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WEIERSTRASS FORMULA FOR MINIMAL SURFACES IN SPECIAL THREE-DIMENSIONAL $\delta-$LORENTZIAN TRANS SASAKIAN MANIFOLD

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ABSTRACT. In this paper, we describe a methode to derive a Weierstrass-type formula for a simply connected minimal surfaces in special three-dimensional $\delta-$Lorentzian trans sasakian manifold $\mathbb{T}$. We consider the $\delta-$Lorentzian metric and use somme results of Levi-Civita coonection. Furthermore, we contruct examples of minimal surfaces in this space.

1. INTRODUCTION

Minimal surface, such as soap film, has zero curvature at every point. And it is well-known that the classical Weierstrass representation formula represents minimal surfaces in $\mathbb{R}^3$ via holomorphic functions. This reprentation of minimal surfaces is a very important notion in mathematics and physics for its applications. Mathematicians are attracted in studying of minimal surfaces that have certain properties, such as completedness and finite total curvature, while scientists are more inclined to periodic minimal surfaces observed in crystals or biosystems such as lipid bilayers.
For the first time Weierstrass representation for conformal immersion of surface into $\mathbb{R}^3$ appeared in the result of variational problem on search of minimal surface [13].
Weierstrass representation for minimal surfaces into Hyperbolic space have been obtained by kokubu in [14] ,Kenmotsu gave a type of weierstrass representation of prescribed mean curvature in [12]. Bonbenko have contributed to construct minimal surfaces and to understand thier propertie [10]. The works of Konopelchenko, Taimanov,Landolfi ([3],[4], [5]), Berdinski and Taimanov [6], Hofiman and Osserman [7], Uhlenbeck [11], and Osserman [18] have helped to build many examples and understand many properties of mimal surfaces.
More recently, Mercuri, Montaldo and Piu have published A Weierstrass representation formula for minimal surfaces $\mathbb{H}_3$ and $\mathbb{H}^2 \times \mathbb{R}$ [14]. Later in [9],Turhan and k$\ddot{o}$pinar described minimal immersion in sol space, Koivogui and Todjihoundé in [17] give a Weierstrass representation for minimal immersions into Damek-Ricci spaces. And more recently Adriana, Francesco and Irene in [1], complex analysis and paracomplex analysis served to discuss a type of Weierstrass representation for minimal surfaces in Lorentzian Heisenberg group and Damek-Ricci spaces.

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In this paper, we applied the general setting on $\delta-$Lorentzian trans sasakian manifold and described a method to derive Weierstrass-type representation formulas for simply connected minimal surfaces into three-dimensional special $\delta-$Lorantzian trans sasakian manifold $\mathbb{T}^3$.

2. Preliminaries

Let $(M;\varphi;\xi;\eta;g)$ be a $2n+1-$dimensional contact metric manifold, where φ is a $(1;1)-$tensor field, ξ a unit vector field and η a smooth $1-$form dual to ξ with respect to the Riemannian metric g satisfying :

$$\varphi^2 = I + \eta \otimes \xi,\ \varphi(\xi) = 0, \eta \circ \varphi = 0, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) + \delta\eta(X)\eta(Y),\ g(X,\varphi Y) = -g(\varphi X, Y), \tag{2.3}$$

$$\delta g(X,\xi) = \eta(X) \tag{2.4}$$

$$\nabla_X \xi = X - \eta(X)\xi \tag{2.5}$$

for all $X, Y \in \mathcal{X}(M)$, where $\mathcal{X}(M)$ is the Lie algebra of smooth vector fields on$M$ [8]. δ is such that $\delta^2 = 1$ so that $\delta = \pm 1$. The above structure $(\varphi, \xi, \eta, g\delta)$ on M is called the δ-Lorentzian structure on M. If $\delta = 1$, this is the usual Lorentzian structure [13], the vector field ξ is the time like[19] , that is M contains a time like vector field. In [16], Tanno classified the connected almosst contact metric manifold in three classes. we define the δ-Lorentzian trans-Sasakian manifolds (see [2]) as follows:

Definition 2.1. *A δ-Lorentzian manifold with structure $(\varphi, \xi, \eta, g, \delta)$ is said to be δ-Lorentzian trans-Sasakian manifold of type (α, β) if it satisfies the condition*

$$(\nabla_X \varphi)Y = \alpha(g(X;Y)\xi - \delta\eta(Y)X) + \beta(g(\varphi X;Y)\xi - \delta\eta(Y)\varphi X). \tag{2.6}$$

3. Weierstrass representation for minimal surfaces in 2n+1-dimensional δ-Lorentzian trans sasakian manifold

3.1. **Minimal surfaces in Riemannian manifold.** The arguments will be essentially local so we will consider, as ambient manifold M, the space $\mathbb{R}^{2n+1}$ with a Riemannian metric $g = (g_{ij})$. We will denote by $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$ a simply connected domain with a complex coordinate $z = u + iv, u, v \in \mathbb{R}$, and by:

$$\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}); \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v})$$

the complex derivatives.
In this situation, the general Weierstrass representation formula can be stated as follows:

Theorem 3.1. *(see* [14]*)*
Let $f : \Omega \to M$ be a conformal minimal immersion and $g = (g_{ij})$ be the induced metric. The complex tangent vector:

$$\frac{\partial f}{\partial \bar{z}} := \phi := \sum_i \phi_i \frac{\partial}{\partial x_i};\ \phi_i : \Omega \to \mathbb{C}$$

has the following properties:

(1) $\sum_{i,j} \phi_i \bar{\phi}_j \neq 0$
(2) $\sum_{i,j} \phi_i \phi_j = 0$

(3) $\frac{\partial \phi_i}{\partial \bar{z}} + \frac{1}{2}\sum_{j,k}\Gamma^i_{j,k}\bar{\phi}_j\phi_k = 0,$

where $\{\Gamma^i_{j,k}\}$ *are the Christoffel symbols of the Riemannian connection. Conversely, given functions:* $\phi_i : \Omega \to \mathbb{C}$ *that verify the above conditions, then the map:*

$$f : \Omega \to M, f_i(z) = 2\Re e\int_{z_0}^{z}\phi_i dz,$$

is a well defined conformal minimal immersion of Ω *into* M *(here* z_0 *is an arbitrary fixed point of* Ω *and the integral is along any curve joining* z_0 *to* z*).*

Remark 3.1. *The first condition of theorem 3.1 tells us that f is an immersion, the second that f is conformal and the last one that f is minimal. The last condition is called the holomorphicity condition since it is the local coordinates version of the condition:* $\tilde{\nabla}_{\frac{\partial}{\partial \bar{z}}}\phi = 0$*, where* $\tilde{\nabla}$ *is the induced connection on the pull-back bundle* $f^{\star}(TM \otimes \mathbb{C})$*. In fact, we have that the section* ϕ *is holomorphic if and only if*

$$\begin{aligned}
\tilde{\nabla}_{\frac{\partial}{\partial \bar{z}}}\phi &= \tilde{\nabla}_{\frac{\partial}{\partial \bar{z}}}\sum_i \phi_i \frac{\partial}{\partial x_i} \\
&= \sum_i\{\frac{\partial \phi_i}{\partial \bar{z}}\frac{\partial}{\partial x_i} + \phi_i \nabla_{\frac{\partial f}{\partial \bar{z}}}\frac{\partial}{\partial x_i}\} \\
&= \sum_i\{\frac{\partial \phi_i}{\partial \bar{z}}\frac{\partial}{\partial x_i} + \phi_i \nabla_{\sum_j \bar{\phi}_j \frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_i}\} \\
&= \sum_i\{\frac{\partial \phi_i}{\partial \bar{z}} + \sum_{j,k}\Gamma^i_{jk}\bar{\phi}_j\phi_k\}\frac{\partial}{\partial x_i}
\end{aligned} \tag{3.1}$$

In general it is quite difficult to produce functions ϕ_i with the above properties since the holomorphic condition is given by partial differential equations with nonconstant coefficients.
If M is a Lie group equipped with a left-invariant metric g and $\{e_i\}$ are orthonormal left-invariant vector fields, we can write

$$\phi = \sum_i \phi_i \frac{\partial}{\partial x_i} = \sum_i \psi_i e_i \quad \psi_i : \Omega \to \mathbb{C}.$$

with $\phi_i = \sum_{ij} A_{ij}\psi_j$ and $A = (A_{ij})$ being an invertible matrix, with function entries A_{ij}. In this case the Weierstrass formula becomes:

Theorem 3.2. *(see* [14]*) Given functions* $\psi_i : \Omega \to \mathbb{C}$ *such that:*

(1) $\sum_i \mid \psi_i \mid^2 \neq 0$
(2) $\sum_{i,j}\psi^2 = 0$
(3) $\frac{\partial \psi_i}{\partial \bar{z}} + \sum_{j,k} L^i_{j,k}\bar{\psi}_j\psi_k = 0,$

where $L^i_{jk} := g(\nabla_{e_j}e_k, e_i)$*, then the map:*

$$f : \Omega \to M, f_i(z) = 2\Re e\int_{z_0}^{z}\sum_{i,j}A_{ij}\psi_j dz, \quad \psi_i : \Omega \to \mathbb{C}$$

defines a conformal minimal immersion.

3.2. **Minimal surfaces in $\delta-$lorentzian trans sasakian manifold.** Let $(\mathbb{T}^{2n+1};\varphi;\xi;\eta;g)$ be a $(2n+1)-$dimensional $\delta-$lorentzian trans sasakian manifold and let $f\Omega \subseteq \mathbb{L} \to M$ be a conformal minimal immersion in an open set $\Omega \subseteq \mathbb{L}$. Let $\{e_1;e_2;...;e_{2n+1}\}$ orthonormal basis such that

$$\begin{cases} g(e_i;e_j)=0, & \text{if } i\neq j;\\ g(e_i;e_i)=1, & \text{if } i\in\{1,2,..,2n\}\\ g(e_{2n+1};e_{2n+1})=-\delta. \end{cases}$$

with $e_{2n+1}=\xi$.
If g is a Lorentzian metric, for the case of spacelike surfaces i.e. the induced metric $f^{\star}g$ is Riemannian, the statement is the same as that the section above. For timelike surfaces i.e. the induced metric $f^{\star}g$ is Lorentzian, let Ω be is an open subset of the paracomplex numbers L and the expression $\frac{\partial\phi_i}{\partial\bar{z}}$ as well as the conjugation, has to be understood in the Lorentz or paracomplex sense. We recall that the algebra of the paracomplex numbers is the algebra $\mathbb{L}=\{a+\tau b; a,b\in\mathbb{R}\}$, where τ is an imaginary unit whith $\tau^2=1$.The internal operations are the obvious ones and this algebra is isomorphic to $\mathbb{R}\oplus\mathbb{R}$ via the map

$$a+\tau b\mapsto \frac{1}{2}(a+b;a-b)$$

The norm of $z\in\mathbb{L}$ is denoted by $\mid z\mid$ and defined by

$$\mid z\mid=\sqrt{\mid z\bar{z}\mid}$$

Also, the set of zero divisors of $\mathbb{L}$ is given by $\mathcal{K}=\{a\pm\tau a; a\neq 0\}$.
The set $\mathbb{L}$ has a natural topology as a 2-dimensional real vector space.

Definition 3.1. *Let $\Omega\subseteq\mathbb{L}$ be an open set and $z_0\in\Omega$. The $\mathbb{L}$-derivative of a function $f:\Omega\to\mathbb{L}$ at z_0 is defined by*

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0},\ \textit{with}\ z-z_0\in\mathbb{L}-K\cup\{0\}$$

if this limit exists. If $f'(z_0)$ exists, we will say that f is $\mathbb{L}-$differentiable at z_0.

Remark 3.2. *$\mathbb{L}$-differentiability in an open set Ω implies usual differentiability in Ω (for more details, see* [13]*).*

We introduce the paracomplex operators :

$$\frac{\partial}{\partial z}:=\frac{1}{2}(\frac{\partial}{\partial u}+\tau\frac{\partial}{\partial v});\quad \frac{\partial}{\partial\bar{z}}:=\frac{1}{2}(\frac{\partial}{\partial u}-\tau\frac{\partial}{\partial v})$$

where $z=u+\tau v$. A differentiable function $f:\Omega\to\mathbb{L}$ is $\mathbb{L}-$differntiable if and only if $\frac{\partial f}{\partial\bar{z}}=0$ and for $z=a+\tau b$, we have $\bar{z}=\overline{a+\tau b}=a-\tau b$.

We can write the paracomplex tangent vectors both in terms of local coordinates $\{x_1;x_2;...;x_{2n+1}\}$ in M and so that

$$\phi:=\frac{\partial f}{\partial\bar{z}}=\sum_{i=1}^{2n+1}\phi_i\frac{\partial}{\partial x_i}=\sum_{i=1}^{2n+1}\psi_i e_i,$$

where the functions ϕ_i and ψ_j for $i, j = 1, ..., 2n+1$ are related by

$$\phi_i = \sum_{i=1}^{2n+1} A_{ij}\psi_j. \tag{3.2}$$

where $A : \Omega \to GL(2n+1, \mathbb{R})$ is a smooth map. In this case, the Theormen 3.1 may be repplace as follows:

Theorem 3.3. *Let $(\mathbb{T}^{2n+1}; \varphi; \xi; \eta; g)$ be a $(2n+1)-$dimensional $\delta-$lorentzian trans sasakian manifold and let $\{e_1; e_2; ...; e_{2n+1}\}$ be an orthonormal frame field. let $f : \Omega \subseteq \mathbb{L} \to \mathbb{T}^{2n+1}$ be a conformal minimal immersion in an open set $\Omega \subseteq \mathbb{L}$. We denote by $\phi \in \Gamma(f^{\star}T(\mathbb{T}^{2n+1}) \otimes \mathbb{L})$ the paracomplex vector*

$$\phi := \frac{\partial f}{\partial \bar{z}} = \sum_{i=1}^{2n+1} \psi_i e_i.$$

Then, the components $\psi_i, i = 1; ..., 2n+1$ of ϕ satisfy the following conditions:

(1) $\psi_1\overline{\psi_1} + ... + \psi_{2n}\overline{\psi_{2n}} - \delta\psi_{2n+1}\overline{\psi_{2n+1}} = 0,$
(2) $\psi_1^2 + ... + \psi_{2n}^2 - \delta\psi_{2n+1}^2 = 0,$
(3) $\frac{\partial\psi_i}{\partial\bar{z}} + \sum_{j,k=1}^{2n+1} L_{j,k}^i \bar{\psi}_j\psi_k = 0,$

where the symbol $L_{ij}^k := g(\nabla_{e_i}e_j; e_k),\ for\ i, j = 1, ...2n+1.$
Conversely, if $\Omega \subseteq \mathbb{L}$ is a simply connected domain and $\psi_i : \Omega \to \mathbb{L},\ i = 1; ...; 2n+1$ are paracomplex functions satisfying the conditions above, then the map $f : \Omega \to M$ which coordinates are given by

$$f_i = 2\Re e \int \sum_{i=1}^{2n+1} A_{ij}\psi_j dz,\ for\ i = 1; ...; 2n+1,$$

is a well-defined conformal minimal immersion

Remark 3.3. *We observe that the formula in Theorem 3.3 is not a direct integration since we have to compute the matrix A_{ij} along a solution. However, as we will see, for particular ambient spaces, this problem may be solved by ad hoc arguments.*

4. The Weierstrass Representation in the special 3-dimensional $\delta-$Lorentzian trans-Sasakian manifold

We consider the 3-dimensional manifold $M = \mathbb{T}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_3 \neq 0\}$, where (x_1, x_2, x_3) are standard coordinates of $\mathbb{R}^3$. The vector fields

$$e_1 = x_3\frac{\partial}{\partial x_1}, \quad e_2 = x_3\frac{\partial}{\partial x_2}, \quad e_3 = \delta x_3\frac{\partial}{\partial x_3}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$\begin{cases} g(e_i; e_j) = 0, & \text{if } i \neq j \\ g(e_i, e_i) = 1, & \text{if } i \in \{1, 2\} \\ g(e_3, e_3) = -\delta. \end{cases}$$

that is, the form of the metric becomes

$$g = \frac{dx_1^2 + dx_2^2 - \delta dx_3^2}{x_3^2}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \mathcal{X}(\mathbb{T}^3)$. Let φ be the $(1,1)-$ *tensor* field defined by

$$\varphi(e_1) = -e_2, \varphi(e_2) = e_1, \varphi(e_3) = 0.$$

Then using the linearity of φ and g, we have

$$\begin{aligned} \eta(e_3) &= -1 \\ \varphi^2 Z &= Z + \eta(Z)e_3 \\ g(\varphi Z, \varphi W) &= g(Z,W) + \delta\eta(Z)\eta(W) \end{aligned}$$

for all $Z, W \in \mathcal{X}(M)$. Then for $e_3 = \xi$, the structure (φ, ξ, η, g) defines an almost contact metric structure on M. Let ∇ be the Levi-Civita connection with respect to metric g. Then we have

$$\begin{aligned} [e_1; e_3] &= e_1e_3 - e_3e_1 \\ &= x_3\frac{\partial}{\partial x_1}(\delta x_3 \frac{\partial}{\partial x_3}) - x_3\frac{\partial}{\partial x_3}(\delta x_3\frac{\partial}{\partial x_1}) \\ &= \delta(x_3^2\frac{\partial^2}{\partial x_1 x_3} - x_3^2\frac{\partial^2}{\partial x_1 x_3} - x_3\frac{\partial}{\partial x_1}) \\ &= -\delta e_1 \end{aligned}$$

similary

$$[e_1; e_2] = 0 \quad and \quad [e_2; e_3] = -\delta e_2.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = &Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) \\ &-g(Y,[X,Z]) + g(Z,[X,Y]) \end{aligned} \tag{4.1}$$

which known as Koszul's formula.

Lemma 4.1. *For the covariant derivatives of the Levi-Civita connection of the metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} \delta e_3 & 0 & -\delta e_1 \\ 0 & \delta e_3 & -\delta e_2 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.2}$$

where the $(i,j) -$ element in the table above equals $\nabla_{e_i} e_j$ for our basis $\{e_1; e_2; e_3\}$.

The manifold $\mathbb{T}^3$ is satisfied for $\alpha = 0$ and $\beta = -1$. Hence the manifold is a transsasakian manifold of type $(0, -1)$. Also, we have that the non zero $L_{ij}^k = g(\nabla_{e_i} e_j; e_k)$ are :

$$L_{11}^3 = -1; L_{13}^1 = -\delta; L_{22}^3 = 1; L_{23}^2 = -\delta \tag{4.3}$$

The harmonicity condition is given by the following system of PDEs:

$$\begin{cases} \frac{\partial \psi_1}{\partial \bar{z}} - \delta\bar{\psi}_1\psi_3 = 0 \\ \frac{\partial \psi_2}{\partial \bar{z}} - \delta\bar{\psi}_2\psi_3 = 0 \\ \frac{\partial \psi_3}{\partial \bar{z}} - \psi_1\bar{\psi}_1 - \psi_2\bar{\psi}_2 = 0 \end{cases} \tag{4.4}$$

Theorem 4.1. *Let $\psi_i, i = 1,2,3$, be three paracomplex-valued functions defined in a simply connected domain $\Omega \subseteq \mathbb{L}$, such that the following conditions are satisfied :*

- $\psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_2 - \delta\psi_3\bar{\psi}_3 \neq 0.$
- $\psi_1^2 + \psi_2^2 - \delta\psi_3^2 = 0.$
- $\psi_i, i = 1,2,3$ *solutions f the system* (4.4).

Then map $f : \Omega \subseteq \mathbb{L} \to \mathbb{T}^3$ given by

$$\begin{cases} f_1 &= 2\Re\int_{z_0}^{z} x_3\psi_1 dz \\ f_2 &= 2\Re\int_{z_0}^{z} x_3\psi_2 dz \\ f_3 &= 2\delta\Re\int_{z_0}^{z} x_3\psi_3 dz \end{cases}$$

defines a conformal minimal immersion.

Proof. Using (3.2), we get

$$\phi_1 = x_3\psi_1,\ \phi_3 = x_3\psi_1\ \phi_3 = x_3\psi_3.$$

From $f_i = 2\Re e\int\sum_{i=1}^{2n+1} A_{ij}\psi_j dz,\ for\ i = 1;2;3$, we have the system (4.1). □

Remark 4.1. *The second equation of the theorem 4.1 for $\delta = -1$ suggests the definition of two new complex functions*

$$G = \sqrt{\frac{1}{2}(\psi_1 - i\psi_2)},\ H = \sqrt{\frac{1}{2}(\psi_1 + i\psi_2)} \tag{4.5}$$

The functions G and H are single-valued complex functions which for suitably chosen square roots, satisfy

$$\begin{aligned} \psi_1 &= G^2 - H^2; \\ \psi_2 &= i(G^2 + H^2) \\ \psi_3 &= 2GH \end{aligned} \tag{4.6}$$

In the following, we note $\frac{\partial G}{\partial \bar{z}}$ and $\frac{\partial G}{\partial \bar{z}}$ by : $\frac{\partial G}{\partial \bar{z}} = G_{\bar{z}}$ and $\frac{\partial H}{\partial \bar{z}} = H_{\bar{z}}$

Lemma 4.2. *If the section $\phi = \psi_1 e_1 + \psi_2 e_2 + \psi_3 e_3$ is holomorphic then*

$$GG_{\bar{z}} - HH_{\bar{z}} = -(\bar{G}^2 - \bar{H}^2)GH \tag{4.7}$$

$$GG_{\bar{z}} + HH_{\bar{z}} = (\bar{G}^2 + \bar{H}^2)GH \tag{4.8}$$

$$HG_{\bar{z}} + GH_{\bar{z}} = \mid G\mid^4 + \mid H\mid^4 \tag{4.9}$$

Proof. Substituting (4.1) into (4.4), we have he result. □

Corollary 4.1. *If G and H are complex functions which satisfy* (4.6) *then*

$$GG_{\bar{z}} = \mid H\mid^2 G\bar{H} \tag{4.10}$$

Corollary 4.2. *If G and H are complex functions which satisfy* (4.6) *then*

$$HH_{\bar{z}} = \mid G\mid^2 H\bar{G} \tag{4.11}$$

Corollary 4.3. *If G and H are complex functions which satisfy* (4.6) *then*

$$GH = \alpha(z) + (\mid G\mid^4 + \mid H\mid^4)\bar{z} \tag{4.12}$$

Theorem 4.2. *Let G and H be complex-valued functions defined on simply connected domain $\Omega \subseteq \mathbb{L}$ such that*

- *G and H are not identically zero;*
- *G and H are solutions of* (4.7),

Then the map $f : \Omega \to (\mathcal{M}, g_{\mathcal{M}})$ *defined by :*

$$\begin{cases} f_1(z) = 2\Re e \int_{z_0}^{z} f_3(G^2 - H^2)dz \\ f_2(z) = 2\Re e \int_{z_0}^{z} i f_3(G^2 + H^2)dz \\ f_3(z) = e^{-4\Re e \int_{z_0}^{z} GHdz}, \end{cases} \tag{4.13}$$

is a conformal spacelike minimal immersion.

Remark 4.2. *Now, we will construct some examples of timelike minimal immersions in the Lorentzian trans manifold* $\mathbb{T}^3$.

Example 4.1. *From* $\psi_1^2 + \psi_2^2 - \psi_3^2 = 0$ *(with* $\delta = 1$ *), the paracomplex functions*

$$\psi_1 = \psi_2 = \frac{\tau}{2u}; \psi_3 = \frac{1}{2u}$$

defined in $\Omega = \{u + \tau v \in \mathbb{L} : u > 0\}$ *satisfy the equation 4.4. Then the map* $f : \Omega \to (\mathbb{T}^3$ *with coordinates:*

$$\begin{cases} f_1(u,v) &= C\frac{v-v_0}{u_0} \\ f_2(u,v) &= C\frac{v-v_0}{u_0} \\ f_3(u,v) &= C(\frac{u}{u_0}), \end{cases} \tag{4.14}$$

is a conformal timelike minimal immersion, with $z_0 = u_0 + \tau v_0$. C*, a real constant.*

Example 4.2. *From* $\psi_1^2 + \psi_2^2 - \psi_3^2 = 0$ *(with* $\delta = 1$ *), we have*

$$\psi_1 = \rho \cos \varpi, \psi_2 = \rho \sin \varpi, \psi_3 = \tau\rho$$

If f satisfy 4.4, we have

$$\begin{cases} \frac{\partial\rho}{\partial\bar{z}} \cos \varpi - \rho \frac{\partial\varpi}{\partial\bar{z}} \sin \varpi &= \tau\rho\bar{\rho}\overline{\cos \varpi} \\ \frac{\partial\rho}{\partial\bar{z}} \sin \varpi - \rho \frac{\partial\varpi}{\partial\bar{z}} \cos \varpi &= \tau\rho\bar{\rho}\overline{\sin \varpi} \\ \tau\frac{\partial\rho}{\partial\bar{z}} &= \rho\bar{\rho}, \end{cases} \tag{4.15}$$

and that implies, if ρ *and* ϖ *are paracomplex-valued functions defined in a simply connected domain* $\Omega \subseteq \mathbb{L}$. *Then the map* $f : \Omega \to \mathbb{T}^3$ *defined by*

$$\begin{cases} f_1(u,v) = 2\Re e \int_{z_0}^{z} x_3\rho \cos \varpi dz \\ f_2(u,v) = 2\Re e \int_{z_0}^{z} x_3\rho \sin \varpi dz \\ f_3(u,v) = 2\Re e \int_{z_0}^{z} \tau x_3\rho dz \end{cases} \tag{4.16}$$

is a conformal timelike minimal immersion, with $z_0 = u_0 + \tau v_0$.

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