Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.11, (2022), Issue 2, pp.198-214

GEOMETRIC INEQUALITIES FOR LAGRANGIAN SUBMANIFOLDS IN STATISTICAL QUATERNIONIC SPACE FORM

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ABSTRACT. In this paper, we establish two geometrics inequalities for statistical Lagrangian submanifolds in quaternionic statistical space form: Wintgen-type inequality and a Chen invariant. We give an example of quaternionic statistical manifold. We use a recent generalized algebraic inequality of [16] to obtain a particular of Chen invariant: $\delta\underbrace{(2, 2, \ldots, 2)}_{p-times}$ in statistical case. Finally we characterize the minimal Lagrangian submanifolds in the statistical quaternionic space forms.

1. INTRODUCTION

A statistical manifold is a natural generalization of statistical model. In 1985, Amari introduced the notion of statistical manifolds via information geometry [1]. These manifolds are equipped with dual connections (torsion free), an analogue to conjugate connections in affine geometry. The quaternions were discovered by William Rowan Hamilton, in 1853, after long unsuccessful attempts to construct "three dimensional complex numbers". The concept of quaternionic manifolds was introduced by Ishihara in 1974, [10]. Later, Vîlcu et al extended this concept to the statistical manifold [21]. The theory of statistical manifold and its statistical submanifolds plays a major role in differential geometry and its related fields. On the other hand, the Wintgen inequality which gives a relation between the square of the mean curvature, normalized scalar curvature and normalized normal scalar curvature has been an important research subject for various submanifolds in various ambient manifolds, for instance see [15] for complex case. Moreover, Wintgen inequality for statistical submanifolds has been obtained in [2, 3, 9, 17]. In 2015, A.D. Vîlcu and G.E. Vîlcu [21], studied statistical manifolds in quaternionic settings and proposed several open problems. While answering one of those open problems Aquib [2], obtained some of the curvature properties of submanifolds and a couple of inequalities for totally real statistical submanifolds of quaternionic Kahler-like staistical space forms. Recently, M. S. Lone and M. A. Lone, [11], obtained a basic inequality for statistical submanifolds of quaternionic Kahler-like manifolds. A.Mihai and I.Mihai, [15] obtained the generalized Wintgen inequality for Langrangian statistical submanifolds of holomorphic statistical manifolds with constant holomorphic sectional curvature c.

2010 *Mathematics Subject Classification.* Primary: 53C05, 53C26, 53C40, 53C25, 53C15, 53D12, 53B35.
Key words and phrases. Statistical Manifolds, Wintgen Inequality, $\delta(2, \ldots, 2)$-Invariant, Quaternionic Kahler-like Statistical Manifolds, Lagrangian Submanifolds, Minimal Submanifolds.

In the present paper, we establish the generalized Wintgen-like inequality for Lagrangian statistical submanifolds of quaternionic Kahler-like statistical space forms, generalizing the results in [13]. Also, We establish a Chen invariant $\delta(2,\ldots,2)$ for such submanifolds and we characterize the minimal Lagrangian submanifolds in the statistical quaternionic space forms, generalizing the results in [11] .

2. Preliminaries

A statistical manifold is a Riemannian manifold $(\overline{M},\bar{g})$ endowed with a pair of torsion-free affine connections $\overline{\nabla}$ and $\overline{\nabla}^{\star}$ satisfying

$$Z.\bar{g}(X,Y)=\bar{g}(\overline{\nabla}_Z X,Y)+\bar{g}(X,\overline{\nabla}^{\star}_Z Y), \tag{2.1}$$

for any X,Y and $Z\in\Gamma(T\overline{M})$. It is denoted by $(\overline{M},\bar{g},\overline{\nabla},\overline{\nabla}^{\star})$. The connections $\overline{\nabla}$ and $\overline{\nabla}^{\star}$ are called dual connections and it is easy to show that $(\overline{\nabla}^{\star})^{\star}=\overline{\nabla}$. The pair $(\overline{\nabla},\bar{g})$ is said to be a statistical structure if

$$(\overline{\nabla}_X\bar{g})(Y,Z)=(\overline{\nabla}_Y\bar{g})(X,Z). \tag{2.2}$$

If $(\overline{\nabla},\bar{g})$ is a statistical structure on $\overline{M}$, then $(\overline{\nabla}^{\star},\bar{g})$ is also statistical structure on $\overline{M}$.
On the other hand, any torsion-free connection $\overline{\nabla}$ always has a dual connection given by

$$\overline{\nabla}+\overline{\nabla}^{\star}=2\overline{\nabla}^{\circ}, \tag{2.3}$$

where $\overline{\nabla}^{\circ}$ is the Levi-Civita connection for $\overline{M}$. Denote by $\overline{R}$ and $\overline{R}^{\star}$ the curvature tensor fields of $\overline{\nabla}$ and $\overline{\nabla}^{\star}$, respectively. Then the curvature tensor fields $\overline{R}$ and $\overline{R}^{\star}$ satisfies

$$\bar{g}(\overline{R}(X,Y)Z,W)=-\bar{g}(\overline{R}^{\star}(X,Y)W,Z). \tag{2.4}$$

Let $(M^n,g,\nabla,\nabla^{\star})$ be a statistical submanifold of $(\overline{M}^m,\bar{g},\overline{\nabla},\overline{\nabla}^{\star})$. In what follows, we will suppose that the induced metric g equal to $\bar{g}$. Then according to [22], the corresponding Gauss formulas are given by:

$$\overline{\nabla}^{\star}_X Y=\nabla^{\star}_X Y+h^{\star}(X,Y), \tag{2.5}$$

$$\overline{\nabla}_X Y=\nabla_X Y+h(X,Y), \tag{2.6}$$

where h and $h^{\star}$ are symmetric and bilinear, called imbedding curvature tensor of M in $\overline{M}$ for $\overline{\nabla}$ and the imbedding curvature tensor of M in $\overline{M}$ for $\overline{\nabla}^{\star}$, respectively. Let us denote the normal bundle of M by $\Gamma(TM^{\perp})$. Since h and $h^{\star}$ are bilinear, we have the linear transformations A_ξ and $A^{\star}_\xi$ defined by:

$$\bar{g}(A_\xi X,Y)=\bar{g}(h(X,Y),\xi), \tag{2.7}$$

$$\bar{g}(A^{\star}_\xi X,Y)=\bar{g}(h^{\star}(X,Y),\xi), \tag{2.8}$$

for any $\xi\in\Gamma(TM^{\perp})$ and $X,Y\in\Gamma(TM)$.
The corresponding Weingarten formulas [22], are:

$$\overline{\nabla}_X\xi=-A_\xi X+\nabla^{\perp}_X\xi, \tag{2.9}$$

$$\overline{\nabla}^{\star}_X\xi = -A^{\star}_{\xi}X + \nabla^{\star\perp}_X\xi, \tag{2.10}$$

for any $\xi \in \Gamma(TM^{\perp})$ and $X \in \Gamma(TM)$. The connections $\nabla^{\perp}$ and $\nabla^{\star\perp}$ given in the above equations are Riemannian dual connections with respect to the induced metric on $\Gamma(TM^{\perp})$.
The corresponding Gauss, Codazzi and Ricci equations [22] are given by the following results

Proposition 2.1. *Let $\overline{\nabla}$ be a dual connection on $\overline{M}$ and ∇ the induced connection on M. Let $\overline{R}$ and R be the curvature tensors of $\overline{\nabla}$ and ∇, respectively. Then,*

$$\bar{g}(\overline{R}(X,Y)Z,W) = \bar{g}(R(X,Y)Z,W) + \bar{g}(h(X,Z),h^{\star}(Y,W)) - \bar{g}(h^{\star}(X,W),h(Y,Z)), \tag{2.11}$$

$$(\overline{R}(X,Y)Z)^{\perp} = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y), \tag{2.12}$$

$$\bar{g}(R^{\perp}(X,Y)\xi,\eta) = \bar{g}(\overline{R}(X,Y)\xi,\eta) + \bar{g}([A^{\star}_{\xi}, A_{\eta}]X,Y), \tag{2.13}$$

where $R^{\perp}$ is the curvature tensor on $TM^{\perp}$, $\xi, \eta \in \Gamma(TM^{\perp})$ and $[A^{\star}_{\xi}, A_{\eta}] = A^{\star}_{\xi}A_{\eta} - A_{\eta}A^{\star}_{\xi}$.

Similarly, for the dual connection $\overline{\nabla}^{\star}$ on $\overline{M}$, we have the following result

Proposition 2.2. *Let $\overline{\nabla}^{\star}$ be a dual connection on $\overline{M}$ and $\nabla^{\star}$ the induced connection on M. Let $\overline{R}^{\star}$ and $R^{\star}$ be the curvature tensors of $\overline{\nabla}^{\star}$ and $\nabla^{\star}$, respectively. Then,*

$$\bar{g}(\overline{R}^{\star}(X,Y)Z,W) = \bar{g}(R^{\star}(X,Y)Z,W) + \bar{g}(h^{\star}(X,Z),h(Y,W)) - \bar{g}(h(X,W),h^{\star}(Y,Z)), \tag{2.14}$$

$$(\overline{R}^{\star}(X,Y)Z)^{\perp} = \nabla^{\star\perp}_X h(Y,Z) - h^{\star}(\nabla^{\star}_X Y, Z) - h^{\star}(\nabla^{\star}_X Z, Y), \tag{2.15}$$

$$\bar{g}(R^{\star\perp}(X,Y)\xi,\eta) = \bar{g}(\overline{R}^{\star}(X,Y)\xi,\eta) + \bar{g}([A_{\xi}, A^{\star}_{\eta}]X,Y), \tag{2.16}$$

where $R^{\star\perp}$ is the curvature tensor for $\nabla^{\star\perp}$ on $TM^{\perp}$. $\xi, \eta \in \Gamma(TM^{\perp})$ and $[A_{\xi}, A^{\star}_{\eta}] = A_{\xi}A^{\star}_{\eta} - A^{\star}_{\eta}A_{\xi}$.

The curvature tensors $\bar{R}$, $\bar{R}^{\star}$, R and $R^{\star}$ are not Riemann curvature tensors. In [17] Opozda defined a curvature tensor which can play the role of the Riemann curvature tensor. The statistical curvature tensors S and $\bar{S}$ on statistical manifolds (M, g, ∇) and $(\bar{M}, \bar{g}, \bar{\nabla})$ respectively, are defined by:

$$\bar{S}(X,Y,Z,W) = \frac{1}{2}\left(\bar{R}(X,Y,Z,W) + \bar{R}^{\star}(X,Y,Z,W)\right), \tag{2.17}$$

$$S(X,Y,Z,W) = \frac{1}{2}\left(R(X,Y,Z,W) + R^{\star}(X,Y,Z,W)\right). \tag{2.18}$$

Clearly $\bar{S}$ is skew-symmetric, but in general Shur's lemma does not hold for the sectional curvature associated to $\bar{S}$. Thus $\bar{S}$ is a Riemannian curvature-like tensor. The sectional curvature associated to $\bar{S}$ is called the sectional $\bar{\nabla}-$curvature of $(\bar{M}, \bar{g}, \bar{\nabla})$ and it is given by:

$$K(\pi) = K(X \wedge Y) = \bar{g}(\bar{S}(X,Y)Y,X) = \bar{S}(X,Y,Y,X),$$

where $\{X, Y\}$ is an orthonormal basis of a non degenerate two dimensional subspace π of the tangent space $T_x\bar{M}$, at a point $x \in \bar{M}$. Let M be an n-dimensional submanifold of $\bar{M}$, $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_xM and $\{\xi_{n+1}, \ldots, \xi_m\}$ be an orthonormal basis of $T_x^{\perp}M$. The scalar curvature τ at x is given by:

$$\tau(x) = \sum_{1\leq i<j\leq n} K(e_i \wedge e_j),$$

where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by e_i and e_j. And the normalized scalar curvature ρ of M at x is defined by:

$$\rho(x) = \frac{2\tau(x)}{n(n-1)}, \tag{2.19}$$

see [12] for more details.
The normal scalar curvature $\tau^{\perp}$ at x is given by :

$$\tau^{\perp}(x) = \left(\sum_{1\leq i<j\leq n}\ \sum_{n+1\leq r<\leq m} (S^{\perp}(e_i, e_j, \xi_r, \xi_s))^2\right)^{\frac{1}{2}}$$

and the normalized normal scalar curvature $\rho^{\perp}$ of M at x is defined by :

$$\rho^{\perp}(x) = \frac{2\tau^{\perp}(x)}{n(n-1)}. \tag{2.20}$$

see [12] for more details.
Chen introduced a sequence of Riemannian invariants $\delta(n_1, \ldots, n_k)$ known as Chen invariants [6]. The Chen first invariant is $\delta_M = \tau - \inf K$, where $(\inf K)(x) = \inf\{K(\pi) : \pi\}$.
The Chen invariant $\delta(n_1, \ldots, n_k)$ is defined by $\delta(n_1, \ldots, n_k)(x) = \tau(x) - \inf\{\tau(M_1) + \cdots + \tau(M_k)\}$, where $M_1, \ldots, M_k$ are k mutually orthogonal subspaces of $T_x\bar{M}$ such that $dimM_j = n_j$, $j = 1 \ldots, k$ and $n_1 + \cdots + n_k < m$. In our work, we will focus on $\delta(2, \ldots, 2)$.

3. Statistical Submanifolds of Quaternionic Kahler-like Statistical Space Form

Let $\overline{M}$ be a $4m$-dimensional smooth manifold and σ be a 3-subbundle of $End(T\overline{M})$ such that a canonical basis $\{J_1, J_2, J_3\}$ exists on a section σ satisfying for all $\alpha \in \{1, 2, 3\}$:

$$J_\alpha^2 = -Id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1}J_\alpha = J_{\alpha+2}, \tag{3.1}$$

where Id denotes the identity tensor field of type $(1, 1)$ on $\overline{M}$ and the indices are taken from $\{1, 2, 3\}$ modulo 3. Then $\{J_1, J_2, J_3\}$ is called a canonical basis of σ and σ is called an almost quaternionnic structure on $\overline{M}$. Moreover, $(\overline{M}, \sigma)$ is said to be almost quaternionic manifold. The dimension of such manifold is $4m$, $m \geq 1$.
A semi-Riemannian metric $\bar{g}$ on $\overline{M}$ is said to be adapted to the almost quaternionic structure σ if it satisfies:

$$\bar{g}(J_\alpha X, J_\alpha Y) = \bar{g}(X, Y), \quad \alpha \in \{1, 2, 3\}, \tag{3.2}$$

for all vectors fields X, Y on $\overline{M}$ and any local basis $\{J_1, J_2, J_3\}$ of σ. Moreover, $(\overline{M}, \sigma, \bar{g})$ is said to be an almost Hermitian quaternionic manifold, [4].

Definition 3.1. *Let $(\overline{M}, \bar{g})$ be a semi-Riemannian manifold endowed with an almost quaternionic structure σ which has for any canonical local basis $\{J_1, J_2, J_3\}$ of σ three other tensor fields $\{J_1^\star, J_2^\star, J_3^\star\}$ of type $(1,1)$ on $\overline{M}$, satisfying:*

$$\bar{g}(J_\alpha X, Y) + \bar{g}(X, J_\alpha^\star Y) = 0, \quad \alpha \in \{1,2,3\}, \tag{3.3}$$

for all vectors fields X, Y on $\overline{M}$. Then $(\overline{M}, \sigma, \bar{g})$ is said to be an almost Hermite-like quaternionic manifold. Moreover, if $(\overline{M}, \sigma, \bar{g})$ is equipped with a torsion-free linear connection $\overline{\nabla}$ such that $\overline{\nabla} g$ is symmetric, then $(\overline{M}, \overline{\nabla}, \sigma, \bar{g})$ is said to be an almost Hermite-like quaternionic statistical manifold.

Note that $\{J_1^\star, J_2^\star, J_3^\star\}$ defined by (3.3) satisfy (3.1) and hence we can consider the subbundle $\sigma^\star$ of $End(T\overline{M})$ locally spanned by $\{J_1^\star, J_2^\star, J_3^\star\}$. Observe that :

$$(J_\alpha^\star)^\star = J_\alpha$$

and

$$\bar{g}(J_\alpha X, J_\alpha^\star Y) = \bar{g}(X, Y), \quad \alpha \in \{1,2,3\}. \tag{3.4}$$

Definition 3.2. *Let $(\overline{M}, \overline{\nabla}, \sigma, \bar{g})$ be an almost Hermite-like quaternionic statistical manifold. Then $(\overline{M}, \overline{\nabla}, \sigma, \bar{g})$ is said to be a quaternionic Kahler-like statistical manifold if for any local basis $\{J_1, J_2, J_3\}$ of σ there exist three locally defined 1-forms $\{\omega_1, \omega_2, \omega_3\}$ on $\overline{M}$ such that for all $\alpha \in \{1,2,3\}$:*

$$(\overline{\nabla}_X J_\alpha) Y = \omega_{\alpha+2}(X) J_{\alpha+1} Y - \omega_{\alpha+1}(X) J_{\alpha+2} Y, \tag{3.5}$$

for all vector fields X, Y on $\overline{M}$, where the indices are taken $\{1,2,3\}$ modulo 3.

Definition 3.3. [19] *Let $(\overline{M}, \overline{\nabla}, \sigma, \bar{g})$ be a quaternionic Kahler-like statistical manifold. If the curvature tensor $\overline{R}$ with respect to $\overline{\nabla}$ satisfies*

$$\begin{aligned} \overline{R}(X,Y)Z &= \frac{c}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \sum_{\alpha=1}^{3}[\bar{g}(Z, J_\alpha Y)J_\alpha X - \bar{g}(Z, J_\alpha X)J_\alpha Y] \qquad (3.6)\\ &+ \sum_{\alpha=1}^{3}[\bar{g}(X, J_\alpha Y)J_\alpha Z - \bar{g}(Y, J_\alpha X)J_\alpha Z]\}, \end{aligned}$$

for all vectors fields X, Y and Z on $\overline{M}$, where c is a real constant , then the statistical manifold $(\overline{M}, \overline{\nabla}, \sigma, \bar{g})$ is said to be of type quaternionic space form.

Proposition 3.1. *The statistical manifold $(\overline{M}, \overline{\nabla}, \sigma, \bar{g})$ is a quaternioninic space form if and only if $(\overline{M}, \overline{\nabla}^*, \sigma^*, \bar{g})$ is.*

Proof. We use the relation $\bar{g}(\overline{R}(X,Y)Z, U) = -\bar{g}(\overline{R}^*(X,Y)U, Z)$ and $\bar{g}(J_\alpha X, Y) = -\bar{g}(X, J_\alpha^* Y)$ to prove it. □

3.1. **Examples of statistical quaternionic space form.** In his thesis [4], Edmond Bonan showed how to define quaternionic structures on four types of Riemannian manifolds. One of the methods is to consider the tangent bundle of an almost Hermitian manifold. This method was used by Vilcu [21] to construct a quaternionic statistical manifold. In the following we give a canonical example and we construct a quaternionic statistical manifold with zero sectional curvature.

Example 3.1. *Any form of quaternionic space form with a flat sectional curvature and a canonical metric is a statistical quaternionic space form , with* $J_\alpha = J_\alpha^\star$, $\alpha = 1, 2, 3$.

Example 3.2. *Let* $(\mathbb{R}_2^4, g)$ *a semi-Riemannian manifold with a quaternionic structure* (J_α), $\alpha = 1, 2, 3$ *such that*

$$J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; J_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}; J_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ . \end{pmatrix}$$

The suitable metric g is given by

$$g = 2dx_1^2 + 2dx_2^2 - dx_3^2 - dx_4^2$$

Using (3.4), *after computations we find:*

$$J_1^\star = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; J_2^\star = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}; J_3^\star = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ . \end{pmatrix}$$

It is clear that, (J_α^*), $\alpha = 1, 2, 3$ *verify* (3.4).
A submanifold M in statistical quaternionic space form $(\bar{M}, \bar{g}, (J_i)_{i=1,2,3})$ *is called totally real if* $(J_i)_{i=1,2,3}$ *maps to each tangent space* T_xM *into its corresponding space* $(T_x^\perp M)$, *i.e ,* $J_iT_xM \subset T_x^\perp M$.

Definition 3.4. *A totally real submanifold of maximal dimension is called Lagrangian submanifold.*

Let $\{e_1, \ldots, e_n\}$ and $\{\xi_{n+1}, \ldots, \xi_{4m}\}$ be a tangent orthonormal frame and a normal orthonormal frame, respectively, on M. The mean curvature vector field is given by

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) \tag{3.7}$$

and

$$H^* = \frac{1}{n} \sum_{i=1}^{n} h^*(e_i, e_i). \tag{3.8}$$

For the Levi-Civita connexion ∇^0, we denote by $2h^0 = h + h^\star$ its second fundamental form, and by $2H^0 = H + H^\star$ its mean curvature vector field of M.
The squared mean curvature of M for ∇ and $\nabla^\star$ are given by

$$\| h \|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) \tag{3.9}$$

and

$$\| h^* \|^2 = \sum_{i,j=1}^{n} g(h^*(e_i, e_j), h^*(e_i, e_j)) \tag{3.10}$$

Definition 3.5. [5] *Let M be a submanifold of a statistical manifold $\overline{M}$. Then M is said to be a*

(i) *totally geodesic with respect to $\overline{\nabla}$ if $h = 0$.*
(i^*) *totally geodesic with respect to $\overline{\nabla}^*$ if $h^* = 0$.*
(ii) *totally tangentially umbilical with respect to $\overline{\nabla}$ if $h(X, Y) = \bar{g}(X, Y)H$.*
(ii^*) *totally tangentially umbilical with respect to $\overline{\nabla}^*$ if $h^*(X, Y) = \bar{g}(X, Y)H^*$.*
(iii) *totally normally umbilical with respect to $\overline{\nabla}$ if $A_\xi X = \bar{g}(H, \xi)X$.*
(iii^*) *totally normally umbilical with respect to $\overline{\nabla}$ if $A^*_\xi X = \bar{g}(H^*, \xi)X$.*

Proposition 3.2. *Let M be a totally real statistical submanifold immersed into quaternionic space forms $\overline{M}(c)$. If M is totally umbilical submanifold with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$, then the sectional curvature $K = \frac{c}{4}$ if and only if any one of the following holds:*

(a) *Both H and H^* are perpendicular to each other;*
(b) *$H = 0$;*
(c) *$H^* = 0$.*

Proof. Suppose that $K = \frac{c}{4}$. Following (2.11) , (2.14), (2.17), (2.18) and defintion 3.8 the sectional curvature K on M is given by

$$K(X, Y) = \frac{c}{4} + g(H, H^\star), \tag{3.11}$$

for any orthonormal vectors $X, Y \in \Gamma(TM)$. If $K = \frac{c}{4}$, then any one of the following holds

(a) Both H and H^* are perpendicular to each other;
(b) $H = 0$;
(c) $H^* = 0$.

The converse part is easy to proof if any one of the above holds. □

4. Generalized Wintgen Inequality

In this section, we prove a generalized Wintgen-like inequality for statistical Lagrangian submanifolds in a statistical quaternionic space forms. And with the help of this inequality we obtain a sufficient condition of existence of a minimal Lagrangian submanifold.

4.1. **Main Result.** Ion Mihai (see [15]) obtained the $DDVV$ inequality, also known as generalized Wintgen inequality for Lagrangian submanifold of a complex space form $\overline{M}^m(4c)$ and Legendrian submanifolds in Sasakian space forms:

$$(\rho^\perp)^2 \leq (\| H \|^2 - \rho + c)^2 + \frac{4}{n(n-1)}(\rho - c) + \frac{2c^2}{n(n-1)}, \tag{4.1}$$

$$(\rho^\perp)^2 \leq (\| H \|^2 - \rho + c)^2 + \frac{4}{n(n-1)}(\rho - \frac{c+3}{4})\frac{c-1}{4} + \frac{(c-1)^2}{8n(n-1)}, \tag{4.2}$$

respectively. In [9], the following theorem is proved

Theorem 4.1. *Let $(\mathbb{R}, dt, \nabla^{\mathbb{R}})$ be trivial statistical manifold and $N(c)$ be a holomorphic statistical space form. If M^n is a Legendrian submanifold of the statistical warped product manifold $\overline{M} = \mathbb{R} \times_f N(c)$ then we have*

$$\rho^{\perp\nabla,\nabla^*} \leq 2\rho^{\nabla,\nabla^*} - 8\rho^\circ + \frac{1}{4f^2}(2f|c| - c + 4(f')^2) + 4 \parallel H^\circ \parallel^2 + \parallel H \parallel^2 + \parallel H^* \parallel^2 . \tag{4.3}$$

Now, we will prove generalized Wintgen inequality for Lagrangian submanifold in statistical quaternionic space form.

Theorem 4.2. *Let M be a n-dimensional statistical Lagrangian submanifold in a $4m$-dimensional statistical quaternionic space form $\overline{M}(c)$. Then the following inequality holds*

$$\begin{aligned} \rho^\perp \;&\leq\; (\frac{3\epsilon}{4} + \frac{7}{2})c \\ &+\; 8 \parallel H^\circ \parallel^2 + 2 \parallel H^* \parallel^2 + 2 \parallel H \parallel^2 \\ &+\; 2\rho - 16\rho^\circ, \end{aligned} \tag{4.4}$$

where $\epsilon = -1$ ou $\epsilon = +1$.

We will compute ρ and $\rho^\perp$ respectively and then use the markup properties to finally get our inequality.

Proof. Let M be a n-dimensional statistical Lagrangian submanifold in a $4m$-dimensional statistical quaternionic space form $\overline{M}(c)$. For $x \in M$, consider $\{e_1, \cdots, e_n\}$ and $\{\xi_{n+1}, \cdots, \xi_{4m}\}$ orthonormal bases of T_xM and $T_x^\perp M$, respectively. The scalar curvature of M related to the sectional curvature K is given by

$$\tau = \frac{1}{2} \sum_{1 \leq i < j \leq n} \big(R(e_i, e_j, e_j, e_i) + R^\star(e_i, e_j, e_j, e_i)\big) \tag{4.5}$$

From equations (2.11), (2.14) and defintion (3.3), we obtain

$$\sum_{1 \leq i < j \leq n} R(e_i, e_j, e_j, e_i) = \frac{n(n-1)c}{8} - \sum_{1 \leq i < j \leq n} (g(h(e_i, e_j), h^\star(e_i, e_j)) + g(h^\star(e_i, e_i), h(e_j, e_j))) \tag{4.6}$$

Similarly, we have

$$\sum_{1 \leq i < j \leq n} R^\star(e_i, e_j, e_j, e_i) = \frac{n(n-1)c}{8} - \sum_{1 \leq i < j \leq n} (g(h(e_i, e_i), h^\star(e_j, e_j)) - g(h^\star(e_i, e_j), h(e_i, e_j))) \tag{4.7}$$

replacing (4.7) and (4.6) in (4.5) we find

$$\tau = \frac{n(n-1)c}{8} + \frac{1}{2} \sum_{n+1 \leq \alpha \leq 4m} \sum_{1 \leq i < j \leq n} (h_{ii}^\alpha h_{jj}^{\star\alpha} + h_{jj}^\alpha h_{ii}^{\star\alpha} - 2h_{ij}^\alpha h_{ij}^{\star\alpha}) \tag{4.8}$$

$$\begin{aligned} \tau \;&=\; \frac{n(n-1)c}{8} + \frac{1}{2} \sum_{n+1 \leq \alpha \leq 4m} \sum_{1 \leq i < j \leq n} (h_{ii}^\alpha + h_{ii}^{\star\alpha})(h_{jj}^\alpha + h_{jj}^{\star\alpha}) - h_{ii}^\alpha h_{jj}^\alpha - h_{ii}^{\star\alpha} h_{jj}^{\star\alpha} \\ &-\; (h_{ij}^\alpha + h_{ij}^{\star\alpha})^2 + (h_{ij}^\alpha)^2 + (h_{ij}^{\star\alpha})^2 \end{aligned} \tag{4.9}$$

From $2h^0 = h + h^\star$, the latter equation becomes

$$\begin{aligned} \tau &= \frac{n(n-1)c}{8} + 2\sum_{n+1\leq\alpha\leq 4m}\sum_{1\leq i<j\leq n}(h^{0\alpha}_{ii}h^{0\alpha}_{jj} - (h^{0\alpha}_{ij})^2) \\ &- \frac{1}{2}\sum_{n+1\leq\alpha\leq 4m}\sum_{1\leq i<j\leq n}((h^{\alpha}_{ii}h^{\alpha}_{jj} - (h^{\alpha}_{ij})^2) - h^{\star\alpha}_{ii}h^{\star\alpha}_{jj} - (h^{\star\alpha}_{ij})^2) \end{aligned} \tag{4.10}$$

Using $(3.7) - (3.10)$, (4.10) becomes

$$\begin{aligned} \tau &= \frac{n(n-1)c}{8} + 2(n^2||H^0||^2 - ||h^0||^2) \\ &- \frac{1}{2}((n^2||H||^2 - ||h||^2) + (n^2||H^\star||^2 - ||h^\star||^2)) \end{aligned} \tag{4.11}$$

By normalizing τ, we get ρ given by

$$\begin{aligned} \rho &= \frac{c}{4} + \frac{4}{n(n-1)}(n^2||H^0||^2 - ||h^0||^2) \\ &- \frac{1}{n(n-1)}((n^2||H||^2 - ||h||^2) + (n^2||H^\star||^2 - ||h^\star||^2)). \end{aligned} \tag{4.12}$$

□

Denote by

$$\begin{aligned} \tau^\circ &= h^\circ - H^\circ g, \\ \tau &= h - Hg, \\ \tau^* &= h^* - H^* g, \end{aligned}$$

the traceless part of second fundamental forms. Then we have

$$\begin{aligned} \| \tau^\circ \|^2 &= \| h^\circ \|^2 - n \| H^\circ \|^2, \\ \| \tau \|^2 &= \| h \|^2 - n \| H \|^2, \\ \| \tau^* \|^2 &= \| h^* \|^2 - n \| H^* \|^2, \end{aligned}$$

and the above relation gives rise to

$$\begin{aligned} \rho &= \frac{c}{4} + 4||H^0||^2 - \frac{4}{n(n-1)}||\tau^0||^2 \\ &- (((||H||^2 - \frac{1}{n(n-1)}||\tau||^2) + (||H^\star||^2 - \frac{1}{n(n-1)}||\tau^\star||^2)). \end{aligned} \tag{4.13}$$

On the other hand, the normalized scalar curvature $\tau^\perp$ is given by

$$\tau^\perp = \{\sum_{n+1\leq r<s\leq 4m}\sum_{1\leq i<j\leq n}[\frac{R^\perp(e_i,e_j,\mathcal{E}_r,\mathcal{E}_s) + R^{*\perp}(e_i,e_j,\mathcal{E}_r,\mathcal{E}_s)}{2}]^2\}^{\frac{1}{2}}. \tag{4.14}$$

From the equation (2.13) , we obtain

$$\sum_{n+1\le r<s\le 4m}\sum_{1\le i<j\le n} R^{\perp}(e_i,e_j,\mathcal{E}_r,\mathcal{E}_s) = \frac{c}{4}(\sum_{n+1\le r<s\le 4m}\sum_{1\le i<j\le n}(g(\xi_r,J_\alpha e_j)g(J_\alpha e_i,\xi_s) - g(\xi_r,J_\alpha e_i g(J_\alpha e_j,\xi_s))) + g([A^\star_{\xi_r},A_{\xi_s}]e_i,e_j)$$

Similarly, from the equation (2.16), we obtain

$$\sum_{n+1\le r<s\le 4m}\sum_{1\le i<j\le n} R^{\star\perp}(e_i,e_j,\mathcal{E}_r,\mathcal{E}_s) = \frac{c}{4}(\sum_{n+1\le r<s\le 4m}\sum_{1\le i<j\le n}(g(\xi_r,J^\star_\alpha e_j g(J^\star_\alpha e_i,\xi_s) - g(\xi_r,J^\star_\alpha e_i g(J^\star_\alpha e_j,\xi_s))) + g([A_{\xi_r},A^\star_{\xi_s}]e_i,e_j)$$

Put $\xi_r = J_\beta e_k$ and $\xi_s = J_\gamma e_l$, and replacing above equation in (4.14), we get

$$\tau^{\perp} = \{[\frac{c}{8}(\delta_{\beta\alpha}\delta_{kj}\delta_{\alpha\gamma}\delta_{il} - \delta_{\beta\alpha}\delta_{ki}\delta_{\alpha j}\delta_{jl} + \delta^\star_{\beta\alpha}\delta_{kj}\delta^\star_{\alpha\gamma}\delta_{il} - \delta^\star_{\gamma\alpha}\delta_{li}\delta^\star_{\alpha\gamma}\delta_{jl}) + g([A^\star_{\xi_r},A_{\xi_s}]e_i,e_j) + g([A_{\xi_r},A^\star_{\xi_s}]e_i,e_j)]^2\}^{\frac{1}{2}}, \tag{4.15}$$

where $\delta_{\beta\alpha}$ is Kronecker's symbol such that $\delta_{\beta\alpha}\delta_{kj} = g(J_\beta e_k, J_\alpha e_j)$ and $\delta^\star_{\beta\alpha}\delta_{kj} = g(J_\beta e_k, J^\star_\alpha e_j)$. For (4.15), we fix $\beta = \alpha = \gamma$, $k = j$ and $i = l$. Then we have

$$\tau^{\perp} = \{\sum_{n+1\le r<s\le 4m}\sum_{1\le i<j\le n}[\frac{3n(n-1)c}{16} + g([A^\star_{\xi_r},A_{\xi_s}]e_i,e_j) + g([A_{\xi_r},A^\star_{\xi_s}]e_i,e_j)]^2\}^{\frac{1}{2}}, \tag{4.16}$$

from $2A_0 = A + A^\star$, the latter equation becomes

$$\tau^{\perp} = \{\sum_{n+1\le r<s\le 4m}\sum_{1\le i<j\le n}[\frac{3n(n-1)c}{16} + 4g([A^0_{\xi_r},A^0_{\xi_s}]e_i,e_j) + g([A^\star_{\xi_r},A^\star_{\xi_s}]e_i,e_j) + g([A_{\xi_r},A_{\xi_s}]e_i,e_j)]^2\}^{\frac{1}{2}}, \tag{4.17}$$

from (2.20), we have $\rho^{\perp}$:

$$\rho^{\perp} = \frac{2}{n(n-1)}\{\sum_{n+1\le r<s\le 4m}\sum_{1\le i<j\le n}[\frac{3n(n-1)c}{16} + 4g([A^0_{\xi_r},A^0_{\xi_s}]e_i,e_j) + g([A^\star_{\xi_r},A^\star_{\xi_s}]e_i,e_j) + g([A_{\xi_r},A_{\xi_s}]e_i,e_j)]^2\}^{\frac{1}{2}}, \tag{4.18}$$

by the Cauchy-Schwartz inequality, we have the algebraic inequality

$$(\lambda+\mu+\nu+\omega)^2 \le 4(\lambda^2+\mu^2+\nu^2+\omega^2), \tag{4.19}$$

$\forall \lambda,\mu,\nu,\omega \in \mathbb{R}$.
We obtain from this inequality that

$$\begin{aligned}
\rho^{\perp} &\leq \frac{4}{n(n-1)}\{\sum_{n+1\leq r<s\leq 4m}\sum_{1\leq i<j\leq n}[(\frac{3n(n-1)c}{16})^2 \\
&+ (4g([A^{\circ}_{\mathcal{E}_r}, A^{\circ}_{\mathcal{E}_s}]e_i, e_j))^2 + (g([A^{*}_{\mathcal{E}_r}, A^{*}_{\mathcal{E}_s}]e_i, e_j))^2 \\
&+ (g([A_{\mathcal{E}_r}, A_{\mathcal{E}_s}]e_i, e_j))^2]\}^{\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
\rho^{\perp} &\leq \frac{4}{n(n-1)}\sum_{r,s=n+1}^{4m}\{\sum_{i,j=1}^{n}[(\frac{3n(n-1)c}{16})^2 \\
&+ \frac{1}{4}((4g([A^{\circ}_{\mathcal{E}_r}, A^{\circ}_{\mathcal{E}_s}]e_i, e_j))^2 + (g([A^{*}_{\mathcal{E}_r}, A^{*}_{\mathcal{E}_s}]e_i, e_j))^2 \\
&+ (g([A_{\mathcal{E}_r}, A_{\mathcal{E}_s}]e_i, e_j))^2)]\}^{\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
\rho^{\perp} &\leq \frac{4}{n(n-1)}\{(\frac{3n(n-1)c}{16})^2 + \\
&+ \sum_{r,s=n+1}^{4m}\frac{1}{4}[\sum_{i,j=1}^{n} 16 \parallel [A^{\circ}_{\mathcal{E}_r}, A^{\circ}_{\mathcal{E}_s}] \parallel^2, + \parallel [A^{*}_{\mathcal{E}_r}, A^{*}_{\mathcal{E}_s}] \parallel^2 \\
&+ \parallel [A_{\mathcal{E}_r}, A_{\mathcal{E}_s}] \parallel^2]\}^{\frac{1}{2}}.
\end{aligned}$$

Now we define the sets $\{S^0_1, \ldots, S^0_n\}$, $\{S_1, \ldots, S_n\}$, $\{S^*_1, \ldots, S^*_n\}$ of symmetric with trace zero operators on T_pM, $p \in M$ by

$$\begin{aligned}
g(S^0_\alpha X, Y) &= g(\tau^0(X, Y), \xi_\alpha) \\
g(S_\alpha X, Y) &= g(\tau(X, Y), \xi_\alpha) \\
g(S^*_\alpha X, Y) &= g(\tau^*(X, Y), \xi_\alpha)
\end{aligned} \quad (4.20)$$

for all $X, Y \in T_pM$. Clearly, we obtain

$$\begin{aligned}
S^0_\alpha &= A^0_{\mathcal{E}_\alpha} - g(H^0, \xi_\alpha)I \\
S_\alpha &= A_{\mathcal{E}_\alpha} - g(H, \xi_\alpha)I \\
S^*_\alpha &= A^*_{\mathcal{E}_\alpha} - g(H^*, \xi_\alpha)I
\end{aligned}$$

and

$$\begin{aligned}
[S^0_\alpha, S^0_\beta] &= [A^0_{\mathcal{E}_\alpha}, A^0_{\mathcal{E}_\beta}] \\
[S_\alpha, S_\beta] &= [A_{\mathcal{E}_\alpha}, A_{\mathcal{E}_\beta}] \\
[S^*_\alpha, S^*_\beta] &= [A^*_{\mathcal{E}_\alpha}, A^*_{\mathcal{E}_\beta}].
\end{aligned} \quad (4.21)$$

Then from (4.21), it is clear that

$$\begin{aligned}\rho^{\perp} &\leq \frac{4}{n(n-1)}\{(\frac{3n(n-1)c}{16})^2 + \\ &+ \sum_{r,s=n+1}^{4m} \frac{1}{4}[16 \parallel [S_r^{\circ}, S_s^{\circ}] \parallel^2, + \parallel [S_r^*, S_s^*] \parallel^2 \\ &+ \parallel [S_r, S_s] \parallel^2]\}^{\frac{1}{2}}.\end{aligned}$$

We are going to use the following theorem from [12].

Theorem 4.3. *Let M^n be Riemannian submanifold of Riemannian space form $\widetilde{M}^{m+n}(c)$. For every $\{B_1, \dots, B_m\}$ of symmetric $(n \times n)-$matrices with trace zero the following inequality holds:*

$$\sum_{1\leq\alpha,\beta\leq m} \parallel [B_\alpha, B_\beta] \parallel^2 \leq (\sum_{1\leq\alpha\leq m} \parallel B_\alpha \parallel^2)^2.$$

By this theorem, we can write

$$\begin{aligned}\rho^{\perp} &\leq \frac{3|c|}{4} \\ &+ \sum_{n+1\leq r\leq 4m} \frac{8}{n(n-1)} \parallel S_r^{\circ} \parallel^2 + \frac{2}{n(n-1)} \sum_{n+1\leq r\leq 4m} \parallel S_r^* \parallel^2 \\ &+ \frac{2}{n(n-1)} \sum_{n+1\leq r\leq 4m} \parallel S_r \parallel^2 .\end{aligned}$$

i.e

$$\begin{aligned}\rho^{\perp} &\leq \frac{3|c|}{4} \\ &+ \frac{8}{n(n-1)} \parallel \tau^{\circ} \parallel^2 + \frac{2}{n(n-1)} \parallel \tau^* \parallel^2 \\ &+ \frac{2}{n(n-1)} \parallel \tau \parallel^2 .\end{aligned}$$

On the other hand, the normalized scalar curvature ρ° of M with respect to Levi-Civita connection ∇° of M can be obtain as

$$\rho^{\circ} = \frac{c}{4} + \frac{1}{n(n-1)}[n^2 \parallel H^{\circ} \parallel^2 - \parallel h^{\circ} \parallel^2]. \tag{4.22}$$

Now, if we set $\parallel \tau^{\circ} \parallel^2 = -n \parallel H^{\circ} \parallel^2 + \parallel h^{\circ} \parallel^2$ in (4.22) then we get

$$\rho^{\circ} = \frac{c}{4} - \frac{1}{n(n-1)} \parallel \tau^{\circ} \parallel^2 + \parallel H^{\circ} \parallel^2 . \tag{4.23}$$

Therefore we have

$$\frac{16}{n(n-1)} \parallel \tau^{\circ} \parallel^2 = 4c + 16 \parallel H^{\circ} \parallel^2 - 16\rho^{\circ} \tag{4.24}$$

and we derive

$$\begin{aligned}\rho^{\perp} &\leq (\frac{3\epsilon}{4}+\frac{7}{2})c \\ &+ 8\parallel H^{\circ}\parallel^2 + 2\parallel H^{*}\parallel^2 + 2\parallel H\parallel^2 \\ &+ 2\rho - 16\rho^{\circ},\end{aligned} \tag{4.25}$$

where $\epsilon=-1$ ou $\epsilon=+1$.

4.2. **Some application.** Chen, [6], has constructed an invariant which allows to find a sufficient condition for the existence of a minimal isometric immersion of a manifold in a space form. We use Wintgen's inequality to obtain a result of non-existence of a minimal Lagrangian submanifold in a statistical quaternionic space form .

Theorem 4.4. *Let M be a n-dimensional statistical Lagrangian submanifold in a $4m$-dimensional statistical quaternionic space form $\overline{M}(c)$. If*

$$(\frac{3\epsilon}{4}+\frac{7}{2})c < \rho^{\perp} - 2\rho + 16\rho^{0},$$

then M is non minimal.

Proof. The proof follows from the previous theorem. □

5. $\delta(2,2,\ldots,2)$-Inequality for statistical Lagrangian submanifold of a statistical quaternionic space form

In this section we obtain a special case of a Chen invariant: $\delta\underbrace{(2,2,\ldots,2)}_{p-times}$. For this purpose, we use a new algebraic inequality of [16]. We also characterise the minimal submanifolds.

Lemma 5.1. *Let $n\geq 2p$ be an integer and let $a_1,a_2,\ldots,a_n$ be n real numbers. Then one has*

$$\sum_{1\leq i<j\leq n} a_i a_j - \sum_{1\leq k\leq p} a_{2k-1}a_{2k} \leq \frac{n-p-1}{2(n-p)}(\sum_{i=1}^{n} a_i)^2. \tag{5.1}$$

Moreover, the equality holds, if and only if $a_{2i-1}+a_{2i}=a_j$, with $1\leq i\leq p$ and $2p+1\leq j\leq n$.

To establish Chen's inequalities, Mihai [16] uses another type of sectional curvature defined by opozda [18]. It is called the K-sectional curvature and is defined by :

$$K(\pi)=\frac{1}{2}(R(X,Y,Y,X)+R^{\star}(X,Y,Y,X)-2R^{0}(X,Y,Y,X)),$$

where $\{X,Y\}$ is an orthonormal basis of a non degenerate two dimensional subspace π of the tangent space T_xM , at a point $x\in M$.
In our context, $\pi_k=span(e_{2k-1},e_{2k})$, $k=1,\ldots,p$. This means that

$$K(\pi_k)=\frac{1}{2}(R(e_{2k-1},e_{2k},e_{2k},e_{2k-1})+R^{\star}(e_{2k-1},e_{2k},e_{2k},e_{2k-1})-2R^{0}(e_{2k-1},e_{2k},e_{2k},e_{2k-1})).$$

By (2.11) and (3.6), we obtain

$$K(\pi_k) = \frac{c}{4} + \frac{1}{2}\sum_{\alpha=n+1}^{4m}(h^{\alpha}_{2k-12k-1}h^{\star\alpha}_{2k2k} + h^{\star\alpha}_{2k-12k-1}h^{\alpha}_{2k2k} - 2h^{\star\alpha}_{2k-12k}h^{\alpha}_{2k-12k}) - K_0(\pi_k).$$

Using $h + h^{\star} = 2h^0$, we get

$$\begin{aligned} K(\pi_k) &= \frac{c}{4} + 2\sum_{\alpha=n+1}^{4m}(h^{0\alpha}_{2k-12k-1}h^{0\alpha}_{2k2k} - (h^{0\alpha}_{2k-12k})^2) \\ &- \frac{1}{2}\sum_{\alpha=n+1}^{4m}((h^{\alpha}_{2k-12k-1}h^{\alpha}_{2k2k} - (h^{\alpha}_{2k-12k})^2) + (h^{\star\alpha}_{2k-12k-1}h^{\star\alpha}_{2k2k} - (h^{\star\alpha}_{2k-12k})^2)) \\ &- K_0(\pi_k). \end{aligned} \tag{5.2}$$

By (2.11), we have $2\sum_{\alpha=n+1}^{4m}(h^{0\alpha}_{2k-12k-1}h^{0\alpha}_{2k2k} - (h^{0\alpha}_{2k-12k})^2) = 2K_0(\pi_k) - 2\hat{K}_0(\pi_k)$. Then

$$\begin{aligned} K(\pi_k) &= \frac{c}{4} + K_0(\pi_k) - 2\hat{K}_0(\pi_k) \\ &- \frac{1}{2}\sum_{\alpha=n+1}^{4m}((h^{\alpha}_{2k-12k-1}h^{\alpha}_{2k2k} - (h^{\alpha}_{2k-12k})^2) + (h^{\star\alpha}_{2k-12k-1}h^{\star\alpha}_{2k2k} - (h^{\star\alpha}_{2k-12k})^2)). \end{aligned} \tag{5.3}$$

$$\begin{aligned} \sum_{k=1}^{p} K(\pi_k) &= p\frac{c}{4} + \sum_{k=1}^{p}(K_0(\pi_k) - 2\hat{K}_0(\pi_k)) \\ &- \frac{1}{2}\sum_{i=2}^{p}\sum_{\alpha=n+1}^{4m}((h^{\alpha}_{2k-12k-1}h^{\alpha}_{2k2k} - (h^{\alpha}_{2k-12k})^2) + (h^{\star\alpha}_{2k-12k-1}h^{\star\alpha}_{2k2k} - (h^{\star\alpha}_{2k-12k})^2)). \end{aligned} \tag{5.4}$$

The scalar curvature corresponding to the sectional $K-$curvature is given by

$$\tau = \frac{1}{2}\sum_{1\le i<j\le n}(R(e_i,e_j,e_j,e_i) + R^{\star}(e_i,e_j,e_j,e_i) - 2R^0(e_i,e_j,e_j,e_i)).$$

Following similar calculations in [11], we obtain

$$\begin{aligned} \tau &= n(n+2)\frac{c}{8} + \tau_0 - 2\hat{\tau}_0 \\ &- \frac{1}{2}\sum_{1\le i<j\le n}\sum_{\alpha=n+1}^{4m}((h^{\alpha}_{ii}h^{\alpha}_{jj} - (h^{\alpha}_{ij})^2) + (h^{\star\alpha}_{ii}h^{\star\alpha}_{jj} - (h^{\star\alpha}_{ij})^2)). \end{aligned} \tag{5.5}$$

Subtracting (5.4) from (5.5), the invariant $\delta(2,2,\ldots,2)$ can be written as :

$$\begin{aligned}
\tau - \sum_{k=1}^{p} K(\pi_k) &= n(n+2)\frac{c}{8} + \tau_0 - 2\hat{\tau}_0 \\
&- \frac{1}{2}\sum_{1\le i<j\le n}\sum_{\alpha=n+1}^{4m}((h_{ii}^{\alpha}h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2) + (h_{ii}^{\star\alpha}h_{jj}^{\star\alpha} - (h_{ij}^{\star\alpha})^2)) \\
&- p\frac{c}{4} - \sum_{k=1}^{p}(K_0(\pi_k) - 2\hat{K}_0(\pi_k)) \\
&+ \frac{1}{2}\sum_{k=1}^{p}\sum_{\alpha=n+1}^{4m}((h_{2k-12k-1}^{\alpha}h_{2k2k}^{\alpha} - (h_{2k-12k}^{\alpha})^2) + (h_{2k-12k-1}^{\star\alpha}h_{2k2k}^{\star\alpha} - (h_{2k-12k}^{\star\alpha})^2))
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
\tau - \sum_{k=1}^{p} K(\pi_k) &\ge (n(n+2) - 2p)\frac{c}{8} + (\tau_0 - \sum_{k=1}^{p}(K_0(\pi_k)) - 2(\hat{\tau}_0 - \sum_{k=1}^{p}\hat{K}_0(\pi_k)) \\
&- \frac{1}{2}\sum_{\alpha=n+1}^{4m}(\sum_{1\le i<j\le n} h_{ii}^{\alpha}h_{jj}^{\alpha} - \sum_{k=2}^{p} h_{2k-12k-1}^{\alpha}h_{2k2k}^{\alpha}) \\
&- \frac{1}{2}\sum_{\alpha=n+1}^{4m}(\sum_{1\le i<j\le n} h_{ii}^{\star\alpha}h_{jj}^{\star\alpha} - \sum_{k=1}^{p} h_{2k-12k-1}^{\star\alpha}h_{2k2k}^{\star\alpha}).
\end{aligned} \tag{5.7}$$

Let H and $H^\star$ denote the mean curvature vectors with respect to the dual connections ∇ and $\nabla^\star$, respectively. Then, lemma 5.1 implies

$$\sum_{\alpha=n+1}^{4m}(\sum_{1\le i<j\le n} h_{ii}^{\alpha}h_{jj}^{\alpha} - \sum_{k=1}^{p} h_{2k-12k-1}^{\alpha}h_{2k2k}^{\alpha}) \le \frac{n-p-1}{2(n-p)}(\sum_{i=1}^{n} h_{ii}^{\alpha})^2 = \frac{n^2(n-p-1)}{2(n-p)}(H^{\alpha})^2$$

$$\sum_{\alpha=n+1}^{4m}(\sum_{1\le i<j\le n} h_{ii}^{\star\alpha}h_{jj}^{\star\alpha} - \sum_{k=1}^{p} h_{2k-12k-1}^{\star\alpha}h_{2k2k}^{\star\alpha}) \le \frac{n-p-1}{2(n-p)}(\sum_{i=1}^{n} h_{ii}^{\star\alpha})^2 = \frac{n^2(n-p-1)}{2(n-p)}(H^{\star\alpha})^2.$$

By summing the two above relations and substituting the result into equation (5.7), we get

Theorem 5.1. *Let M be a n-dimensional statistical Lagrangian submanifold in a $4m$-dimensional statistical quaternionic space form $\overline{M}(c)$. Then for any $x \in M$, and any plane sections π_k, $k = 1, \dots, p$ at x and $n \ge 2p$ we have*

$$\begin{aligned}
\delta(2,2,\dots,2) &\ge \delta_0(2,2,\dots,2) + \frac{c}{8}((n+2)n - 2p) \\
&- \frac{n^2}{4}\frac{n-p-1}{(n-p)}\|H\|^2 - \frac{n^2}{4}\frac{n-p-1}{(n-p)}\|H^\star\|^2 - 2\hat{\delta}_0(2,2,\dots,2).
\end{aligned}$$

Moreover, the equality holds if and only if for any $r \in \{n+1,\dots,4m\}$

$$\begin{aligned}
h_{2k-12k-1}^{r} + h_{2k2k}^{r} &= h_{jj}^{r}, \\
h_{2k-12k-1}^{\star r} + h_{2k2k}^{\star r} &= h_{jj}^{\star r},
\end{aligned}$$

with $1 \leq k \leq p$ *and* $2p+1 \leq j \leq n$.

$$h^r_{il} = h^{\star r}_{il} = 0,$$

for any $1 \leq i < l \leq n$, $(i, l) \notin \{(1, 2), \ldots, (2k\text{-}1, 2k)\}$.

Remark 5.1. *Theorem* 5.1 *generalises the mains results of* [11], *namely basic Chen inequalities for Lagrangian submanifolds in quaternionic Kahler-like statistical manifolds. Indeed, if in the statement of Theorem* 5.1, *one particularizes the Chen inequality by putting* $p = 1$ *or* $p = 2$, *we obtain the mains results of* [11].

Theorem 5.2. *Let M be a n-dimensional statistical Lagrangian submanifold in a* 4*m-dimensional statistical quaternionic space form* $\overline{M}(c)$. *If there exist* $x \in M$ *and p mutually orthogonal plane sections* $\pi_1, \ldots, \pi_p$ *at x such that*

$$\delta(2,2,\ldots,2) - \delta_0(2,2,\ldots,2) < \frac{c}{8}((n+2)n - 2p) - 2\hat{\delta}_0(2,2,\ldots,2), \tag{5.8}$$

then M is non minimal.

6. Perspective

Todjihounde, [23], has shown that the projection of a dualistic structure defined on a warped product space induces dualistic structures on the base and the fiber manifold. And conversely, dualistic structures on the base and the fiber induces a dualistic structure on the warped product space. Using idea of Todjihounde, Furuhata, [24], has shown that a Kenmotsu statistical manifold is locally obtained as the warped product of a holomorphic statistical manifold and a line. The co-quaternionic manifolds are the odd dimensional analogue of the quaternionic manifolds. They are of dimension $4n+3$. They are provided with three almost metric contact structures verifying specific conditions. Udriste, [25], has shown that a quaternionic manifold can be decomposed into a co-quaternionic manifold and a line. A natural question would be to know under which condition a co-quaternionic manifold can be provided with a statistical structure?

Acknowledgments. The author is very thankful to Professors Gabriel-Eduard Vîlcu and Cyriaque Atindogbé for their valuable comments and suggestions while preparing of this manuscript.

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