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SZABÓ CONDITION ON WARPED PRODUCT MANIFOLDS

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ABSTRACT. A Riemannian manifold (M, g) is called Szabó, if the eigenvalues of the Szabó operators are constant on the unit sphere bundle SM of (M, g). Let $(M_1 \times_f M_2, g)$ the warped product of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) equipped withe the Riemannian metric $g = g_1 \oplus f^2 g_2$, where f is a positive function on M_1. In this note we study the behavior of the Szabó condition on a Riemannian warped product manifold.

1. INTRODUCTION

In 1969, R. L. Bishop and B. O'Neil [3] introduced the concept of warped products, which were used to construct a large class of complete Riemannian manifolds with negative sectional curvature. Warped product have significant applications, in general relativity [2], in the studies related to solutions of Einstein's equations [5]. Besides general relativity, warped product structures have also generated interest in many areas of geometry, especially due to their role in construction of new examples with interesting curvatures and symmetry properties [6, 7].

Let (M, g) be a Riemannian manifold of dimension m with a metric tensor g. Let T_pM be the tangent space at a point $p \in M$ and let S_pM the set of unit vectors in T_pM, i.e. $S_pM := \{u \in T_pM, \|g(u,u)\| = 1\}$. Let $\mathcal{F}(M)$ be the algebra of all smooth functions on M and $\mathfrak{X}(M)$ be the $\mathcal{F}(M)$-module of all smooth vector fields over V. Let also ∇ be the Levi-Civita connection of (M, g) and $\mathcal{R}$ be the $(1, 3)$ curvature tensor which is defined by

$$\mathcal{R}(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.1}$$

where $X, Y, Z \in \mathfrak{X}(M)$. We define

$$R(X, Y, Z, U) := g(\mathcal{R}(X, Y)Z, U), \tag{1.2}$$

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to be the associated $(0,4)$ Riemann curvature tensor which satisfied the following algebraic properties

$$\begin{array}{rcl} R(X,Y,Z,U) & = & -R(Y,X,Z,U), \\ R(X,Y,Z,U) & = & -R(X,Y,U,Z), \\ R(X,Y,Z,U) & = & R(Z,U,X,Y). \end{array}$$

In Riemannian geometry the following algebraic equality is true:

$$R(X,Y,Z,U)+R(Y,Z,X,U)+R(Z,X,Y,U)=0, \quad \text{(first Bianchi identity)} \tag{1.3}$$

and also the following differential equality is true:

$$(\nabla_X\mathcal{R})(Y,Z,W)+(\nabla_Y\mathcal{R})(Z,X,W)+(\nabla_Z\mathcal{R})(X,Y,W)=0, \quad \text{(second Bianchi identity)} \tag{1.4}$$

where

$$(\nabla_X\mathcal{R})(Y,Z,W):=\nabla_X(\mathcal{R}(Y,Z)W)-\mathcal{R}(\nabla_XY,Z)W-\mathcal{R}(Y,\nabla_XZ)W-\mathcal{R}(Y,Z)\nabla_XW, \tag{1.5}$$

and $\nabla_X\mathcal{R}$ is the covariant derivative of the $(1,3)$ curvature tensor $\mathcal{R}$ with respect to $X,Y,Z\in\mathfrak{X}(M)$. By Ric we denote the Ricci tensor of (M,g). One has

$$\mathrm{Ric}(X,Y)=\sum_{i=1}^{m}g(\mathcal{R}(e_i,X)Y,e_i) \tag{1.6}$$

where $\{e_1,e_2,\ldots,e_m\}$ are orthonormal basis vector fields in the tangent bundle TM. Let X and Y be two linearly independent vectors at a point $p\in M$ and $\pi(X,Y)$ be the plane section spanned by X and Y. The sectional curvature $K(\pi)$ for π is defined by

$$K(\pi)=\frac{g(\mathcal{R}(Y,X)X,Y)}{g(X,X)g(Y,Y)-g(X,Y)^2}. \tag{1.7}$$

It is easy to see $K(\pi)$ is uniquely determined by the plane section π and is independent the choice of X and Y on M. If $K(\pi)$ is a constant for all plane sections π in the tangent space T_pM at p and for all points $p\in M$, then M is called a space of constant curvature or we say that it has constant sectional curvature. Furthermore, a Riemannian manifold of constant sectionnel curvature is said to be elliptic, hyperbolic of flat, according as the constant sectional curvature is positive, negative or zero, respectively. A Riemannian manifold (M,g) is called locally symmetric if its curvature tensor is parallel . Analogically, (M,g) is said to be flat if its curvature tensor vanish [12].

For a vector field $X\in\mathfrak{X}(M)$ the divergence of X is defined as the trace of ∇X, that is, $\mathrm{div}X=\sum_i g(\nabla_{E_i}X,E_i)$. For a map $f:(M,g)\to\mathbb{R}$, the gradient of f is determined by $g(\nabla f,X)=df(X)=X(f)$, for all $X\in\mathfrak{X}(M)$. The linear map $h_f(X)=\nabla_X\nabla f$ is called the Hessian tensor of f on (M,g), and $H^f(X,Y)=g(h_f(X),Y)$ is called the Hessian form of f on (M,g). Finally, the Laplacian of f on (M,g) is defined by $\Delta f=-div\nabla f$, and it satisfies $\Delta f=-\mathrm{trace}(H^f)$.

Let $\mathcal{R}$ the curvature tensor and let $\mathcal{S}:T_xM\to T_xM$ be the Szabó operator which is defined by :

$$\mathcal{S}(X)U:=\nabla_X\mathcal{R}(U,X)X; \tag{1.8}$$

where $X, U \in T_xM$ and $x \in M$. The Szabó operator is symmetric with $\mathcal{S}(X)X = 0$. It plays an important role in the study of totally isotropic manifolds. Since $\mathcal{S}(cX) = c^3\mathcal{S}(X)$, the domains of $\mathcal{S}(X)$ are the unit bundles $S(M, g)$ of (M, g). A Riemannian manifold (M, g) is called Szabó, if the eigenvalues of the Szabó operators are constant on the unit sphere bundle SM of (M, g). Szabó [13] used techniques from algebraic topology to show that any Szabó Riemannian manifold is locally symmetric. He used this observation to give a simple proof that any two point homogeneous space is either flat or is a rank one symmetric space. Subsequently Gilkey and Stavrov [9] extended his results to show that any Szabó Lorentzian manifold has constant sectional curvature.

Also, we say that a Riemannian manifold (M, g) is nilpotent Szabó of order n if $\mathcal{S}^n(X) = 0$ for $X \in T_pM$ and if there exists an point $p \in M$ and a tangent vector $X \in T_pM$ so that $\mathcal{S}^{n-1}(X) \neq 0$. We say that (M, g) is nilpotent Szabó if (M, g) is nilpotent Szabó of order n for some n. Note that (M, g) is nilpotent Szabó if and only if 0 is the only eignevalue of $\mathcal{S}(X)$; consequently any nilpotent Szabó manifolds is Szabó manifolds. If (M, g) is nilpotent Szabó of order 1, then $\mathcal{S}(X) = 0$ for all $X \in TM$. This implies that $\nabla\mathcal{R} = 0$ so (M, g) is a local symmetric space. Gilkey, Ivanova and Zhang [10] have constructed pseudo-Riemannian manifolds which are nilpotent Szabó of order 2. See [8] and reference therein for more details.

The aim of the present paper is to study Riemannian product and warped product manifolds products satisfying the Szabó condition. In Section 3 wo prove that if a Riemannian product $(M_1 \times M_2, g_1 + g_2)$ is Szabó then manifold s (M_1, g_1) and (M_2, g_2) are also Szabó. Moreover, Theorem 2 gives neoessary and sufficient conditions in order that a Riemannian product be Szabó. Section 4 i s devoted to warped product. We prove that if $M_1 \times_f M_2$ is Szabó then (M_1, g_1) and (M_2, g_2) are also Szabó. manifolds.

Throughout this paper, by a manifold we mean a connected paracompact manifold of class $\mathcal{C}^\infty$. By abuse of notation, concerning Riemannian manifolds we often write M instead of (M, g).

2. Riemannian product manifolds

Let $(M_1, \mathcal{A}_1), (M_2, \mathcal{A}_2)$ be two manifolds of class $\mathcal{C}^\infty$, endowed with two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ of dimensions m_1, m_2 respectively. Then the product $\mathcal{A}_1 \times \mathcal{A}_2$ given by:

$$\mathcal{A}_1 \times \mathcal{A}_2 = \{(U_1 \times U_2, \varphi_1 \times \varphi_2) | \quad (U_1, \varphi_1) \in \mathcal{A}_1, (U_2, \varphi_2) \in \mathcal{A}_2\}$$

where

$$\begin{aligned} \varphi_1 \times \varphi_2 : U_1 \times U_2 &\rightarrow \varphi_1(U_1) \times \varphi_2(U_2) \\ (x_1, x_2) &\mapsto (\varphi(x_1), \varphi(x_2)) \end{aligned}$$

is an atlas on $M_1 \times M_2$ of dimension $m_1 + m_2$ and of class $\mathcal{C}^\infty$. The manifold $(M_1 \times M_2, \mathcal{A}_1 \times \mathcal{A}_2)$ is called product manifold of M_1 and M_2. From product coordinate system on $M_1 \times M_2$, it is easy to show that

(1) The two projections $\pi : M_1 \times M_2 \rightarrow M_1$ and $\eta : M_1 \times M_2 \rightarrow M_2$ are submersions.
(2) For all $(x, y) \in M_1 \times M_2$ the subspace $M_1 \times \{y\}$ and $\{x\} \times M_2$ are submanifolds of $M_1 \times M_2$.

(3) For each $(x,y) \in M_1 \times M_2$.
- $\pi \mid_{M_1\times\{y\}}$ is a diffeomorphism from $M_1 \times \{y\}$ to M_1;
- $\eta \mid_{\{x\}\times M_2}$ is a diffeomorphism from $\{x\} \times M_2$ to M_2.

(4) For all $(x,y) \in M_1 \times M_2$ one have:

$$T_{(x,y)}(M_1 \times M_2) \cong T_x M_1 \times T_y M_2.$$

(5) The tangent spaces

$$T_{(x,y)}M_1 \equiv T_{(x,y)}(M_1 \times \{y\}) \quad \text{and} \quad T_{(x,y)}M_2 \equiv T_{(x,y)}(\{x\} \times M_2),$$

are subspaces of the tangent space to $M_1 \times M_2$ at (x,y).

(6) Let X and Y be two vector fields over M_1 and M_2 respectively. The pair (X,Y) defined by

$$\begin{array}{rcl}(X,Y): M_1 \times M_2 & \to & TM_1 \times TM_2 \\ (x,y) & \mapsto & (X_x, Y_y),\end{array}$$

is a vector field on the product manifold $M_1 \times M_2$.

Lemma 2.1. *The tangent space $T_{(x,y)}(M_1 \times M_2)$ is the direct sum of its subspaces $T_{(x,y)}M_1$ and $T_{(x,y)}M_2$; that is, each element of $T_{(x,y)}(M_1 \times M_2)$ has a unique expression as*

$$u + v, \text{ where } u \in T_{(x,y)}M_1 \text{ and } v \in T_{(x,y)}M_2.$$

To relate the calculus of $M_1 \times M_2$ to that of its factors the crucial notion is that of lifting, as follows:
- If $f \in \mathcal{C}^\infty(M_1)$ the lift of f to $M_1 \times M_2$ is $\tilde{f} = f \circ \pi \in \mathcal{C}^\infty(M_1 \times M_2)$.
- If $u \in T_x(M_1)$ and $y \in M_2$ then the lift $\tilde{u}$ of u to (x,y) is the unique vector in $T_{(x,y)}(M_1)$ such that $d\pi(\tilde{u}) = u$.
- If $X \in \mathfrak{X}(M_1)$ the lift of X to $M_1 \times M_2$ is the vector field $\tilde{X}$ whose value at each (x,y) is the lift of X_x to (x,y).

Product coordinate systems show that $\widetilde{X}$ is smooth. Thus the lift of $X \in \mathfrak{X}(M_1)$ to $M_1 \times M_2$ is the unique element of $\mathfrak{X}(M_1 \times M_2)$ that is π-related to X and σ-related to the zero vector field on N.

Functions, tangent vectors, and vector fields on M_2 are lifted to $M_1 \times M_2$ in the same way using the projection η. We will denote by $\mathfrak{L}(M_1)$ the set of all horizontal lifts $\tilde{X}$ and by $\mathfrak{L}(M_2)$ the set of all vertical lifts $\tilde{U}$. Note that $\mathfrak{L}(M_1)$ and symmetrically the vertical lifts $\mathfrak{L}(M_2)$ are vector subspaces of $\mathfrak{X}(M_1 \times M_2)$ but (except in trivial cases) neither is invariant under multiplication by arbitrary functions $f \in F(M_1 \times M_2)$.

Corollary 2.1. [12] *Let $M_1 \times M_2$ be a product manifold. Let $\mathfrak{L}(M_1)$, the set of all horizontal lifts and let $\mathfrak{L}(M_2)$ the set of all vertical lifts. We have:*

(1) If $\tilde{X}, \tilde{Y} \in \mathfrak{L}(M_1)$ then $[\tilde{X}, \tilde{Y}] = [X,Y]^\sim \in L(M)$, and similarly for $\mathfrak{L}(M_2)$.
(2) If $\tilde{X} \in \mathfrak{L}(M_1)$ and $\tilde{V} \in \mathfrak{L}(M_2)$, then $[\tilde{X}, \tilde{V}] = 0$.

Definition 2.1. *Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of dimension m_1 and m_2 respectively. We define the Riemannian metric produced on $M_1 \times M_2$ by*

$$g = \pi^* g_1 + \eta^* g_2$$

where $\pi : M_1 \times M_2 \to M_1$ and $\eta : M_1 \times M_2 \to M_2$ denote the first and the second canonical projection.

To see how the geometry of a Riemannian product manifold $M_1 \times M_2$ depends on that of M_1 and M_2, an essential tool is the notion of lift as discussed in Chapter 1. If $X \in \mathfrak{X}(M_1)$ we will use the same notation for its horizontal lift $X \in \mathfrak{L}(M_1) \subset \mathfrak{X}(M_1 \times M_2)$; similarly for the vertical lift $V \in \mathfrak{L}(M_2)$ of $V \in \mathfrak{X}(M_2)$.

Lemma 2.2. [12] *If $X, Y \in \mathfrak{L}(M_1)$ and $V, W \in \mathfrak{L}(M_2)$, then*

(1) $\nabla_X Y$ is the lift of ${}^{M_1}\nabla_X Y \in \mathfrak{X}(M_1)$.
(2) $\nabla_V W$ is the lift of ${}^{M_2}\nabla_V W \in \mathfrak{X}(M_2)$.
(3) $\nabla_V X = 0 = \nabla_X V$.

Lemma 2.3. [12] *On $M_1 \times M_2$ if $X, Y, Z \in \mathfrak{L}(M_1)$ and $U, V, W \in \mathfrak{L}(M_2)$, then:*

(1) $R_{XY}Z$ is the lift of ${}^{M_1}R_{XY}Z$ on M_1.
(2) $R_{VW}U$ is the lift on ${}^{M_2}R_{VW}U$ on M_2.
(3) R is zero on any other choices from $X, \ldots, W$.

These tensor results are valid for individual tangent vectors. It follows that the sectional curvature of a nondegenerate horizontal plane is the same as that of its projection into M_1, and analogously for a vertical plane. A nondegenerate plane spanned by a vertical and a horizontal vector has $K = 0$ since $R_{xv} = 0$. Thus there is always some flatness in a semi-Riemannian product manifold.

In 2003, Atceken and Keles [1] provide a brief survey on some well-known results on products of Riemannian manifolds.
[1] Let $(M_1 \times M_2, g)$ be a Riemannian product manifold of the Riemannian manifolds (M_1, g_2) and (M_2, g_2). Then the Riemannian product manifold $(M_1 \times M_2, g)$ has constant sectional curvature if and only if the Riemannian manifolds (M_1, g_2) and (M_2, g_2) have constant sectional curvatures.
[1] Let $(M_1 \times M_2, g)$ be a Riemannian product manifold of the Riemannian manifolds (M_1, g_2) and (M_2, g_2) with $g = g_1 + g_2$. Then the Riemannian product manifold $(M_1 \times M_2, g)$ is a locally symmetric manifold if and only if (M_1, g_2) and (M_2, g_2) are locally symmetric manifolds.
[1] Let $(M_1 \times M_2, g)$ be a Riemannian product manifold of the Riemannian manifolds (M_1, g_2) and (M_2, g_2) with $g = g_1 + g_2$. Then the Riemannian product manifold $(M_1 \times M_2, g)$ is flat if and only if (M_1, g_2) and (M_2, g_2) are flat.
[1] Let $(M_1 \times M_2, g)$ be a Riemannian product manifold of the Riemannian manifolds (M_1, g_2) and (M_2, g_2) with $g = g_1 + g_2$. Then the Riemannian product manifold $(M_1 \times M_2, g)$ is a Ricci flat manifold if and only if (M_1, g_2) and (M_2, g_2) are Ricci flat manifolds.
The geometry of submanifolds of a Riemannian product manifold has been studied by many geometers. In particularly, Matsumoto [11] proved that a submanifold (M, g) is a locally Riemannian product manifold of Riemannian manifolds (M_1, g_1) and (M_2, g_2), if (M, g) is an invariant submanifold of a Riemannian product manifold $(\tilde{M}_1 \times \tilde{M}_2, \tilde{g}_1 + \tilde{g}_2)$. Then, Senlin and Yilong [14] updated of the Matsumoto's Theorem and proved that (M_1, g_1) and (M_2, g_2) are pseudo-umbilical submanifolds of $(\tilde{M}_1, \tilde{g}_1)$ and $(\tilde{M}_2, \tilde{g}_2)$, respectively, if (M, g) is an invariant pseudo-umbilical submanifold of $(\tilde{M}_1 \times \tilde{M}_2, \tilde{g}_1 +$

$\tilde{g}_2$). They also demonsrated that M is isometric to the production of its two totally-geodesic submanifolds (M_1, g_1) and (M_2, g_2) which are submanifolds of $(\tilde{M}_1, \tilde{g}_1)$ and $(\tilde{M}_2, \tilde{g}_2)$, respectively.
Let $(M = M_1 \times M_2, g)$ be a Riemannian product manifold of the Riemannian manifolds (M_1, g_2) and (M_2, g_2) with $g = g_1 + g_2$. Then the Riemannian product manifold $(M_1 \times M_2, g)$ is Szabó if and only if (M_1, g_2) and (M_2, g_2) are Szabó.

Proof. Let $p \in M$ be an arbitrary point. Set $m_1 = \dim M_1$ and $m_2 = \dim M_2$. On T_pM we can choose an orthonormal bases $\{e_1, \ldots, e_{m_1}, f_1, \cdots, f_{m_2}\}$ with $\{e_i\} \subset T_xM_1$ and $\{f_i\} \subset T_yM_2$ which diagonalises the Ricci tensor. Note that

$$\begin{aligned} S(e_i)e_j &= \nabla_{e_i}\mathcal{R}(e_j, e_i)e_i \subset T_{p_1}M_1 \\ S(f_i)e_j &= \nabla_{e_j}\mathcal{R}(e_j, f_i)f_i \subset T_{p_1}M_1 \end{aligned}$$

and

$$\begin{aligned} S(f_i)f_j &= \nabla_{f_i}\mathcal{R}(f_j, f_i)f_i \subset T_{p_2}M_2 \\ S(e_i)f_j &= \nabla_{e_i}\mathcal{R}(f_j, e_i)e_i \subset T_{p_2}M_2 \end{aligned}$$

Indeed, from the expression of the curvature in Lemma, we compute

$$S(e_i)e_j = S^1(e_i)e_j \quad \text{and} \quad S(f_i)f_j = S^2(f_i)f_j.$$

Therefore, f_j is an eigenvector for every $S(f_i)$ (analogously, e_j is an eigenvector for every $S(e_i)$. Also notice that mixed terms of the curvature tensor vanish, that is, $\mathcal{R}(ej, f_i)f_i = 0$ and $\mathcal{R}(f_j, e_i)e_i = 0$. □

3. Riemannian warped product manifolds

One of the most fruitful generalizations of the direct product of two Riemannian manifolds is the warped product defined in [3]. The concept of warped products appeared in the mathematical and physical literature long before [3], e.g. warped products were called semi-reducible (pseudo) Riemannian spaces in [4]. The notion of warped products plays very important roles in differential geometry as well as in mathematical physics, especially in general relativity.

Many basic solutions of the Einstein field equations are warped products. For instance, both Schwarzschild's and Robertson-Walker's models in general relativity are warped products. Schwarzschild's spacetime 1 is the best relativistic model that describes the outer space around a massive star or a black hole and the Robertson-Walker model describes a simply-connected homogeneous isotropic expanding or contracting universe. Basic properties on warped products can be found in [3, 4] and [12].

Let $(M_1 g_1)$ et (M_2, g_2) be two Riemannian manifolds of dimensions m_1 and m_2 respectively and f a function strictly positive on M_1. Consider the product manifold $M_1 \times M_2$ with its canonical projections $\pi : M_1 \times M_2 \to M_1$ and $\eta : M_1 \times M_2 \to M_2$. The warped product is the product manifold $M_1 \times M_2$ equipped with the Riemannian metric such that

$$g = \pi^* g_1 + f^2 \eta^* g_2, \tag{3.1}$$

for any tangent vector $X \in TM$. Thus we have

$$g = g_1 \oplus f^2 g_2. \tag{3.2}$$

We denote the warped product of Riemannian manifolds (M_1, g_1) and (M_2, g_2) by $M_1 \times_f M_2$ and we refer to (M_1, g_1) and (M_2, g_2) as the base and the fiber of the product, respectively. The function f is called the warping function. If the warping function f is constant then the warped product $M_1 \times_f M_2$ is a direct product, which we call as trivial warped product.

As in the case of Riemannian product, it is easy to see that the fibers $\{x\} \times M_2 = \pi^{-1}(x)$ and the leaves $M_1 \times \{y\} = \eta_2^{-1}(y)$ are submanifolds of $M_1 \times M_2$, and the warped metric is characterized by

(1) for each $y \in M_2$, the map $\pi|_{M_1 \times \{y\}}$ is a isometry onto M_1;
(2) for each $x \in M_1$, the map $\eta|_{\{x\} \times M_2}$ is a positive homothety onto M_2;
(3) for each $(x, y) \in M$, the leaf $M_1 \times \{y\}$ and the fiber $\{x\} \times M_2$ are orthogonal at (x, y).

Vectors tangent to leaves are called horizontal and those tangent to fibers are called verticals. We denote by $\mathcal{H}$ the orthogonal projection of $T_{(x,y)}M$ onto its horizontal subspace $T_{(x,y)}(M_1 \times \{q\})$, and by $\mathcal{V}$ the projection onto the vertical subspace $T_{(x,y)}(\{p\} \times M_2)$.

Now, we denote by tan for the projection $\mathcal{V}$ onto $T_{(x,y)}\big(\{x\} \times M_2\big)$ and by nor for the projection onto $T_{(x,y)}\big(M_1 \times \{y\}\big) = \Big(T_{(x,y)}\big(\{x\} \times M_2\big)\Big)^{\perp}$.

If $u \in T_x M_1, x \in M_1$ and $y \in M_2$, then the lift $\bar{u}$ of u to (x, y) is the unique vector in $T_{(x,y)}M$ such that $d\pi_1(\bar{u}) = u$. For a vector field $X \in \mathfrak{X}(M_1)$, the lift of X to M is the vector field $\bar{X}$ whose value at each (x, y) is the lift of X_x to (x, y). The set of all horizontal lifts is denoted by $\mathcal{L}(B)$. Similarly, we denote by $\mathcal{L}(F)$ the set of all vertical lifts.

The relation of a warped product to the base M_1 is almost as simple as in the special case of a Riemannian product. However, the relation to the fiber M_2 often involves the warping function f.

Lemma 3.1. [12] *If $h \in \mathcal{F}(M_1)$, then the gradient of the lift $h \circ \pi$ of h to $M = M_1 \times_f M_2$ is the lift to M of the gradient of h on M_1.*

Thus there should be no confusion if we simplify the notation by writing h for $h \circ \pi$ and grad h for grad$(h \circ \pi)$. The Levi-Civita connection of $M = M_1 \times_f M_2$ is related to those M_1 and M_2 as follows.

Lemma 3.2. [12] *Let $M = M_1 \times_f M_2$ be a warped product manifold. Denote by ∇, ∇^1 and ∇^2 the Levi-Civita connections on M, M_1 and M_2, respectively. Then, for any $X, Y, Z \in \mathcal{L}(M_1)$ and $U, V, W \in \mathcal{L}(M_2)$, we have:*

(1) $\nabla_X Y$ is the lift of $\nabla^1_X Y$.

(2) $\nabla_X V = \nabla_V X = (X \cdot f/f)V$.
(3) The component of $\nabla_V W$ *normal to the fibers is* $-\big(g(V,W)/f\big)\mathrm{grad} f$.
(4) The component of $\nabla_V W$ *tangent to the fibers is the lift of* $\nabla^2_V W$.

Lemma 3.3. [12] *Let* $M = M_1 \times_f M_2$ *be a warped product. Let* R, R^1 *and* R^2 *be the Riemannian curvature tensors with respect to the Levi-Civita connections* ∇, ∇^1 *and* ∇^2 *on* M, M_1 *and* M_2*, respectively. Let* $X, Y, Z \in \mathcal{L}(M_1)$ *and* $U, V, W \in \mathcal{L}(M_2)$*. The Riemannian curvature operator* R *of* M *is given by:*

(1) $R(X,Y)Z = R^1(X,Y)Z$,
(2) $R(X,U)Y = f^{-1}\nabla^2 f(X,Y)U$,
(3) $R(X,Y)U = R(U,V)X = 0$,
(4) $R(U,X)V = f^{-1}g_2(U,V)\nabla_X \nabla f$,
(5) $R(U,V)W = R^2(U,V)W - f^{-2}g_B(\nabla f, \nabla f)\Big[g_2(U,W)V - g_F(V,W)U\Big]$.

Lemma 3.4. [12] *Let* $M = M_1 \times_f M_2$ *be a warped product. Let* $\mathrm{Ric}, \mathrm{Ric}^1$ *and* Ric^2 *be the Ricci curvature tensors with respect to the Levi-Civita connections* ∇, ∇^1 *and* ∇^2 *on* M, M_1 *and* M_2*, respectively. Let* $X, Y, Z \in \mathcal{L}(M_1)$ *and* $U, V, W \in \mathcal{L}(M_2)$*. The Ricci curvature* Ric *of the warped product* $M = M_1 \times_f M_2$ *with* $m_2 = \dim M_2$ *satisfies*

(1) $\mathrm{Ric}(X,Y) = \mathrm{Ric}^1(X,Y) - \frac{m_2}{f}H^f(X,Y)$;
(2) $\mathrm{Ric}(X,U) = 0$;
(3) $\mathrm{Ric}(U,V) = \mathrm{Ric}^2(U,V) - g_2(U,V)\Big[\frac{\Delta f}{f} + \frac{m_2-1}{f^2}g_1(\nabla f, \nabla f)\Big]$;

for any $X, Y \in \mathfrak{X}(M_1)$ *and* $U, V \in \mathfrak{X}(M_2)$*, where* H^f *and* Δf *denote the Hessian of* f *and the Laplacian of* f *given by* $-\mathrm{trace}(H^f)$*, respectively.*

Lemma 3.5. *The warped product metric is in the conformal class of a direct product metric.*

Proof. By definition:

$$g = g_1 + f^2 g_2 = f^2\Big(\frac{1}{f^2}g_1 + g_2\Big) = (\tilde{g}_1 + g_2),$$

where $\tilde{g}_1 = \frac{1}{f^2}g_1$, that is, the warped product can be expressed as conformal to a direct product, where the conformal factor is f^2. □

Let $(M_1 \times_f M_2, g)$ be a Riemannian warped product manifold of the Riemannian manifolds (M_1, g_2) and (M_2, g_2) with $g = g_1 + g_2$. Then the Riemannian warped product manifold $(M_1 \times M_2, g)$ is Szabó if and only if (M_1, g_2) and (M_2, g_2) are Szabó.

Proof. It follows immediately from Theorem 2 just using that any warped product metric is in the conformal class of a product metric. □

A Riemannian Warped product $M_1 \times_f M_2$ is Szabó if and only if it is a space of constant sectional curvature

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