differentiating of (5.16) again with respect to y_1^l and then contracting it with y_1^l and using this result to (5.16), We obtain the desired result. This completes the proof of theorem. □

The converse of the above theorem is established by placing an additional condition.

Theorem 5.2. *Let $F = F_1 \times_f F_2$ be a warped product Finsler space. The warped product Finsler metric F is a Ricci-constant metric of constant K if the following conditions are satisfied:*

$$\begin{aligned} Kf^2\check{g}_{\mu\nu} &= \check{Ric}_{\mu\nu} + \check{g}_{\mu\nu}[-\frac{1}{2}\frac{\partial \hat{g}^{ih}}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^h} - \frac{1}{2}\hat{g}^{ih}\frac{\partial^2 f^2}{\partial x_1^i \partial x_1^h} - \frac{1}{2}\hat{G}^j\frac{\partial^2 g^{is}}{\partial y_1^j \partial y_1^i}\frac{\partial f^2}{\partial x_1^s} \\ &-\frac{1}{2}\hat{g}^{jh}\frac{\partial f^2}{\partial x_1^h}\hat{G}^i_{ji} + \frac{1}{4}\frac{1}{f^2}\hat{g}^{ih}\frac{\partial f^2}{\partial x_1^i}\frac{\partial f^2}{\partial x_1^h} + \frac{1}{4}\hat{G}^i_j\frac{\partial g^{js}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s} + \frac{1}{4}\hat{G}^j_i\frac{\partial g^{ih}}{\partial y_1^j}\frac{\partial f^2}{\partial x_1^h}] + y_1^h\frac{\partial f^2}{\partial x_1^h}\check{G}^\alpha_{\mu\nu\alpha}, \end{aligned} \tag{5.17}$$

$$K\hat{g}_{kl} = \hat{Ric}_{kl} + \frac{1}{2}\frac{1}{f^2}\frac{\partial f^2}{\partial x_1^j}\hat{G}^j_{kl}, \tag{5.18}$$

$$\begin{aligned} &\frac{\partial g^{is}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s}\check{G}^\beta\frac{\partial F_2^2}{\partial y_2^\beta} - \frac{1}{2}y_2^\beta\frac{\partial g^{ih}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^h}\frac{\partial F_2^2}{\partial x_2^\beta} = F_2^4[\frac{1}{4}\hat{g}^{jh}\frac{\partial^2 \hat{g}^{is}}{\partial y_1^j \partial y_1^i} - \frac{1}{8}\frac{\partial \hat{g}^{ih}}{\partial y_1^j}\frac{\partial \hat{g}^{js}}{\partial y_1^i}] \\ &+F_2^2[\frac{1}{2}y_1^j\frac{\partial^2 g^{ih}}{\partial x_1^j \partial y_1^i}\frac{\partial f^2}{\partial x_1^h} + \frac{1}{2}y_1^j\frac{\partial g^{ih}}{\partial y_1^i}\frac{\partial^2 f^2}{\partial x_1^h \partial x_1^j} - \frac{1}{f^2}y_1^h\frac{\partial f^2}{\partial x_1^h}\frac{\partial g^{is}}{\partial y_1^i}\frac{\partial f^2}{\partial x_1^s}]. \end{aligned} \tag{5.19}$$

Proof. Let us different (5.19) with respect to y_1^k and y_2^ν and different (5.17) with respect to y_1^k, then substitu these results to (5.7), we obtain $Ric_{k\nu} = 0$.
If we different (5.17) with respect to y_1^k and y_1^l and then contract it with $\check{g}^{\mu\nu}$ and different also of (5.19) with respect to y_1^k and y_1^l, then apply these results to (5.6), we get $Ric_{kl} = K\hat{g}_{kl}$.
Finally, differentiating (5.19) with respect to y_2^μ and y_2^ν and using (5.17) and (5.8) we obtain $Ric_{\mu\nu} = Kf^2\check{g}_{\mu\nu}$.
Therefore the warped product Finsler metric F is a Ricci-constant metric of constant K. □

In Finsler geometry, the warped product Finsler space is Riemmanian if and only if the fiber and the base space are Riemmanian [1]. Thus we can state the following corollary.

Corollary 5.1. *Let F_1 and F_2 be Riemmanian spaces. The warped product Finsler space $F = F_1 \times_f F_2$ is a Ricci-constant of constant K if and only if the fiber space is a Ricci-constant of constant K_2 i.e. $\check{Ric}_{\mu\nu} = K_2\check{g}_{\mu\nu}$ and the followings hold:*

$$\hat{Ric}_{kl} = K\hat{g}_{kl} - \frac{1}{2}\frac{1}{f^2}\frac{\partial f^2}{\partial x_1^j}\hat{G}^j_{kl},$$

$$Kf^2 - K_2 = \frac{1}{4}\frac{1}{f^2}\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2) - \frac{1}{2}\hat{\Delta}_h f^2.$$

Applying Theorems 5.1 and 5.2 and the spectral lemma of the horizontal Laplacian on Finsler manifolds, we may derive new results about special warped product Finsler spaces.

Example 5.1. *The warped product Finsler space $F = F_1 \times_f F_2$ when the base space is a Riemannian and the fiber space is a locally Minkowski space, is Ricci-constant of constant K if and only if the following equations are satisfied*

$$K\hat{g}_{kl} = \hat{Ric}_{kl} + \frac{1}{2}\frac{1}{f^2}\frac{\partial f^2}{\partial x_1^j}\hat{G}^j_{kl},$$

$$K = -\frac{1}{2}\frac{1}{f^2}\hat{\Delta}_h f^2 + \frac{1}{4}\frac{1}{f^4}\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2).$$

Proof. Using (4.1), (4.2), (5.18), (5.17) and (5.19) when the base space is Riemannian and the fiber space is locally Minkowski space. □

Corollary 5.2. *Consider the base space be Riemannian and the fiber space be Berwald space. The warped product Finsler space $F = F_1 \times_f F_2$ is a Ricci-constant of constant K if and only if F_2 is Ricci-constant i.e. $\check{Ric}_{\mu\nu} = K_2\check{g}_{\mu\nu}$ and the followings hold:*

$$K\hat{g}_{kl} = \hat{Ric}_{kl} + \frac{1}{2}\frac{1}{f^2}\frac{\partial f^2}{\partial x_1^j}\hat{G}^j_{kl},$$

$$K = \frac{1}{f^2}K_2 - \frac{1}{2}\frac{1}{f^2}\hat{\Delta}_h f^2 + \frac{1}{4}\frac{1}{f^4}\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2).$$

Proof. Applying Theorems 5.1 and 5.2, and also (4.1), (5.1) whereby the base space is Riemmanian and the fiber space is Berwald. We know that in this case $\hat{g}_{ij}$ and $\check{G}^i_{jk}$ are independent of y_1 and y_2, respectively. □

Now we consider the case of the base space is locally Minkowski.

Lemma 5.1. *Let the base space F_1 be locally Minkowski space. The Ricci-constant warped product Finsler space $F = F_1 \times_f F_2$ must be Ricci flat.*

Proof. We know that $\hat{G}^j = 0$ when the base space is locally Minkowski, so by (5.18) we implies that $K = 0$. □

An immediate consequence of this Lemma is the following.

Corollary 5.3. *Let F_1 and F_2 be locally Minkowski space and the warped product Finsler space $F = F_1 \times_f F_2$ be Ricci flat. Then we have*

$$\hat{\Delta}_h f^2 = \frac{1}{2}\frac{1}{f^2}\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2).$$

Remark 5.3. *Note that if F_1 and F_2 are locally Minkowski space, the warped product Finsler $F = F_1 \times_f F_2$ is not necessary locally Minkowski.*

By Lemma 5.1, we can classify Ricci flat warped product Finsler space F whereby the base space is locally Minkowski space.

Corollary 5.4. *Let the warped product Finsler space $F = F_1 \times_f F_2$ be a Ricci-constant whereby F_1 be a locally Minkowski space and F_2 be a Berwald (or Riemmanian) space. Then the fiber space F_2 is a Ricci-constant with constant*

$$K_2 = \frac{1}{2}\hat{\Delta}_h f^2 - \frac{1}{4}\frac{1}{f^2}\hat{G}(\hat{\nabla}_h f^2, \hat{\nabla}_h f^2).$$

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