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FEUERBACH TANGENT CIRCLE - GENERALIZATION IN E^n

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ABSTRACT. We will show that if we define a Feuerbach tangent circle that is internally tangent to the inscribed circle and externally tangent to each of the circles which touch the sides of the triangle externally, then it can be generalized in any space E^n $(n \geq 2)$.

1. INTRODUCTION AND MOTIVATIONS

The nine-point circle of the triangle contains the feet of the altitudes, the midpoints of the sides and the midpoints of the segments that connect the vertices with the orthocenter. This circle is sometimes called the Euler circle [3]. The circle which is internally tangent to the inscribed circle and externally tangent to each of the circles which touch the sides of the triangle externally is called the Feuerbach circle. In the plane, these two circles coincide. Feuerbach discovered the nine-point circle of a triangle [2]. But this incorrectly attributes the result. We point out that these terms can be considered congruent only in a plane.

Let us consider the three-dimensional generalization of the previous concepts in the following way. A tetrahedron is a three-dimensional triangle. Let Euler's sphere contain all Euler's circles of its two-dimensional sides, while Feuerbach's should be tangent to all: inscribed and four externally escribed spheres of a tetrahedron. (The escribed sphere touches all sides of the tetrahedron and is located opposite with one of its sides in relation to the inscribed sphere.) Adhering to the previous definitions, we can ask ourselves whether there are Euler's and Feuerbach's spheres in the tetrahedron? We will show that in a three-dimensional generalization at the starting setup, previous two spheres may represent two different objects. Furthermore Euler sphere appears only in the orthocentric tetrahedron where there is the Feuerbach sphere but as a different object. Feuerbach tangent sphere exists in each tetrahedron. In addition, increasing the dimension of space in the last statements does not change the previous claim. A completely different generalization of these terms can be found in papers [5]-[7]. The generalization of the Feuerbach sphere is based on points of intersection with altitudes and medians, and not on touching with escribed and inscribe spheres. Except in the plane, such objects differ from each other.

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In a regular tetrahedron the Euler and Feuerbach spheres are not considered as new objects. Both spheres coincide with the inscribed sphere. In every other tetrahedron there are at least one of them as a new object.
The natural dimensional generalization of a triangle is n-simplex. In this paper, we will show that for $n \geq 2$, the Feuerbach tangent sphere exists in every n-simplex. The Euler sphere exists only in orthocentric n-simplexes. These two spheres in the same n-simplex are different objects for $n > 2$. This property justifies the different names that coincide for $n = 2$.
A generalization based on the tangent property of the Feuerbach circle can rarely be found in the literature. That was the decisive motive for this work. Proof of the existence of such a sphere requires solving the nonlinear system of $n+2$ equations with $n+1$ unknowns. Several key features and procedures applied to this system allow finding solutions in general form, which is contained in this paper.

2. Preliminaries

If we make the deformation on the regular tetrahedron so that all the vertices are moved, along the straight lines, determined by the altitude of the initial regular tetrahedron, the obtained tetrahedron does not lose its orthocentric property [1]. In such a tetrahedron, there are both the Feuerbach tangent sphere and Euler sphere. The centre of Euler's sphere is the midpoint of the segment with endpoints the orthocentre and centre of a circumscribed sphere of the tetrahedron. This centre coincides with the centroid of the tetrahedron. In the orthocentric tetrahedron, the Feuerbach sphere is the separate object which centre and radius do not coincide with the corresponding Euler sphere. We will distinguish two cases when the tetrahedron is orthocentric and when it is not.

2.1. **Tetrahedron is orthocentric.** In that case its vertices can be points: $A(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$ and $D(0,0,1)$ [1]. Now we find that it is orthocenter $H(0,0,0)$, centroid $T(\frac{1}{4},\frac{1}{4},\frac{1}{4})$ and centre of circumscribe sphere $O(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. The equation of the circumscribed sphere of the tetrahedron is $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2+\left(z-\frac{1}{2}\right)^2=\frac{3}{4}$. There is a sphere (Euler sphere) that contains all the Euler circles of all the two-dimensional sides of the tetrahedron. The equation of that sphere is $\left(x-\frac{1}{4}\right)^2+\left(y-\frac{1}{4}\right)^2+\left(z-\frac{1}{4}\right)^2=\frac{3}{16}$. The centre of that sphere is in the centroid of the tetrahedron. The radius of the Euler sphere is exactly one half of the radius of the circumscribed sphere of the tetrahedron. Substituting one after the other: $x=0$, $y=0$ and $z=0$, in previous equation we obtain the equations of all Euler circles of two-dimensional sides of a tetrahedron. The center of the inscribed sphere is $S\left(\frac{3-\sqrt{3}}{6},\frac{3-\sqrt{3}}{6},\frac{3-\sqrt{3}}{6}\right)$, and radius $r=\frac{3-\sqrt{3}}{6}$. The centres of the escribed spheres are: $S_A\left(\frac{3+\sqrt{3}}{6},\frac{3+\sqrt{3}}{6},\frac{3+\sqrt{3}}{6}\right)$, and radius $r_A=\frac{3+\sqrt{3}}{6}$; $S_B\left(\frac{1-\sqrt{3}}{2},\frac{\sqrt{3}-1}{2},\frac{\sqrt{3}-1}{2}\right)$, radius $r_B=\frac{\sqrt{3}-1}{2}$; $S_C\left(\frac{\sqrt{3}-1}{2},\frac{1-\sqrt{3}}{2},\frac{\sqrt{3}-1}{2}\right)$, radius $r_C=\frac{\sqrt{3}-1}{2}$ and $S_D\left(\frac{\sqrt{3}-1}{2},\frac{\sqrt{3}-1}{2},\frac{1-\sqrt{3}}{2}\right)$ with radius $r_D=\frac{\sqrt{3}-1}{2}$. (The indices A, B, ... denote the sphere opposite that vertex). Let us now determine the equation of the sphere that touches all the previous five. Let the centre of that sphere be at point (f_1,f_2,f_3) and the radius f. The required values are the solution of the following system.

$$\left(f_1-\frac{3-\sqrt{3}}{6}\right)^2+\left(f_2-\frac{3-\sqrt{3}}{6}\right)^2+\left(f_3-\frac{3-\sqrt{3}}{6}\right)^2=\left(f-\frac{3-\sqrt{3}}{6}\right)^2$$
$$\left(f_1-\frac{3+\sqrt{3}}{6}\right)^2+\left(f_2-\frac{3+\sqrt{3}}{6}\right)^2+\left(f_3-\frac{3+\sqrt{3}}{6}\right)^2=\left(f+\frac{3+\sqrt{3}}{6}\right)^2$$
$$\left(f_1-\frac{1-\sqrt{3}}{2}\right)^2+\left(f_2-\frac{\sqrt{3}-1}{2}\right)^2+\left(f_3-\frac{\sqrt{3}-1}{2}\right)^2=\left(f+\frac{\sqrt{3}-1}{2}\right)^2$$
$$\left(f_1-\frac{\sqrt{3}-1}{2}\right)^2+\left(f_2-\frac{1-\sqrt{3}}{2}\right)^2+\left(f_3-\frac{\sqrt{3}-1}{2}\right)^2=\left(f+\frac{\sqrt{3}-1}{2}\right)^2$$
$$\left(f_1-\frac{\sqrt{3}-1}{2}\right)^2+\left(f_2-\frac{\sqrt{3}-1}{2}\right)^2+\left(f_3+\frac{\sqrt{3}-1}{2}\right)^2=\left(f+\frac{\sqrt{3}-1}{2}\right)^2$$

The solution of this system is: $f_1 = f_2 = f_3 = \frac{\sqrt{3}}{9}$ and $f = \frac{1-\sqrt{3}}{3}$. Accordingly the required equation is $\left(x-\frac{\sqrt{3}}{9}\right)^2+\left(y-\frac{\sqrt{3}}{9}\right)^2+\left(z-\frac{\sqrt{3}}{9}\right)^2=\left(\frac{1-\sqrt{3}}{3}\right)^2$. The resulting equation is the tangent sphere of all five inscribed. It is quite clear that Feuerbach's and Euler's spheres are two different objects in three-dimensional space in the same tetrahedron.

2.2. **Tetrahedron is not orthocentric.** i) There is no sphere that would contain all the Euler circles of the two-dimensional sides. For example, let a tetrahedron be given with vertices in the points: $A(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$ and $D(1,1,1)$. The sphere is uniquely defined by four non-planar points. Euler circles in the sides contain points: $(0,0,0)$, $(\frac{1}{2},0,0)$, $(0,\frac{1}{2},0)$, $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $(0,0,1)$, etc. The first four uniquely define the sphere $\left(x-\frac{1}{4}\right)^2+\left(y-\frac{1}{4}\right)^2+\left(z-\frac{1}{4}\right)^2=\frac{3}{16}$, which does not contain $(0,0,1)$ point.
ii) Let us prove that in every tetrahedron there is a Feuerbach sphere. For the coordinates of an arbitrary tetrahedron (3-simplex), it is justified to observe its vertices in points: $A(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$ and $D(a,b,1)$, $a,b\in R$, (for more details and justification see [1]). The two-dimensional sides of this tetrahedron are in the following planes: ABC on $z=0$; ABD on $y-bz=0$; ACD on $x-az=0$ and BCD on $-x-y-(1-a-b)z+1=0$. The equations are written in such a way that the interior of the tetrahedron is positively oriented. If the centre of the inscribed circle is point (u,v,w) and radius r, then it is valid

$$r=w=\frac{v-bw}{\sqrt{1+b^2}}=\frac{u-aw}{\sqrt{1+a^2}}=\frac{-u-v-(1-a-b)w+1}{\sqrt{2+(1-a-b)^2}}$$

For simplicity, we introduce substitutions: $k_1=\sqrt{1+a^2}$, $k_2=\sqrt{1+b^2}$ and $k_3=\sqrt{2+(1-a-b)^2}$. By solving the previous system we find: $u=(a+k_1)r$, $v=(b+k_2)r$, $w=r$, and $r=\frac{1}{1+k_1+k_2+k_3}$. In a similar way, we find the centres and radii of all four spheres inscribed on the outside. The centre of the escribed sphere which is opposite the vertex A, is the point $((a+k_1)r_A,(b+k_2)r_A,r_A)$ and $r_A=\frac{1}{1+k_1+k_2-k_3}$; Opposite the vertex

B, the centre is the point $((a-k_1)r_B,(b+k_2)r_B,r_B)$ and $r_B=\frac{1}{1-k_1+k_2+k_3}$; Opposite the vertex C, the centre is the point $((a+k_1)r_C,(b-k_2)r_C,r_C)$ and $r_C=\frac{1}{1+k_1-k_2+k_3}$; and opposite the vertex D, the centre is the point $((a-k_1)r_D,(b-k_2)r_D,-r_D)$ and $r_D=\frac{1}{k_1+k_2+k_3-1}$. (Only r_D deviates from the previous rule because it appears on the account with a negative sign. The last sphere is in the negative part of the z axis.)
Let the point (f_1,f_2,f_3) be the centre of the sphere touching all the previous five and have the radius f. Due to the touch conditions, it is possible to write the following five quadratic equations with four unknowns:

$$\begin{aligned}
(f_1-(a+k_1)r)^2+(f_2-(b+k_2)r)^2+(f_3-r)^2&=(f-r)^2\\
(f_1-(a+k_1)r_A)^2+(f_2-(b+k_2)r_A)^2+(f_3-r_A)^2&=(f+r_A)^2\\
(f_1-(a-k_1)r_B)^2+(f_2-(b+k_2)r_B)^2+(f_3-r_B)^2&=(f+r_B)^2\\
(f_1-(a+k_1)r_C)^2+(f_2-(b-k_2)r_C)^2+(f_3-r_C)^2&=(f+r_C)^2\\
(f_1-(a-k_1)r_D)^2+(f_2-(b-k_2)r_D)^2+(f_3+r_D)^2&=(f+r_D)^2
\end{aligned}$$

Multiplying the first equation by -1, and adding the others we obtain a linear system of four equations with four unknowns. For simplification, it is convenient to convert sums and differences of radii into products. For example, $r_A-r=-2k_3rr_A$, $(a+k_1)r-(a-k_1)r_D=2k_1(S-a-k_1)r\cdot r_D$, etc., where $S=1+k_1+k_2+k_3$. That is how we obtain the system

$$\begin{aligned}
(a+k_1)f_1+(b+k_2)f_2+f_3+\frac{(S-k_3)}{k_3}f&=\alpha_1\\
(S-a-k_1)f_1-(b+k_2)f_2-f_3-\frac{(S-k_1)}{k_1}f&=\alpha_2\\
-(a+k_1)f_1+(S-b-k_2)f_2-f_3-\frac{(S-k_2)}{k_2}f&=\alpha_3\\
(aS-a-k_1)f_1+(bS-b-k_2)f_2+(S-1)f_3-(S-1)f&=\alpha_4
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1&=\left((a+k_1)^2+(b+k_2)^2\right)(S-k_3)\frac{1}{S(S-2k_3)}\\
\alpha_2&=\left(-(S-a-k_1)(aS-ak_1-k_1^2)+(b+k_2)^2(S-k_1)\right)\frac{1}{S(S-2k_1)}\\
\alpha_3&=\left((a+k_1)^2(S-k_2)-(S-b-k_2)(bS-bk_2-k_2^2)\right)\frac{1}{S(S-2k_2)}\\
\alpha_4&=\left(-(k_2S-b-k_2)(bS-b-k_1)-(k_1S-a-k_1)(aS-a-k_1)\right)\frac{1}{S(S-2)}
\end{aligned}$$

If we add the first equation to all the others, we have

$$\begin{aligned}
(a+k_1)f_1+(b+k_2)f_2 \quad & +f_3+\frac{S-k_3}{k_3}f && =\alpha_1\\
S\cdot f_1 \quad & +\left(\frac{S-k_3}{k_3}-\frac{S-k_1}{k_1}\right)f && =\alpha_2+\alpha_1\\
S\cdot f_2 \quad & +\left(\frac{S-k_3}{k_3}-\frac{S-k_2}{k_2}\right)f && =\alpha_3+\alpha_1\\
aS\cdot f_1+bS\cdot f_2 \quad & +S\cdot f_3-\frac{1-k_3}{k_3}Sf && =\alpha_4+\alpha_1
\end{aligned}$$

By multiplying the second equation with $-\frac{k_1}{S}$, the third with $-\frac{k_2}{S}$ and fourth with $-\frac{1}{S}$ and add to the first equation, we immediately get that

$$f=\frac{k_3\alpha_1-k_1\cdot\alpha_2-k_2\cdot\alpha_3-\alpha_4}{3\cdot S}.$$

It is clear that such a sphere always exists.

3. The Feuerbach tangent sphere ($n>3$)

An orthocentric 4-simplex in four-dimensional space consists of five tetrahedrons, every four with a common vertex, each three with a common triangle face and each two with a common edge. In orthocentric 4-simplex exist the Euler hypersphere containing all five of its three-dimensional Euler sphere. In other words, four-dimensional Euler hypersphere cutting through three-dimensional space of its three-dimensional faces at the respective Euler spheres. A regular four-dimensional simplex does not have the Euler or Feuerbach tangent hypersphere as special objects. The Euler and Feuerbach tangent hypersphere exists and coincides with the inscribed hyperspheres of a regular 4-simplex. If we make deformation to a regular 4-simplex moving some of its vertices in such a manner that 4-simplex remains orthocentric (moving a vertex along the altitudes of initial 4-simplex), we obtained an orthocentric 4-simplex which has both: the Euler and the Feuerbach hypersphere which are different objects. In non-orthocentric simplexes there is only the Feuerbach tangent hypersphere. We can formulate and prove the following theorem.

Theorem 1. In each Euclidean space E^n ($n\geq 2$) and in any n-simplex there is Feuerbach tangent hypersphere which touches inscribed and all $n+1$ escribed hyperspheres.

Proof. Without diminishing the generality, we can set up the vertices of the n-dimensional simplex at the points: $A_1(0,0,0,...,0)$, $A_2(1,0,0,...,0)$, $A_3(0,1,0,...,0)$,..., $A_n(0,0,0,...,1,0)$, $A_{n+1}(a_1,a_2,a_3,...,a_n,1)$, in a coordinate system with perpendicular axes: $x_1,x_2,\ldots,x_n$.

General equations for the $(n-1)$-dimensional facet are:

$$
\begin{aligned}
&A_1A_2\ldots A_n : x_n = 0;\\
&A_1A_3A_4\ldots A_{n+1} : x_1 - a_1x_n = 0;\\
&A_1A_2A_4\ldots A_{n+1} : x_1 - a_1x_n = 0;\\
&A_1A_2A_3A_5\ldots A_{n+1} : x_2 - a_2x_n = 0;\\
&\quad\vdots\\
&A_1A_2\ldots A_{n-1}A_{n+1} : x_{n-1} - a_{n-1}x_n = 0;\\
&A_2A_3A_4\ldots A_{n+1} : -x_1 - x_2 - \ldots - x_{n-1} - (1 - a_1 - a_2 - \ldots - a_{n-1})x_n + 1 = 0.
\end{aligned}
$$

Note that with these equations of its $n+1$ facets, the interior of an n-simplex is positively oriented. The centres of the inscribed and all the escribed hyperspheres are points equidistant from all $(n-1)$-dimensional facets. Let the centre of the inscribed hypersphere be the point $I(u_1, u_2, \ldots, u_n)$ and its radius equals r, then the unknown values can be determined from the following system of equations:

$$
\begin{aligned}
r = u_n = \frac{u_1 - a_1u_n}{\sqrt{1+a_1^2}} = \frac{u_2 - a_2u_n}{\sqrt{1+a_2^2}} = \cdots = \frac{u_{n-1} - a_{n-1}u_n}{\sqrt{1+a_{n-1}^2}}\\
= \frac{-u_1 - u_2 - \cdots - u_{n-1} - (1 - a_1 - \cdots - a_{n-1})u_n + 1}{\sqrt{n-1+(1-a_1-a_2-\cdots-a_{n-1})^2}}.
\end{aligned}
$$

Simplifying the denominators with: $k_1 = \sqrt{1+a_1^2}, k_2 = \sqrt{1+a_2^2}, \ldots, k_{n-1} = \sqrt{1+a_{n-1}^2}$, $k_n = \sqrt{n-1+(1-a_1-a_2-\cdots-a_{n-1})^2}$, and then solving the system, we can find that the centre and radius of the inscribed hypersphere are given by:

$$
I((a_1+k_1)r, (a_2+k_2)r, (a_3+k_3)r, \ldots, (a_{n-1}+k_{n-1})r, r);
$$

$$
r = \frac{1}{1+k_1+k_2+k_3+\ldots+k_n}.
$$

Similarly, we can set and solve a system of equations to find the coordinates of the centres and the corresponding radii of other escribed hyperspheres. It is necessary to consider the orientation of each hypersphere concerning equations of the facets it touches. For the hypersphere opposite the vertex A_1 with centre at the point $I_1(v_1, v_2, ..., v_n)$ and with the radius r_1, it is necessary to solve the following system of equations:

$$
\begin{aligned}
r_1 = v_n = \frac{v_1 - a_1v_n}{k_1} = \frac{v_2 - a_2v_n}{k_2} = \cdots = \frac{v_{n-1} - a_{n-1}v_n}{k_{n-1}}\\
= \frac{v_1 + v_2 + \cdots + v_{n-1} + (1 - a_1 - \cdots - a_{n-1})v_n - 1}{k_n}.
\end{aligned}
$$

This system differs from the previous one only in the sign of the hyper-plane given by the equation of the facet $A_2A_3\ldots A_{n+1}$. So, we find the equation of the hypersphere externally inscribed with respect to the outside of vertex A_1:

$$I_{A_1}((a_1+k_1)r_1, (a_2+k_2)r_1, (a_3+k_3)r_1, \ldots, (a_{n-1}+k_{n-1})r_1, r_1);$$
$$r_1 = \frac{1}{1+k_1+k_2+k_3+\ldots-k_n}.$$

In a similar way we can find the equations of the other externally inscribed hyperspheres:

$$A_2 : I_{A_2}((a_1-k_1)r_2, (a_2+k_2)r_2, (a_3+k_3)r_2, \ldots, (a_{n-1}+k_{n-1})r_2, r_2);$$
$$r_2 = \frac{1}{1-k_1+k_2+k_3+\ldots+k_n}.$$

For $2 \leq i \leq n$, we have:

$$A_i : I_{A_i}((a_1+k_1)r_i, (a_2+k_2)r_i, \ldots, (a_{i-1}-k_{i-1})r_i, \ldots, (a_{n-1}+k_{n-1})r_i, r_i);$$
$$r_i = \frac{1}{(1+k_1+k_2+\ldots-k_{i-1}+\ldots+k_n)}.$$

We can notice that only r_{n+1} deviates from the established rules, taking the negative sign. It is the only radius that has the opposite direction with respect to the base hyper-plane given by $x_n = 0$. The centre of the sphere opposite compared to A_{n+1}

$$I_{A_{n+1}}((a_1-k_1)r_{n+1}, (a_2-k_2)r_{n+1}, (a_3-k_3)r_{n+1}, \ldots, (a_{n-1}-k_{n-1})r_{n+1}, -r_{n+1});$$
$$r_{n+1} = \frac{1}{k_1+k_2+k_3+\ldots+k_n-1}.$$

Now, we will consider the hypersphere with centre $F(f_1, f_2, f_3, \ldots, f_n)$ and radius f which is tangent for all previous. Knowing that this unknown hypersphere touches all the inscribed and escribed hyperspheres, we obtain the non-linear system of $n+2$ equations with $n+1$ unknowns. The first equation we form from the condition that the difference between the radius of the unknown hypersphere and the radius of the corresponding inscribed hypersphere is equal to the distance between their centres. All other equations are obtained from the condition that the sum of the radii of the unknown Feuerbach tangent hypersphere and the corresponding escribed hyperspheres equals the distance between their centres.

In that way, we obtain the following system of equations:

$$\begin{aligned}
&(f_1-(a_1+k_1)r)^2+(f_2-(a_2+k_2)r)^2+\cdots+(f_{n-1}-(a_{n-1}+k_{n-1})r)^2+(f_n-r)^2 \\
&\quad=(f-r)^2 \\
&(f_1-(a_1+k_1)r_1)^2+(f_2-(a_2+k_2)r_1)^2+\cdots+(f_{n-1}-(a_{n-1}+k_{n-1})r_1)^2+(f_n-r_1)^2 \\
&\quad=(f+r_1)^2 \\
&(f_1-(a_1-k_1)r_2)^2+(f_2-(a_2+k_2)r_2)^2+\cdots+(f_{n-1}-(a_{n-1}+k_{n-1})r_2)^2+(f_n-r_2)^2 \\
&\quad=(f+r_2)^2 \\
&\quad\vdots \\
&(f_1-(a_1+k_1)r_n)^2+(f_2-(a_2+k_2)r_n)^2+\cdots+(f_{n-1}-(a_{n-1}-k_{n-1})r_n)^2+(f_n-r_n)^2 \\
&\quad=(f+r_n)^2 \\
&(f_1-(a_1-k_1)r_{n+1})^2+(f_2-(a_2-k_2)r_{n+1})^2+\cdots+(f_{n-1}-(a_{n-1}-k_{n-1})r_{n+1})^2 \\
&\quad+(f_n+r_{n+1})^2=(f+r_{n+1})^2 \qquad (3.1)
\end{aligned}$$

Although the previous system is not linear and has $n+2$ equations with $n+1$ unknowns, it can be solved in the following way. By multiplying the first equation by -1 and adding to all the other equations, we obtain the linear system of $n+1$ equations with $n+1$ unknowns. Then, each obtained equation can be further simplified by converting the sums and differences of the radii into the product in the following way.

$$\begin{aligned}
r-r_1&=\frac{1}{(1+k_1+k_2+\cdots+k_n)}-\frac{1}{(1+k_1+\cdots+k_{n-1}-k_n)}=-2k_n\cdot r\cdot r_1, \\
r-r_i&=2k_{i-1}\cdot r\cdot r_i, i=1,2,3,\ldots,n, \\
r-r_{n+1}&=-2r\cdot r_{n+1}, \\
r+r_i&=2(S-k_i)\cdot r\cdot r_i, i=1,2,3,\ldots,n, \\
r+r_{n+1}&=2(S-1)\cdot r\cdot r_{n+1}, \\
(a_i+k_i)r-(a_i-k_i)r_i&=2(k_i(S-k_{i-1})-a_ik_{i-1})\cdot r\cdot r_i, i=2,3,\ldots,n, \\
(a_i+k_i)r-(k_i-a_i)r_{n+1}&=2(a_i(S-1)-k_i)\cdot r\cdot r_{n+1}, \qquad (3.2)
\end{aligned}$$

where S is sum $1+k_1+k_2+\ldots+k_n$. We will show the simplification process only in the first equation. After multiplying the first equation of the system (3.1) by -1 and adding to the second with the elementary separation of the known from the unknown, we get the equation.

$$2\sum_{i=1}^{n-1}(a_i+k_i)(r-r_1)f_i+2(r-r_1)f_n-2(r+r_1)f=\left(r^2-r_1^2\right)\sum_{i=1}^{n-1}(a_i+k_i)^2.$$

Using formula (3.2), and canceling with $-4k_n\cdot rr_1$ we get

$$\sum_{i=1}^{n-1}(a_i+k_i)f_i+f_n-\frac{S-k_n}{k_n}f=(S-k_n)rr_1\sum_{i=1}^{n-1}(a_i+k_i)^2.$$

By acting similarly with all other equations, we get a system

$$
\begin{aligned}
&(a_1+k_1)f_1+(a_2+k_2)f_2+\cdots+(a_{n-1}+k_{n-1})f_{n-1}+f_n+\frac{S-k_n}{k_n}f=\alpha_1\\
&(S-a_1-k_1)f_1+(a_2+k_2)f_2+\cdots+(a_{n-1}+k_{n-1})f_{n-1}-f_n-\frac{(S-k_1)}{k_1}f=\alpha_2\\
&-(a_1+k_1)f_1+(S-a_2-k_2)f_2+\cdots+(a_{n-1}+k_{n-1})f_{n-1}-f_n-\frac{S-k_2}{k_2}f=\alpha_3\\
&\vdots\\
&-(a_1+k_1)f_1-(a_2+k_2)f_2-\cdots+(S-a_{n-1}-k_{n-1})f_{n-1}-f_n-\frac{S-k_{n-1}}{k_{n-1}}f=\alpha_n\\
&(a_1S-a_1-k_1)f_1+(a_2S-a_2-k_2)f_2+\cdots+(a_{n-1}S-a_{n-1}-k_{n-1})f_{n-1}\\
&\qquad+(S-1)f_n-(S-1)f=\alpha_{n+1}
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_1&=\left((a_1+k_1)^2+(a_2+k_2)^2+\ldots+(a_{n-1}+k_{n-1})^2\right)(S-k_n)\,r\cdot r_1\\
\alpha_2&=(-(S-a_1-k_1)(a_1S-a_1k_1-k_1^2)+(a_2+k_2)^2(S-k_1)+(a_3+k_3)^2(S-k_1)\\
&\qquad\cdots+(a_{n-1}+k_{n-1})^2(S-k_1))r\cdot r_2\\
\alpha_3&=((a_1+k_1)^2(S-k_2)-(S-a_2-k_2)(a_2S-a_2k_2-k_2^2)+(a_3+k_3)^2(S-k_2)\\
&\qquad\cdots+(a_n+k_n)^2(S-k_2))\cdot rr_3\\
&\vdots\\
\alpha_n&=\Big((a_1+k_1)^2(S-k_n)+(a_2+k_2)^2(S-k_n)\\
&\qquad\cdots-(S-a_n-k_n)\left(a_nS-a_nk_n-k_n^2\right)\Big)\cdot rr_n\\
\alpha_{n+1}&=\big(-(S-a_1-k_1)\left(a_1S-a_1k_1-k_1^2\right)-(S-a_2-k_2)\left(a_2S-a_2k_2-k_2^2\right)\\
&\qquad-\cdots-(S-a_n-k_n)\left(a_nS-a_nk_n-k_n^2\right)\big)\cdot rr_{n+1}.
\end{aligned}
$$

Multiply the second equation by $-\frac{k_1}{S}$, the third by $-\frac{k_2}{S},\ldots$, the penultimate by $-\frac{k_n}{S}$ and the last by $-\frac{1}{S}$ and add it all to the first. That is how we get the system which is possible to calculate all the unknowns relatively easily. In the first equation, all unknowns disappear except f. The determinant of this system is easy to calculate, which shows that the system always has a solution. First we calculate f from the first in the following way

$$
\begin{aligned}
&\left(\frac{S-k_{n+1}}{k_{n+1}}-\frac{k_1}{S}\left(\frac{S-k_{n+1}}{k_{n+1}}-\frac{S-k_1}{k_1}\right)-\cdots-\frac{k_n}{S}\left(\frac{S-k_{n+1}}{k_{n+1}}-\frac{S-k_1}{k_1}\right)\right.\\
&\qquad\left.+\frac{1-k_n}{k_n}\right)f=\alpha_1-\frac{k_1}{S}(\alpha_2+\alpha_1)-\cdots-\frac{k_n}{S}(\alpha_n+\alpha_1)+\alpha_{n+1}.
\end{aligned}
$$

Notice that it is:

$$\frac{S-k_{n+1}}{k_{n+1}}-\frac{k_1}{S}\left(\frac{S-k_{n+1}}{k_{n+1}}-\frac{S-k_1}{k_1}\right)-\cdots-\frac{k_n}{S}\left(\frac{S-k_{n+1}}{k_{n+1}}-\frac{S-k_1}{k_1}\right)+\frac{1-k_n}{k_n}=n,$$
$$\alpha_1-\frac{k_1}{S}(\alpha_2+\alpha_1)-\cdots-\frac{k_n}{S}(\alpha_n+\alpha_1)+\alpha_{n+1}=\frac{1}{S}(k_n\alpha_1-k_1\alpha_2-\cdots-k_{n-1}\alpha_n-\alpha_{n+1}),$$

therefore,

$$f=\frac{1}{n\cdot S}\left(k_n\alpha_1-k_1\alpha_2-\cdots-k_{n-1}\alpha_n-\alpha_{n+1}\right).$$

We get the unknowns: $f_1, f_2, \ldots$ respectively from the remaining equations by replacing the previous ones already found:

$$f_1=\frac{1}{S}\left(\alpha_2+\alpha_1+\left(\frac{S-k_n}{k_n}-\frac{S-k_2}{k_2}\right)f\right),$$
$$f_2=\frac{1}{S}\left(\alpha_3+\alpha_1+\left(\frac{S-k_n}{k_n}-\frac{S-k_3}{k_3}\right)f\right),$$
$$\ldots$$

This completes finding all the unknowns. Also, from the last system it is possible to find its determinant.

$$Det=(-1)^{n-1}n\cdot S^{n-1}\neq 0.$$

Its value is obviously different from zero, and the system always has a solution. This means that there is always a hypersphere that touches all the above. Therefore, we have proved that there is always a Feuerbach hypersphere in each n-simplex. □

Conclusion We have shown that the generalization of a mathematical concept is closely related to its definition. It would be very interesting if we locate new points for which it can be claimed in advance that they are contained in the Feuerbach tangent sphere in $E^n (n \geq 2)$. That is the next task.

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