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CUBIC POLYNOMIALS WHOSE ROOTS FORM A TRIANGLE IN WHICH THE SIEBECK-MARDEN AND EULER LINES COINCIDE OR ARE PERPENDICULAR

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ABSTRACT. Siebeck-Marden theorem says that the roots of the derivative of a cubic polynomial, with complex coefficients, are the foci of the Steiner inellipse of the triangle whose vertices are the roots of that polynomial. Using Vieta's formulas, it is not hard to see that the midpoint, of the two complex roots of the derivative, coincides with the center of mass of the triangle whose vertices are the roots of the polynomial. Therefore, if we call the major axis of the Steiner inellipse, Siebeck-Marden line, since Euler line of a triangle also contains the center of mass, the Siebeck-Marden line and Euler line intersect at the center of mass. We describe all cubic polynomials with complex coefficients, whose roots form a triangle in which Siebeck-Marden line and Euler line coincide or ar perpendicular.

1. INTRODUCTION AND MOTIVATIONS

In this section we present the motivation and some well known results related to the paper.
The classic Rolle theorem says that if f is a real-valued differentiable function on an interval I, and x_1 and x_2 are two roots of f, with $x_1 < x_2$ in I, then there exists a root of f' in the interval (x_1, x_2). Let us observe that $[x_1, x_2]$ is the convex covering of the set $\{x_1, x_2\}$. Building up on this idea, Gauss-Lucas theorem states that given a polynomial $P(z)$, of degree $n \geq 2$, with complex coefficients, and complex roots $z_1, z_2, \ldots, z_n$, which exist by the Fundamental Theorem of Algebra, the roots of the derivative, $P'(z)$, belong to the convex covering of the set $\{z_1, z_2, \ldots, z_n\}$. When $n = 3$, the roots w_1 and w_2 of $P'(z)$ are not only in the interior of the triangle with vertices at z_1, z_2, and z_3, as Gauss-Lucas theorem says, but Siebeck-Marden theorem gives us the precise location of these roots, namely, they are the foci of the ellipse that passes through the midpoints of the sides of this triangle and is tangent to each of these sides, see [1], [4], and [10]. This ellipse is called the Steiner inellipse. If we call the line passing through w_1 and w_2, the Siebeck-Marden line, then this line passes through the centroid (i.e., the point of intersection of the medians) of the triangle determined by the roots. However, there is another important line passing to the centroid, namely Euler line, which passes through the circumcenter (i.e., point of intersection of the perpendicular bisectors of the sides), centroid, and orthocenter (i.e., point of intersection of the altitudes of the triangle), see [2] and [7]. Thus, Siebeck-Marden and Euler lines share the centroid of the triangle as a common

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point. It is naturally to ask for which cubic polynomials the Siebeck-Marden and Euler lines coincide, and for which cubic polynomials they are perpendicular. The purpose of this paper is to answer these questions. Our treatment of the problem is an application of complex numbers to Geometry. Moreover, since it is an Algebra problem concerning cubic polynomials, the results must ultimately be expressed in terms of symmetric polynomials of three variables, so that, using the Fundamental Theorem of Symmetric Polynomials, we can use the Vieta relations and express our condition in terms of the coefficients of our given cubic polynomial.
In the next section, we show the proof of Siebeck-Marden theorem using complex numbers and describe the cubic polynomials for which the Siebeck-Marden and Euler lines either coincide or are perpendicular.

2. MAIN RESULTS

In this section we present the main results related to the paper.
Let $P(z) = az^3 + bz^2 + cz + d$ be a cubic polynomial with complex coefficients, such that its three roots z_1, z_2, and z_3 correspond to three non-collinear points, A, B, and C, respectively. Dividing $P(z)$ by a, we may assume that the leading coefficient of P is 1, so that:

$$P(z) = z^3 + bz^2 + cz + d, \tag{2.1}$$

where b, c, and d are complex numbers.
Vieta formulas imply:

$$\begin{aligned} z_1 + z_2 + z_3 &= -b, &(2.2)\\ z_1z_2 + z_2z_3 + z_3z_1 &= c, &(2.3)\\ z_1z_2z_3 &= -d. &(2.4) \end{aligned}$$

The derivative of this polynomial is:

$$P'(z) = 3z^2 + 2bz + c. \tag{2.5}$$

If w_1 and w_2 denote the complex roots of P', then Vieta relations imply:

$$\begin{aligned} w_1 + w_2 &= -\frac{2b}{3}, &(2.6)\\ w_1w_2 &= \frac{c}{3}. &(2.7) \end{aligned}$$

Let us observe first that the midpoint of w_1 and w_2 coincides with the centroid of z_1, z_2, and z_3. Indeed, from the above Vieta relations, we have:

$$\begin{aligned} \frac{z_1 + z_2 + z_3}{3} &= -\frac{b}{3}\\ &= \frac{1}{2} \cdot \left(\frac{-2b}{3}\right)\\ &= \frac{1}{2}\left(w_1 + w_2\right). &(2.8) \end{aligned}$$

In fact, a more general result holds, as explained below (see also [5]):

Remark 2.1. *Given any polynomial* $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, *of degree* $n \geq 2$, *with complex coefficients, the center of mass of the roots of P coincides with the center of mass of the roots of its derivative,* P'.

Proof. Indeed, we have:

$$P'(w) = na_n w^{n-1} + (n-1)a_{n-1}w^{n-1} + \cdots + a_1. \quad (2.9)$$

If $z_1, z_2, \ldots, z_n$ denote the roots of P, and $w_1, w_2, \ldots, w_{n-1}$ the roots of P', then using Vieta relations we have:

$$\begin{aligned} \frac{w_1 + w_2 + \cdots + w_{n-1}}{n-1} &= \frac{1}{n-1} \cdot \left[-\frac{(n-1)a_{n-1}}{na_n} \right] \\ &= \frac{1}{n} \cdot \left(-\frac{a_{n-1}}{a_n} \right) \\ &= \frac{z_1 + z_2 + \cdots + z_n}{n}. \end{aligned}$$

□

Let us come back to our cubic polynomial $P(z)$, and let F_1 and F_2 be the points in the complex plane corresponding to the complex numbers w_1 and w_2 (the roots of P'). Let M_3 be the point corresponding to $(z_1 + z_2)/2$, that means, M_3 is the midpoint of the segment AB.

We compute below the square of the sum of the distance from M_3 to F_1 and the distance from M_3 to F_2. We have:

$$\begin{aligned} &\left(\overline{M_3F_1} + \overline{M_3F_2}\right)^2 \\ = &\left(\left| \frac{z_1 + z_2}{2} - w_1 \right| + \left| \frac{z_1 + z_2}{2} - w_2 \right| \right)^2 \\ = &\left| \frac{z_1 + z_2}{2} - w_1 \right|^2 + \left| \frac{z_1 + z_2}{2} - w_2 \right|^2 + 2 \left| \frac{z_1 + z_2}{2} - w_1 \right| \left| \frac{z_1 + z_2}{2} - w_2 \right|. \quad (2.10) \end{aligned}$$

Since the leading coefficient of P' is 3, and the roots of P' are w_1 and w_2, we have:

$$\begin{aligned} &\left(\frac{z_1 + z_2}{2} - w_1 \right) \left(\frac{z_1 + z_2}{2} - w_2 \right) \\ = &\frac{1}{3} P' \left(\frac{z_1 + z_2}{2} \right) \\ = &\frac{1}{3} \frac{d}{dz} \left[(z - z_1)(z - z_2)(z - z_3) \right] |_{z=(z_1+z_2)/2} \\ = &\frac{1}{3} \left[(z - z_2)(z - z_3) + (z - z_1)(z - z_3) + (z - z_1)(z - z_2) \right] |_{z=(z_1+z_2)/2} \\ = &\frac{1}{3} \left[(z - z_1)(z - z_2) + (2z - z_1 - z_2)(z - z_3) \right] |_{z=(z_1+z_2)/2} \\ = &\frac{1}{3} \left[\left(\frac{z_1 + z_2}{2} - z_1 \right) \left(\frac{z_1 + z_2}{2} - z_2 \right) + 0 \right] \\ = &-\frac{1}{12} (z_2 - z_1)^2. \quad (2.11) \end{aligned}$$

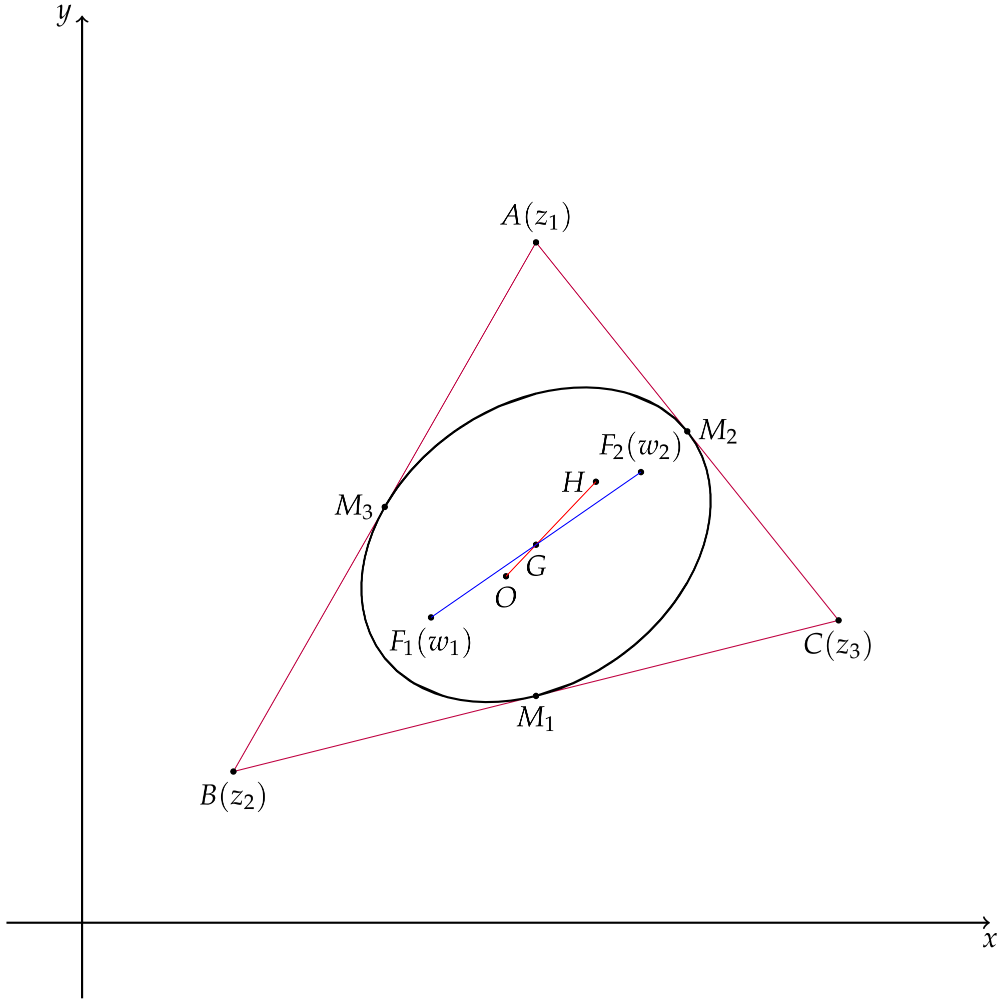

Figure 1. Siebeck-Marden and Euler lines

Substituting this relation into formula (2.10) and using the parallelogram identity:

$$2\left(|u|^2+|v|^2\right) \quad = \quad |u+v|^2+|u-v|^2, \tag{2.12}$$

for $u := (z_1+z_2)/2 - w_1$ and $v := (z_1+z_2)/2 - w_2$, we obtain:

$$
\begin{aligned}
& \left(\overline{M_3F_1} + \overline{M_3F_2}\right)^2 \\
= & \left|\frac{z_1+z_2}{2} - w_1\right|^2 + \left|\frac{z_1+z_2}{2} - w_2\right|^2 + 2\left|\left(\frac{z_1+z_2}{2} - w_1\right)\left(\frac{z_1+z_2}{2} - w_2\right)\right| \\
= & \frac{1}{2}\left[\left|\frac{z_1+z_2}{2} - w_1 + \frac{z_1+z_2}{2} - w_2\right|^2 + \left|\frac{z_1+z_2}{2} - w_1 - \frac{z_1+z_2}{2} + w_2\right|^2\right] + \frac{1}{6}|z_2 - z_1|^2 \\
= & \frac{1}{2}|z_1 + z_2 - (w_1+w_2)|^2 + \frac{1}{2}|w_2 - w_1|^2 + \frac{1}{6}|z_2 - z_1|^2 \\
= & \frac{1}{2}\left|z_1 + z_2 - 2\left(\frac{z_1+z_2+z_3}{3}\right)\right|^2 + \frac{1}{2}|w_2 - w_1|^2 + \frac{1}{6}|z_2 - z_1|^2 \\
= & \frac{1}{18}|z_1 + z_2 - 2z_3|^2 + \frac{1}{6}|z_2 - z_1|^2 + \frac{1}{2}|w_2 - w_1|^2 \\
= & \frac{1}{18}\left[|(z_1 - z_3) + (z_2 - z_3)|^2 + |(z_1 - z_3) - (z_2 - z_3)|^2\right] + \frac{1}{9}|z_2 - z_1|^2 + \frac{1}{2}|w_2 - w_1|^2.
\end{aligned}
$$

Applying the parallelogram identity one more time for $u := z_1 - z_3$ and $v := z_2 - z_3$, we obtain:

$$
\begin{aligned}
& \left(\overline{M_3F_1} + \overline{M_3F_2}\right)^2 \\
= & \frac{1}{9}|z_1 - z_3|^2 + \frac{1}{9}|z_2 - z_3|^2 + \frac{1}{9}|z_2 - z_1|^2 + \frac{1}{2}|w_2 - w_1|^2 \\
= & \frac{1}{9}\left(\overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2\right) + \frac{1}{2}\overline{F_1F_2}^2. \qquad (2.13)
\end{aligned}
$$

Since the last expression is symmetric with respect to z_1, z_2, and z_3, we conclude that:

$$\overline{M_3F_1} + \overline{M_3F_2} = \overline{M_2F_1} + \overline{M_2F_2} = \overline{M_1F_1} + \overline{M_1F_2},$$

where M_1 and M_2 are the points in the complex plane corresponding to $(z_2+z_3)/2$ and $(z_3+z_1)/2$, respectively.

Therefore, the midpoints M_1, M_2, and M_3 are on an ellipse with foci F_1 and F_2.

It remains to show that this ellipse is tangent to each of the three sides of the triangle ABC.

To show that the ellipse is tangent to AB at the point M_3, it is enough to show that:

$$\sphericalangle AM_3F_1 \equiv \sphericalangle BM_3F_2, \qquad (2.14)$$

since it is known that the tangent line at a point T, of an ellipse with foci F_1 and F_2, is perpendicular to the bisector of the angle $\sphericalangle F_1TF_2$. Denoting:

$$\alpha := m(\sphericalangle AM_3F_1) \qquad (2.15)$$

and

$$\beta := m(\sphericalangle BM_3F_2), \qquad (2.16)$$

we have:

$$\begin{aligned}
&\frac{w_1-(z_1+z_2)/2}{z_1-(z_1+z_2)/2} \div \frac{z_2-(z_1+z_2)/2}{w_2-(z_1+z_2)/2} \\
= \ & \left|\frac{w_1-(z_1+z_2)/2}{z_1-(z_1+z_2)/2}\right| \exp(i\alpha) \div \left[\left|\frac{z_2-(z_1+z_2)/2}{w_2-(z_1+z_2)/2}\right| \exp(i\beta)\right] \\
= \ & \left\{\left|\frac{w_1-(z_1+z_2)/2}{z_1-(z_1+z_2)/2}\right| \div \left|\frac{z_2-(z_1+z_2)/2}{w_2-(z_1+z_2)/2}\right|\right\} \exp(i(\alpha-\beta)). \qquad (2.17)
\end{aligned}$$

Thus, to show that $\alpha = \beta$, we have to prove that:

$$\frac{w_1-(z_1+z_2)/2}{z_1-(z_1+z_2)/2} \div \frac{z_2-(z_1+z_2)/2}{w_2-(z_1+z_2)/2} \quad \in \quad \mathbb{R}_+.$$

Indeed, using formula (2.11), we have:

$$\begin{aligned}
\frac{w_1-(z_1+z_2)/2}{z_1-(z_1+z_2)/2} \div \frac{z_2-(z_1+z_2)/2}{w_2-(z_1+z_2)/2} &= \frac{[w_1-(z_1+z_2)/2]\cdot[w_2-(z_1+z_2)/2]}{[z_1-(z_1+z_2)/2]\cdot[z_2-(z_1+z_2)/2]} \\
&= \frac{-(z_1-z_2)^2/12}{-(z_1-z_2)^2/4} \\
&= \frac{1}{3}, \qquad (2.18)
\end{aligned}$$

which is a positive real number.

Therefore, the points F_1 and F_2, corresponding to the roots w_1 and w_2 of P', are the foci of the Steiner inellipse of the triangle ABC, with vertices corresponding to the roots z_1, z_2, and z_3 of P. This is the statement of the Siebeck-Marden theorem.

We solve now the following two problems:

Problem 1. Find all monic cubic polynomials $P(z) = z^3 + bz^2 + cz + d$, with complex coefficients, whose roots z_1, z_2, and z_3 satisfy the following conditions:

- The points A, B, and C in the complex plane corresponding to z_1, z_2, and z_3 are non-collinear.
- The roots w_1 and w_2 of the derivative P' are distinct.
- The Siebeck-Marden and Euler lines of the triangle ABC coincide.

Problem 2. Find all monic cubic polynomials $P(z) = z^3 + bz^2 + cz + d$, with complex coefficients, whose roots z_1, z_2, and z_3 satisfy the following conditions:

- The points A, B, and C in the complex plane corresponding to z_1, z_2, and z_3 are non-collinear.
- The roots w_1 and w_2 of the derivative P' are distinct.
- The Siebeck-Marden and Euler lines of the triangle ABC are perpendicular.

To solve these two problems, let us find first the formula of the complex number z_0 corresponding to the circumcenter O of the triangle ABC. Since $\overline{OA} = \overline{OB}$, we have:

$$|z_0 - z_1|^2 \quad = \quad |z_0 - z_2|^2. \qquad (2.19)$$

This is equivalent to:

$$(z_0 - z_1)\left(\overline{z_0} - \overline{z_1}\right) \quad = \quad (z_0 - z_2)\left(\overline{z_0} - \overline{z_2}\right), \qquad (2.20)$$

which is further equivalent to:

$$(\overline{z_2} - \overline{z_1})\, z_0 + (z_2 - z_1)\, \overline{z_0} \quad = \quad |z_2|^2 - |z_1|^2 . \tag{2.21}$$

Similarly, $\overline{OA} = \overline{OC}$ implies:

$$(\overline{z_3} - \overline{z_1})\, z_0 + (z_3 - z_1)\, \overline{z_0} \quad = \quad |z_3|^2 - |z_1|^2 . \tag{2.22}$$

We can think of (2.21) and (2.22) as forming a linear system in the unknowns z_0 and $\overline{z_0}$. Multiplying first equation (2.21) by $z_1 - z_3$, and equation (2.22) by $z_2 - z_1$, and then adding the two resulting equations, we eliminate $\overline{z_0}$ and solve for z_0 as:

$$z_0 \quad = \quad \frac{\left(|z_2|^2 - |z_3|^2\right) z_1 + \left(|z_3|^2 - |z_1|^2\right) z_2 + \left(|z_1|^2 - |z_2|^2\right) z_3}{\overline{z_2}z_1 - \overline{z_1}z_2 + \overline{z_3}z_2 - \overline{z_2}z_3 + \overline{z_1}z_3 - \overline{z_3}z_1}. \tag{2.23}$$

Let us apply a translation by the vector $-z_G := -(z_1 + z_2 + z_3)/3 = b/3$, where G is the centroid of the triangle ABC whose vertices correspond to the roots, z_1, z_2, and z_3 of P, to the polynomial $P(z)$ and its roots. Since the translation and differentiation operators commute, we also have $(w_1 + w_2)/2 = 0$, and the angle made by the Siebeck-Marden and Euler lines does not change.
So, we may assume that $z_1 + z_2 + z_3 = w_1 + w_2 = 0$, and thus $z_G = 0$.
Now, the vector $\overrightarrow{GO}$, that gives the direction of the Euler line, corresponds to the complex number:

$$z_0 - z_G \quad = \quad \frac{\left(|z_2|^2 - |z_3|^2\right) z_1 + \left(|z_3|^2 - |z_1|^2\right) z_2 + \left(|z_1|^2 - |z_2|^2\right) z_3}{\overline{z_2}z_1 - \overline{z_1}z_2 + \overline{z_3}z_2 - \overline{z_2}z_3 + \overline{z_1}z_3 - \overline{z_3}z_1}. \tag{2.24}$$

The vector $\overrightarrow{F_1F_2}$, that gives the direction of the Siebeck-Marden line, corresponds to the complex number $w_2 - w_1$. We make the following observations:

- The Siebeck-Marden and Euler lines coincide if and only if the vectors $\overrightarrow{F_1F_2}$ and $\overrightarrow{GO}$ are collinear. This is equivalent to:

$$\frac{w_2 - w_1}{z_0 - z_G} \quad \in \quad \mathbb{R}, \tag{2.25}$$

  which in turn is equivalent to:

$$\frac{(w_2 - w_1)^2}{(z_0 - z_G)^2} \quad \in \quad \mathbb{R}_+. \tag{2.26}$$

- The Siebeck-Marden and Euler lines are perpendicular if and only if:

$$\frac{w_2 - w_1}{z_0 - z_G} \quad \in \quad i\mathbb{R}, \tag{2.27}$$

  which in turn is equivalent to

$$\frac{(w_2 - w_1)^2}{(z_0 - z_G)^2} \quad \in \quad \mathbb{R}_-. \tag{2.28}$$

Therefore, we can see, from relations (2.26) and (2.28), that our two problems can be joined into one larger problem, namely, to describe the cubic polynomials, for which:

$$\frac{(w_2 - w_1)^2}{(z_0 - z_G)^2} \quad \in \quad \mathbb{R}. \tag{2.29}$$

After solving this problem, we may worry about the sign of this real number ratio, and decide when it is $+$ and when it is $-$.

Let us make the important observation that the denominator $\overline{z_2}z_1 - \overline{z_1}z_2 + \overline{z_3}z_2 - \overline{z_2}z_3 + \overline{z_1}z_3 - \overline{z_3}z_1$ of $z_0 - z_G$, from formula (2.24), is a purely imaginary number. Indeed, we have:

$$\begin{aligned}\overline{\overline{z_2}z_1 - \overline{z_1}z_2 + \overline{z_3}z_2 - \overline{z_2}z_3 + \overline{z_1}z_3 - \overline{z_3}z_1} &= z_2\overline{z_1} - z_1\overline{z_2} + z_3\overline{z_2} - z_2\overline{z_3} + z_1\overline{z_3} - z_3\overline{z_1} \\ &= -\left(\overline{z_2}z_1 - \overline{z_1}z_2 + \overline{z_3}z_2 - \overline{z_2}z_3 + \overline{z_1}z_3 - \overline{z_3}z_1\right).\end{aligned}$$

Thus, in the condition (2.29), when we square the denominator of $z_0 - z_G$, we obtain a negative real number. So, we can omit the square of the denominator of $z_0 - z_G$, in the condition (2.29), and rewrite this condition as:

$$\frac{(w_2 - w_1)^2}{\left[\left(|z_2|^2 - |z_3|^2\right) z_1 + \left(|z_3|^2 - |z_1|^2\right) z_2 + \left(|z_1|^2 - |z_2|^2\right) z_3\right]^2} \quad \in \quad \mathbb{R}. \tag{2.30}$$

Let us define the numbers:

$$p_1 := |z_2|^2 - |z_3|^2, \quad p_2 := |z_3|^2 - |z_1|^2, \quad \text{and} \quad p_3 := |z_1|^2 - |z_2|^2.$$

Then p_1, p_2, and p_3 are real numbers, such that:

$$p_1 + p_2 + p_3 \quad = \quad 0. \tag{2.31}$$

Now, condition (2.30) becomes:

$$\frac{(w_2 - w_1)^2}{(p_1 z_1 + p_2 z_2 + p_3 z_3)^2} \quad \in \quad \mathbb{R}. \tag{2.32}$$

Since we translated the roots to have $z_1 + z_2 + z_3 = 0$, we can substitute z_3 by $-z_1 - z_2$ in condition (2.32), and obtain the new condition:

$$\frac{(w_2 - w_1)^2}{[(p_1 - p_3) z_1 + (p_2 - p_3) z_2]^2} \quad \in \quad \mathbb{R}. \tag{2.33}$$

This relation is equivalent to:

$$\frac{(w_1 + w_2)^2 - 4w_1 w_2}{[(p_1 - p_3) z_1 + (p_2 - p_3) z_2]^2} \quad \in \quad \mathbb{R}. \tag{2.34}$$

Using Vieta relations, we have $w_1 + w_2 = 0$ and $w_1 w_2 = c/3 = (z_1 z_2 + z_2 z_3 + z_3 z_1)/3$. Thus, there exists $t \in \mathbb{R}$, such that:

$$\frac{z_1 z_2 + z_2 z_3 + z_3 z_1}{\alpha^2 z_1^2 + 2\alpha\beta z_1 z_2 + \beta^2 z_2^2} \quad = \quad t, \tag{2.35}$$

where $\alpha := p_1 - p_3$ and $\beta := p_2 - p_3$ are real numbers. Substituting z_3 by $-z_1 - z_2$, and multiplying both sides of (2.35) by $\alpha^2 z_1^2 + 2\alpha\beta z_1 z_2 + \beta^2 z_2^2$, we obtain:

$$\left(\alpha^2 t + 1\right) z_1^2 + (2\alpha\beta t + 1)\, z_1 z_2 + \left(\beta^2 t + 1\right) z_2^2 \quad = \quad 0. \tag{2.36}$$

At least one of the two roots z_1 and z_2 is not zero. Suppose $z_2 \neq 0$. Dividing both sides of equation (2.36) by z_2^2, we obtain:

$$\left(\alpha^2 t + 1\right)\left(\frac{z_1}{z_2}\right)^2 + (2\alpha\beta t + 1)\frac{z_1}{z_2} + \left(\beta^2 t + 1\right) = 0. \tag{2.37}$$

The ratio z_1/z_2 cannot be a real number, since if $z_1 = sz_2$, for some real number s, then because $z_1 + z_2 + z_3 = 0$, we obtain $z_3 = -(1+s)z_2$, and thus the three roots z_1, z_2, and z_3 correspond to three collinear points, which we have assumed not to be the case. Therefore, the complex non-real number $\tau := z_1/z_2$ is a root of the quadratic polynomial with real coefficients:

$$g(X) := \left(\alpha^2 t + 1\right) X^2 + (2\alpha\beta t + 1) X + \left(\beta^2 t + 1\right). \tag{2.38}$$

On the other hand, since τ is not real, the minimal monic polynomial with real coefficients that τ satisfies is:

$$\begin{aligned} h(X) &:= (X - \tau)(X - \overline{\tau}) \\ &= X^2 - (\tau + \overline{\tau}) X + |\tau|^2. \end{aligned} \tag{2.39}$$

Since the minimal polynomial of τ, over $\mathbb{R}$, divides any other polynomial with real coefficients for which τ is a root, we conclude that there exists a real constant λ, such that:

$$\alpha^2 t + 1 = \lambda \cdot 1, \quad 2\alpha\beta t + 1 = -\lambda \cdot (\tau + \overline{\tau}), \quad \text{and} \quad \beta^2 t + 1 = \lambda \cdot |\tau|^2. \tag{2.40}$$

These three relations can be interpreted as the fact that the homogenous system

$$\begin{cases} 1 \cdot x_1 + \alpha^2 \cdot x_2 + 1 \cdot x_3 = 0 \\ 1 \cdot x_1 + 2\alpha\beta \cdot x_2 - (\tau + \overline{\tau}) \cdot x_3 = 0 \\ 1 \cdot x_1 + \beta^2 \cdot x_2 + |\tau|^2 \cdot x_3 = 0 \end{cases} \tag{2.41}$$

has the non-trivial solution $(x_1, x_2, x_3) = (1, t, -\lambda) \neq (0, 0, 0)$. This implies that the following determinant is zero:

$$\begin{vmatrix} 1 & \alpha^2 & 1 \\ 1 & 2\alpha\beta & -\tau - \overline{\tau} \\ 1 & \beta^2 & |\tau|^2 \end{vmatrix} = 0. \tag{2.42}$$

Let us subtract the first row, of the above determinant, from the other two rows. We obtain:

$$\begin{vmatrix} 1 & \alpha^2 & 1 \\ 0 & \alpha(2\beta - \alpha) & -\tau - \overline{\tau} - 1 \\ 0 & (\beta - \alpha)(\beta + \alpha) & |\tau|^2 - 1 \end{vmatrix} = 0. \tag{2.43}$$

Developing this determinant after the first column, we get:

$$\begin{vmatrix} \alpha(2\beta - \alpha) & -\tau - \overline{\tau} - 1 \\ (\beta - \alpha)(\beta + \alpha) & |\tau|^2 - 1 \end{vmatrix} = 0. \tag{2.44}$$

The last relation is equivalent to:

$$\alpha(2\beta - \alpha)\left(|\tau|^2 - 1\right) = (\beta - \alpha)(\beta + \alpha)(-\tau - \overline{\tau} - 1). \tag{2.45}$$

Replacing first τ by z_1/z_2 and then multiplying both sides by $|z_2|^2 = z_2\overline{z_2}$, the last equation becomes:

$$\alpha(2\beta - \alpha)\left(|z_1|^2 - |z_2|^2\right) = (\beta - \alpha)(\beta + \alpha)\left(-z_1\overline{z_2} - z_2\overline{z_1} - |z_2|^2\right). \tag{2.46}$$

In the right-hand side of this equation, let us replace z_1 by $-z_2 - z_3$, and z_2 by $-z_1 - z_3$. We obtain:

$$\alpha(2\beta-\alpha)\left(|z_1|^2-|z_2|^2\right) \;=\; (\beta-\alpha)(\beta+\alpha)\left[(z_2+z_3)\,\overline{z_2}+(z_1+z_3)\,\overline{z_1}-|z_2|^2\right].$$

This is equivalent to:

$$\alpha(2\beta-\alpha)\left(|z_1|^2-|z_2|^2\right) \;=\; (\beta-\alpha)(\beta+\alpha)\left[|z_1|^2+z_3\left(\overline{z_1}+\overline{z_2}\right)\right].$$

Finally, replacing $\overline{z_1}+\overline{z_2}$ by $-\overline{z_3}$, in the right-hand side of the last relation, we obtain:

$$\alpha(2\beta-\alpha)\left(|z_1|^2-|z_2|^2\right) \;=\; (\beta-\alpha)(\beta+\alpha)\left(|z_1|^2-|z_3|^2\right).$$

Recalling now that $\alpha = p_1 - p_3$, $\beta = p_2 - p_3$, $|z_1|^2 - |z_2|^2 = p_3$, and $|z_1|^2 - |z_3|^2 = -p_2$, we obtain:

$$(p_1-p_3)\,(2p_2-p_3-p_1)\,p_3 \;=\; -\,(p_2-p_1)\,(p_1+p_2-2p_3)\,p_2. \qquad (2.47)$$

Remembering that $p_1 + p_2 + p_3 = 0$, we can replace $-p_3 - p_1$ in the left by p_2, and $p_1 + p_2$ in the right by $-p_3$. Thus, we obtain:

$$(p_1-p_3)\,(3p_2)\,p_3 \;=\; -\,(p_2-p_1)\,(-3p_3)\,p_2. \qquad (2.48)$$

Dividing first both sides of this equation by 3 and then moving all terms to the left, we obtain:

$$(2p_1-p_2-p_3)\,p_2p_3 \;=\; 0. \qquad (2.49)$$

Finally, replacing $-p_2 - p_3$ by p_1, we obtain:

$$3p_1p_2p_3 \;=\; 0. \qquad (2.50)$$

Thus, we obtain that:

$$p_1 = 0 \quad \text{or} \quad p_2 = 0 \quad \text{or} \quad p_3 = 0.$$

Let us see what happens if $p_1 = 0$. Since $p_1 = |z_2|^2 - |z_3|^2$, we conclude that $|z_2| = |z_3|$. That means $|0 - z_2| = |0 - z_3|$. Because we made the translation to have $z_1 + z_2 + z_3 = 0$, the origin 0 corresponds to the centroid, G, of the triangle ABC. Thus, we obtain:

$$\overline{GB} \;=\; \overline{GC}. \qquad (2.51)$$

This means, the centroid G is equally far away from the margins of the segment BC. Therefore, the centroid G is on the perpendicular bisector of the segment BC, but the centroid is on the median m_A corresponding to the vertex A of the triangle ABC. Thus, the median and perpendicular bisector of the side BC coincide, and so, the triangle ABC is isosceles with the vertex at A. Another way to conclude that the triangle ABC is isosceles, with the vertex at A, is to use the fact that since $|0 - z_2| = \overline{GB} = \overline{GC} = |0 - z_3|$, $\overline{GB} = (2/3)m_B$, and $\overline{GC} = (2/3)m_C$. Hence, we conclude that the medians m_B and m_C corresponding to the vertices B and C, respectively, of the triangle ABC are congruent. Thus, the triangle ABC is isosceles with the vertex at A.

Similarly, the condition $p_2 = 0$ implies that triangle ABC is isosceles with the vertex at B, while the condition $p_3 = 0$ implies that triangle ABC is isosceles with the vertex at C. Therefore, we have proven that if the Siebeck-Marden and Euler lines coincide or are perpendicular, then the triangle ABC must be isosceles. We exclude from our discussion the equilateral triangles since in that case the two foci w_1 and w_2 of Steiner inellipse coincide, and so the Siebeck-Marden line is not properly (uniquely) defined in that case.

The Euler line is also not properly defined in that case.
We will show below the other implication, that for an isosceles non-equilateral triangle the Siebeck-Marden and Euler lines coincide or are perpendicular, and discuss precisely the condition for the two lines to coincide and the two lines to be perpendicular.

We will prove now that if the points A, B, and C, corresponding to the roots of a cubic polynomial, form an isosceles triangle, then the Siebeck-Marden and Euler lines are either perpendicular or coincide.
Indeed, by rotating and translating eventually the isosceles triangle ABC, we may assume that the axis of symmetry of this isosceles triangle is AO, where $O(0, 0)$ is the origin of the axes, and $A(0, a)$ is on the y–axis, while $B(b, 0)$ and $C(-b, 0)$ are on the x–axis symmetrically displaced around the origin, for a and b positive real numbers. Since the circumcenter, centroid, and orthocenter belong to the axis of symmetry, the Euler line of our isosceles triangle is the y–axis. The complex numbers associated to A, B, and C are $z_1 = ia$, $z_2 = b$, and $z_3 = -b$. The monic cubic polynomial having these roots is:

$$\begin{aligned} f(z) &= (z - ia)(z - b)(z + b) \\ &= z^3 - (ia + b - b)z^2 + \left(iab - b^2 - iab\right)z - (ia)b(-b) \\ &= z^3 - iaz^2 - b^2 z + iab^2. \end{aligned} \tag{2.52}$$

The derivative of f is:

$$f'(w) = 3w^2 - 2iaw - b^2. \tag{2.53}$$

Using the quadratic formula, the roots of f' are:

$$w_{1,2} = \frac{ia \pm \sqrt{-a^2 + 3b^2}}{3}. \tag{2.54}$$

We can see from here that we have two cases:
Case 1. If $a > b\sqrt{3}$, then

$$w_{1,2} = i \cdot \frac{a \pm \sqrt{a^2 - 3b^2}}{3}. \tag{2.55}$$

So w_1 and w_2 are purely imaginary numbers, which implies that the Siebeck-Marden line coincides with the y-axis, which is the same as the Euler line of the triangle ABC.
Case 2. If $a < b\sqrt{3}$, then:

$$w_{1,2} = \pm\frac{\sqrt{3b^2 - a^2}}{3} + i \cdot \frac{a}{3}. \tag{2.56}$$

In this case, the roots w_1 and w_2 are symmetrically displaced about the y–axis. Therefore, the Siebeck-Marden line is perpendicular to the y–axis, which is the Euler line.
In the case when $a = b\sqrt{3}$, the triangle ABC is equilateral since $\overline{AB} = \overline{BC} = \overline{CA} = 2b$. In this case, both the Euler and Siebeck-Marden lines are not well defined, since all the points: circumcenter, centroid, orthocenter, w_1, and w_2 coincide. We are excluding this case from our discussion.

We can see that if the triangle ABC is isosceles, the fact that $a > b\sqrt{3}$ or $a < b\sqrt{3}$ is equivalent to $\tan(A/2) < 1/\sqrt{3}$ or $\tan(A/2) > 1/\sqrt{3}$, which in turn means $m(\sphericalangle A) < 60^\circ$ or $m(\sphericalangle A) > 60^\circ$. When looking at an isosceles triangle ABC with the vertex A,

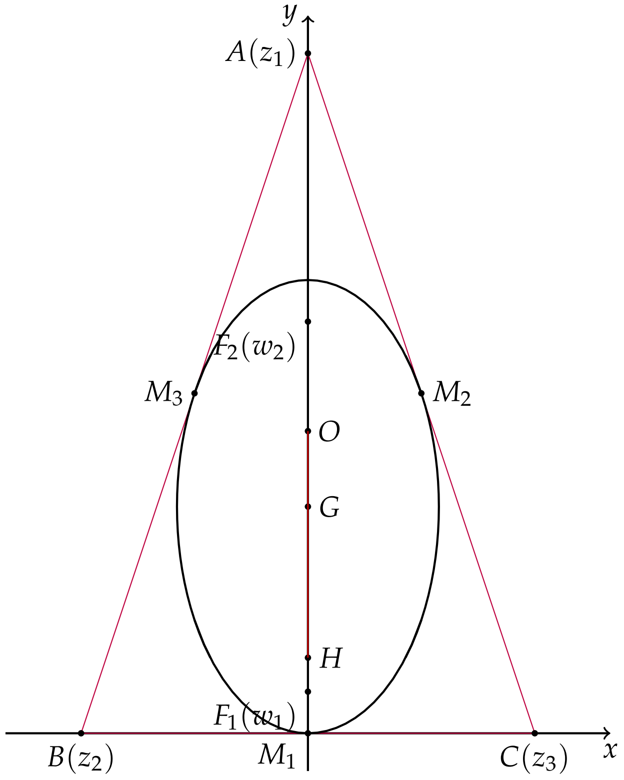

Figure 2. $a > b\sqrt{3}$

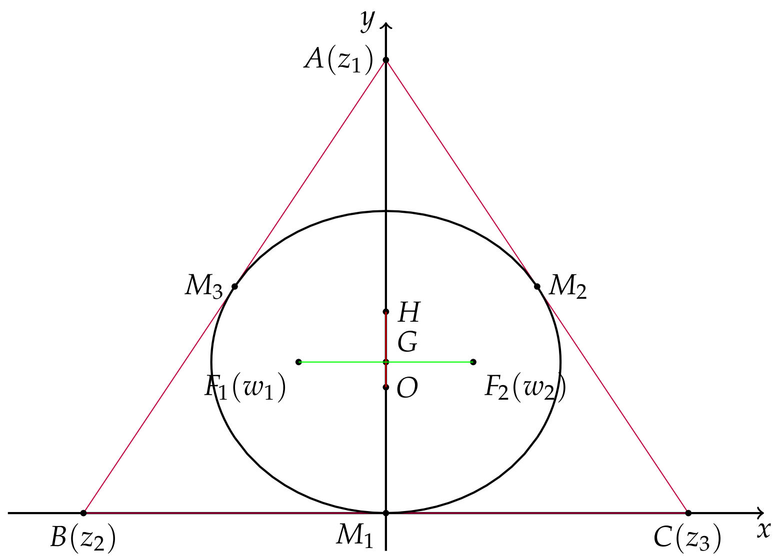

Figure 3. $a < b\sqrt{3}$

for which $m(\sphericalangle A) < 60^\circ$, the triangle $\triangle ABC$ appears to be taller than the equilateral triangle with the same base. When looking at an isosceles triangle ABC with the vertex A, for which $m(\sphericalangle A) > 60^\circ$, the triangle $\triangle ABC$ appears to be shorter than the equilateral triangle with the same base. For this reason, we introduce the following definition:

Definition 2.1. *Let ABC be an isosceles triangle with $\sphericalangle B = \sphericalangle C$.*

- *If $m(\sphericalangle A) < 60^\circ$, then we say that the triangle ABC is an* over-equilateral *isosceles triangle.*
- *If $m(\sphericalangle A) > 60^\circ$, then we say that the triangle ABC is an* under-equilateral *isosceles triangle.*

We can re-state our result in the following way:

Proposition 2.1.
- *The Euler and Siebeck-Marden lines of a non-equilateral triangle coincide if and only if the triangle is over-equilateral isosceles.*
- *The Euler and Siebeck-Marden lines of a non-equilateral triangle are perpendicular if and only if the triangle is under-equilateral isosceles.*

We also have the following important proposition:

Proposition 2.2. *Given a monic cubic polynomial $f(z) = z^3 + bz^2 + cz + d$, with complex coefficients, that has simple roots, z_1, z_2, and z_3, then the expression:*

$$E = \frac{(z_2 - z_1)^2(z_3 - z_2)^2(z_1 - z_3)^2}{(\bar{z} - z_1)^2(\bar{z} - z_2)^2(\bar{z} - z_3)^2}, \tag{2.57}$$

where $\bar{z} := (z_1 + z_2 + z_3)/3$, is invariant with respect to every translation and rotation of the roots z_1, z_2, z_3. Moreover, if the triangle ABC, whose vertices are the roots z_1, z_2, and z_3, is isosceles, then E is a negative real number, and we have three possibilities:

(1) If the triangle ABC is equilateral, then:

$$E = -27. \tag{2.58}$$

(2) If the triangle ABC is over-equilateral isosceles, then:

$$-27 < E < 0. \tag{2.59}$$

(3) If the triangle ABC is under-equilateral isosceles, then:

$$E < -27. \tag{2.60}$$

Proof. It is clear that if we replace z_i by $e^{i\theta}z_i + \xi$, for any $\theta \in [0, 2\pi)$ and $\xi \in \mathbb{C}$, for all $i \in \{1, 2, 3\}$, then the value of E does not change.
Let us assume now that the triangle ABC with vertices z_1, z_2, and z_3 is isosceles. Since E is invariant under rotations and translations, as in the second part of the proof of our previous theorem, we may assume that $z_1 = ia$, $z_2 = b$, and $z_3 = -b$, for a and b positive

real numbers. Thus we have:

$$\begin{aligned} E &= \frac{(z_2-z_1)^2(z_3-z_2)^2(z_1-z_3)^2}{(\bar{z}-z_1)^2(\bar{z}-z_2)^2(\bar{z}-z_3)^2} \\ &= \left[\frac{(b-ia)(-b-b)(ia+b)}{(ia/3-ia)(ia/3-b)(ia/3+b)}\right]^2 \\ &= \left[\frac{-2b(b^2+a^2)}{(2ia/3)(b^2+(a^2/9))}\right]^2 \\ &= -\left[\frac{27b(a^2+b^2)}{a(a^2+9b^2)}\right]^2 \qquad (2.61) \end{aligned}$$

If we divide both the numerator and denominator of the last fraction by b^3 and define $x := a/b$, then we have:

$$E = -3^6\left[\frac{x^2+1}{x(x^2+9)}\right]^2. \qquad (2.62)$$

Let us observe that the function:

$$h(x) := \frac{x^2+1}{x(x^2+9)} \qquad (2.63)$$

is decreasing on $(0, \infty)$, since its derivative is:

$$\begin{aligned} h'(x) &= \frac{2x(x^3+9x)-(3x^2+9)(x^2+1)}{x^2(x^2+9)^2} \\ &= -\frac{x^4-6x^2+9}{x^2(x^2+9)^2} \\ &= -\frac{(x^2-3)^2}{x^2(x^2+9)^2} \qquad (2.64) \\ &\leq 0. \qquad (2.65) \end{aligned}$$

We have seen in the second part of the proof of the previous proposition that:

- If the triangle ΔABC is equilateral, then $a = b\sqrt{3}$. This is equivalent to $x = a/b = \sqrt{3}$. In that case we have:

$$\begin{aligned} E &= -3^6\left[h(\sqrt{3})\right]^2 \\ &= -3^6\left[\frac{(\sqrt{3})^2+1}{(\sqrt{3})((\sqrt{3})^2+9)}\right]^2 \\ &= -3^6\frac{4^2}{3\cdot 12^2} \\ &= -\frac{3^6}{3\cdot 3^2} \\ &= -27. \qquad (2.66) \end{aligned}$$

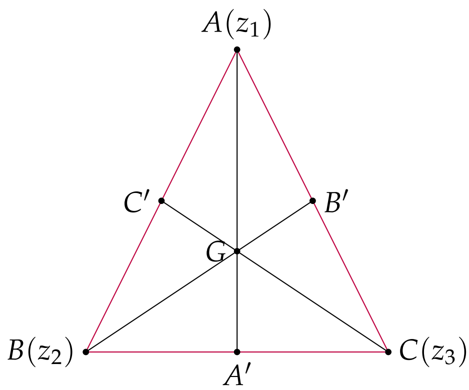

Figure 4. centroid

- If the triangle ΔABC is over-equilateral isosceles, then $x = a/b > \sqrt{3}$. Since h is a decreasing function, we have $h(x) < h(\sqrt{3})$. Thus, since $E = -3^6 h(x)$, we obtain:

$$\begin{aligned} E &= -3^6 \left[h(x)\right]^2 \\ &> -3^6 \left[h(\sqrt{3})\right]^2 \\ &= -27. \end{aligned} \tag{2.67}$$

- If the triangle ΔABC is under-equilateral isosceles, then $x = a/b < \sqrt{3}$. Since h is a decreasing function, we have $h(x) > h(\sqrt{3})$. Thus, since $E = -3^6 h(x)$, we obtain:

$$\begin{aligned} E &< -3^6 \left[h(\sqrt{3})\right]^2 \\ &= -27. \end{aligned} \tag{2.68}$$

□

So, geometrically speaking, the condition that the Siebeck-Marden and Euler lines, of a non-equilateral triangle, coincide or are perpendicular, is equivalent to the fact that the triangle is isosceles. However, our problem was to express this condition in terms of the complex coefficients a, b, c, and d of a cubic polynomial. To do this we need the following result.

Proposition 2.3. *Let ABC be a triangle and G its centroid. Then the triangle ABC is isosceles if and only if:*

$$m(\sphericalangle GAB) + m(\sphericalangle GBC) + m(\sphericalangle GCA) = 90^\circ. \tag{2.69}$$

Proof. ($\Leftarrow$) Let us assume that the triangle ABC is isosceles with $\overline{AB} = \overline{AC}$. Denoting by A', B', and C' the midpoints of the sides BC, CA, and AB, respectively, from the congruence of triangles $\Delta BAA'$ and $\Delta CAA'$, we can see that $m(\sphericalangle GAB) = m(\sphericalangle GAC) =: \alpha$. Furthermore, from the congruence of the triangle ΔCAG and ΔBAG, we can see that

$m(\sphericalangle GCA) = m(\sphericalangle GBA) =: \gamma$ and $\overline{GC} = \overline{GB}$. Since the triangle ΔGBC is isosceles, we conclude that $m(\sphericalangle GBC) = m(\sphericalangle GCB) =: \beta$. Thus we obtain:

$$\begin{aligned} m(\sphericalangle GAB) + m(\sphericalangle GBC) + m(\sphericalangle GCA) &= m(\sphericalangle GAC) + m(\sphericalangle GCB) + m(\sphericalangle GBA) \\ &= \alpha + \beta + \gamma. \end{aligned} \tag{2.70}$$

Since, we also have:

$$\begin{aligned} & [m(\sphericalangle GAB) + m(\sphericalangle GBC) + m(\sphericalangle GCA)] \\ & + [m(\sphericalangle GAC) + m(\sphericalangle GCB) + m(\sphericalangle GBA)] \\ = \ & m(\sphericalangle BAC) + m(\sphericalangle ABC) + m(\sphericalangle BCA) \\ = \ & 180^\circ, \end{aligned} \tag{2.71}$$

we conclude that:

$$\begin{aligned} m(\sphericalangle GAB) + m(\sphericalangle GBC) + m(\sphericalangle GCA) &= \frac{1}{2} \cdot 180^\circ \\ &= 90^\circ. \end{aligned}$$

($\Rightarrow$) Let us suppose now that in the triangle ΔABC with the centroid G, we have:

$$m(\sphericalangle GAB) + m(\sphericalangle GBC) + m(\sphericalangle GCA) = 90^\circ. \tag{2.72}$$

Let $\alpha := m(\sphericalangle GAB)$, $\beta := m(\sphericalangle GBC)$, and $\gamma := m(\sphericalangle GCA)$. Let us also associate to the points A, B, and C the complex numbers z_1, z_2, and z_3, respectively, where to each point M of coordinates (x, y), we associate the complex number $z = x + iy$. Then the complex number associated to the centroid G, of the triangle ΔABC, is $\bar{z} = (z_1 + z_2 + z_3)/3$. Since the vectors $\overrightarrow{GA}$ and $\overrightarrow{AB}$ are associated with the complex numbers $z_1 - \bar{z}$ and $z_2 - z_1$, we have:

$$\frac{z_2 - z_1}{z_1 - \bar{z}} = \frac{|z_2 - z_1|}{|z_1 - \bar{z}|} e^{i\alpha}. \tag{2.73}$$

Similarly, we have:

$$\frac{z_3 - z_2}{z_2 - \bar{z}} = \frac{|z_3 - z_2|}{|z_2 - \bar{z}|} e^{i\beta} \tag{2.74}$$

and

$$\frac{z_1 - z_3}{z_3 - \bar{z}} = \frac{|z_1 - z_3|}{|z_3 - \bar{z}|} e^{i\gamma}. \tag{2.75}$$

Multiplying relations (2.73), (2.74), and (2.75) together, we obtain:

$$\frac{z_2 - z_1}{z_1 - \bar{z}} \cdot \frac{z_3 - z_2}{z_2 - \bar{z}} \cdot \frac{z_1 - z_3}{z_3 - \bar{z}} = \frac{|z_2 - z_1|}{|z_1 - \bar{z}|} \cdot \frac{|z_3 - z_2|}{|z_2 - \bar{z}|} \cdot \frac{|z_1 - z_3|}{|z_3 - \bar{z}|} \cdot e^{i(\alpha+\beta+\gamma)}. \tag{2.76}$$

Since $\alpha + \beta + \gamma = 90^\circ$, we conclude from here that the complex number:

$$w_{\Delta ABC} := \frac{(z_2 - z_1)(z_3 - z_2)(z_1 - z_3)}{(z_1 - \bar{z})(z_2 - \bar{z})(z_3 - \bar{z})} \tag{2.77}$$

is a purely imaginary number, that means, $w_{\Delta ABC} \in i\mathbb{R}$.
Let us observe that if we apply a rotation by the angle $\theta \in [0, 2\pi)$ to the triangle ΔABC and its centroid G, then the complex numbers z_1, z_2, z_3, and $\bar{z}$, of the transformed triangle, $A'B'C'$, become $e^{i\theta}z_1$, $e^{i\theta}z_2$, $e^{i\theta}z_3$, and $e^{i\theta}\bar{z}$, respectively, and thus both the numerator

and denominator of $w_{\Delta ABC}$ get multiplied by $e^{i\theta}$. That means, $w_{\Delta A'B'C'} = w_{\Delta ABC}$.
Let us also observe that if we apply a translation $z \mapsto z + z_0$ to the triangle ΔABC and its centroid G, then the differences $z_i - z_j$ and $z_j - \bar{z}$ do not change. Thus, if $\Delta A'B'C'$ is the image of ΔABC via this translation, then we have $w_{\Delta A'B'C'} = w_{\Delta ABC}$.
We can apply first a rotation to the triangle ΔABC, such that the image triangle $\Delta A'B'C'$ will have the side BC parallel to the $x-$axis. After this, we can apply a translation to the triangle $\Delta A'B'C'$ such that the vertex A'', of the image triangle $\Delta A''B''C''$, will lie on the $y-$axis, and the side $B''C''$ will be included in the $x-$axis. Since the triangles ΔABC, $\Delta A'B'C'$, and $\Delta A''B''C''$ are congruent, to prove that the triangle ΔABC is isosceles is equivalent to proving that the triangle $\Delta A''B''C''$ is isosceles.
Therefore, we may assume that $z_1 = ia$, $z_2 = b$, and $z_3 = c$, where a, b, and $c \in \mathbb{R}$, $a \neq 0$, and $b \neq c$. Then $\bar{z} = (b + c + ia)/3$, and the condition that $w_{\Delta ABC} \in i\mathbb{R}$ becomes:

$$\frac{(b - ia)(c - b)(ia - c)}{[ia - (b + c + ia)/3][b - (b + c + ia)/3][c - (b + c + ia)/3]} \quad \in \quad i\mathbb{R}. \tag{2.78}$$

Since $c - b \in \mathbb{R} \setminus \{0\}$, the above condition is equivalent to:

$$\frac{(b - ia)(ia - c)}{(2ai - b - c)(2b - c - ai)(2c - b - ai)} \quad \in \quad i\mathbb{R}. \tag{2.79}$$

Multiplying the numerator by -1 and the denominator by $1/2$, the above condition is equivalent to:

$$\frac{(ia - b)(ia - c)}{[ia - (b + c)/2](ia - 2b + c)(ia - 2c + b)} \quad \in \quad i\mathbb{R}. \tag{2.80}$$

Finally, multiplying both the numerator and denominator of the above fraction by the conjugate of the denominator, since $z \cdot \bar{z} \in \mathbb{R}$, for all complex numbers z, we conclude that:

$$(ia - b)(ia - c)[ia + (b + c)/2](ia + 2b - c)(ia + 2c - b) \quad \in \quad i\mathbb{R}. \tag{2.81}$$

To multiply the factors from the above formula, we take $x := ia$, and define the polynomial:

$$f(x) \quad := \quad (x - b)(x - c)\left(x + \frac{b + c}{2}\right)(x + 2b - c)(x + 2c - b), \tag{2.82}$$

whose roots are $x_1 := b$, $x_2 := c$, $x_3 := -(b+c)/2$, $x_4 = -2b+c$, and $x_4 = -2c+b$. Thus we have:

$$\begin{aligned} f(x) &= (ia-b)(ia-c)[ia+(b+c)/2](ia+2b-c)(ia+2c-b) \\ &= x^5 - \left(\sum_{1\le j\le 5} x_j\right) x^4 + \left(\sum_{1\le j<k\le 5} x_j x_k\right) x^3 \\ &\quad - \left(\sum_{1\le j<k<l\le 5} x_j x_k x_l\right) x^2 + \left(\sum_{1\le j<k<l<r\le 5} x_j x_k x_l x_r\right) x - x_1x_2x_3x_4x_5 \\ &= (ia)^5 - \left(\sum_{1\le j\le 5} x_j\right)(ia)^4 + \left(\sum_{1\le j<k\le 5} x_j x_k\right)(ia)^3 \\ &\quad - \left(\sum_{1\le j<k<l\le 5} x_j x_k x_l\right)(ia)^2 + \left(\sum_{1\le j<k<l<r\le 5} x_j x_k x_l x_r\right)(ia) - x_1x_2x_3x_4x_5. \end{aligned}$$

For $f(ia)$ to belong to $i\mathbb{R}$, the real part of $f(ia)$ must be zero. Since the roots x_1, x_2, x_3, x_4, and x_5 are all real numbers, we can see from the above expression that this means:

$$0 = -\left(\sum_{1\le j\le 5} x_j\right) a^4 + \left(\sum_{1\le j<k<l\le 5} x_j x_k x_l\right) a^2 - x_1x_2x_3x_4x_5. \tag{2.83}$$

We have:

$$\begin{aligned} \sum_{1\le j\le 5} x_j &= b+c-\frac{b+c}{2} - 2b + c - 2c + b \\ &= -\frac{b+c}{2}. \end{aligned} \tag{2.84}$$

We also have:

$$\sum_{1\le j<k<l\le 5} x_j x_k x_l = x_1x_2\left(x_3+x_4+x_5\right) + x_3\left(x_1+x_2\right)\left(x_4+x_5\right) + x_4x_5\left(x_1+x_2+x_3\right).$$

Since $x_3+x_4+x_5 = -(b+c)/2 - 2b + c - 2c + b = -3(b+c)/2$, $x_1+x_2 = b+c$, $x_4+x_5 = -2b+c-2c+b = -(b+c)$, and $x_1+x_2+x_3 = b+c-(b+c)/2 = (b+c)/2$, the above equality becomes:

$$\begin{aligned} \sum_{1\le j<k<l\le 5} x_j x_k x_l &= -\frac{3}{2}bc(b+c) + \frac{1}{2}(b+c)^3 + \frac{1}{2}(-2b+c)(-2c+b)(b+c) \\ &= \frac{1}{2}(b+c)\left[-3bc + (b+c)^2 + (-2b+c)(-2c+b)\right] \\ &= -\frac{1}{2}(b+c)\left[\left(b^2-2bc\right) + \left(c^2-2bc\right)\right]. \end{aligned} \tag{2.85}$$

Finally, we have:

$$\begin{aligned} x_1x_2x_3x_4x_5 &= -\frac{1}{2}(b+c)\left[b(-2c+b)\right]\left[c(-2b+c)\right] \\ &= -\frac{1}{2}(b+c)\left(b^2-2bc\right)\left(c^2-2bc\right). \end{aligned} \tag{2.86}$$

Substituting (2.84), (2.85), and (2.86) in (2.83), we obtain:

$$\begin{aligned} 0 &= \frac{1}{2}(b+c)a^4 - \frac{1}{2}(b+c)\left[\left(b^2-2bc\right)+\left(c^2-2bc\right)\right]a^2 + \frac{1}{2}(b+c)\left(b^2-2bc\right)\left(c^2-2bc\right) \\ &= \frac{1}{2}(b+c)\left\{a^4 - \left[\left(b^2-2bc\right)+\left(c^2-2bc\right)\right]a^2 + \left(b^2-2bc\right)\left(c^2-2bc\right)\right\} \\ &= \frac{1}{2}(b+c)\left[a^2-\left(b^2-2bc\right)\right]\left[a^2-\left(c^2-2bc\right)\right]. \end{aligned}$$

For the above product to be 0, one of its factors must be zero. That means, $b+c=0$, or $a^2-(b^2-2bc)=0$, or $a^2-(c^2-2bc)=0$. Thus, we have the following three cases:
Case 1. If $b+c=0$, then $b=-c$, and so, we have:

$$\begin{aligned} \overline{AB} &= |z_2-z_1| \\ &= |b-ai| \\ &= \sqrt{b^2+a^2} \\ &= \sqrt{c^2+a^2} \\ &= |c-ai| \\ &= |z_2-z_1| \\ &= \overline{AC}. \end{aligned} \quad (2.87)$$

Therefore, the triangle ΔABC is isosceles with the vertex at A.
Case 2. If $a^2-(b^2-2bc)=0$, then $a^2=b^2-2bc$, and adding c^2 to both sides of this equality, we obtain $a^2+c^2=(b-c)^2$. This means,

$$\begin{aligned} \overline{CA} &= |z_3-z_1| \\ &= |c-ai| \\ &= \sqrt{c^2+a^2} \\ &= \sqrt{(b-c)^2} \\ &= |b-c| \quad (2.88) \\ &= |z_2-z_3| \\ &= \overline{CB}. \quad (2.89) \end{aligned}$$

Therefore, the triangle ΔABC is isosceles with the vertex at C.
Case 3. If $a^2-(c^2-2bc)=0$, then similarly to **Case 2** we have $\overline{BA}=\overline{BC}$, and so, the triangle ΔABC is isosceles with the vertex at B. □

Therefore, we have obtained the following result:

Lemma 2.1. *The roots of a cubic polynomial z_1, z_2, and z_3 form an isosceles (possible equilateral) triangle if and only if the number:*

$$E := \left[\frac{(z_2-z_1)(z_3-z_2)(z_2-z_1)}{(\bar{z}-z_1)(\bar{z}-z_2)(\bar{z}-z_3)}\right]^2, \quad (2.90)$$

where $\bar{z}=(z_1+z_2+z_3)/3$ is negative and real.

The fact, that the above number is negative, excludes the possibility that the roots z_1, z_2, and z_3 are collinear, since in that case $\sqrt{E}$ is a real number, and therefore, $E = (\sqrt{E})^2$ will be positive. Contradiction.
Since both the numerator and denominator of E are symmetric polynomials of the roots z_1, z_2, and z_3, by the Fundamental Theorem of Symmetric polynomials, they can both be written as polynomials of the sums $z_1 + z_2 + z_3$, $z_1z_2 + z_2z_3 + z_3z_1$, and $z_1z_2z_3$. Because the last three sums can be expressed, via Vieta relations, in terms of the coefficients a, b, c, and d of the cubic polynomials $f(z) = az^3 + bz^2 + cz + d$, whose roots are z_1, z_2, and z_3, we should be able now to answer the initial question that we posed at the beginning of this work. Therefore, let us compute E in terms of a, b, c, and d. We have:

$$\begin{aligned} E &:= \left[\frac{(z_2 - z_1)(z_3 - z_2)(z_2 - z_1)}{(\bar{z} - z_1)(\bar{z} - z_2)(\bar{z} - z_3)}\right]^2 \\ &= -\frac{1}{a} \cdot \frac{[a(z_1 - z_2)(z_1 - z_3)][a(z_2 - z_3)(z_2 - z_1)][a(z_3 - z_1)(z_3 - z_2)]}{\{a[(z_1 + z_2 + z_3)/3 - z_1][(z_1 + z_2 + z_3)/3 - z_2][(z_1 + z_2 + z_3)/3 - z_3]\}^2} \\ &= -\frac{1}{a} \cdot \frac{f'(z_1)f'(z_2)f'(z_3)}{f^2((z_1 + z_2 + z_3)/3)}. \end{aligned} \tag{2.91}$$

Using Vieta relations, we have:

$$\frac{z_1 + z_2 + z_3}{3} = -\frac{b}{3a}. \tag{2.92}$$

Since the leading coefficient of f' is $3a$ and its roots are denoted by w_1 and w_2, we have:

$$\begin{aligned} f'(z_1)f'(z_2)f'(z_3) &= [3a(z_1 - w_1)(z_1 - w_2)]\,[3a(z_2 - w_1)(z_2 - w_2)]\,[3a(z_3 - w_1)(z_3 - w_2)] \\ &= 27a\,[a(w_1 - z_1)(w_1 - z_2)(w_1 - z_3)]\,[a(w_2 - z_1)(w_2 - z_2)(w_2 - z_3)] \\ &= 27af(w_1)f(w_2) \\ &= 27a\left[q(w_1)f'(w_1) + r(w_1)\right]\left[q(w_1)f'(w_2) + r(w_2)\right] \\ &= 27ar(w_1)r(w_2), \end{aligned} \tag{2.93}$$

where q and r represent the quotient and remainder, respectively of f when divided by f'. We have:

$$\begin{aligned} f(z) &= az^3 + bz^2 + cz + d \\ &= \frac{1}{3}z\left(3az^2 + 2bz + c\right) + \frac{1}{3}\left(bz^2 + 2cz + 3d\right) \\ &= \frac{1}{3}z\left(3az^2 + 2bz + c\right) + \frac{1}{3} \cdot \frac{b}{3a}\left[3az^2 + 2bz + c\right] \\ &\quad + \frac{1}{9a}\left[2\left(3ac - b^2\right)z + (9ad - bc)\right] \\ &= \left[\frac{1}{3}z + \frac{b}{9a}\right]\left(3az^2 + 2bz + c\right) \\ &\quad + \frac{1}{9a}\left[2\left(3ac - b^2\right)z + (9ad - bc)\right]. \end{aligned} \tag{2.94}$$

From here it follows that:

$$q(z) = \frac{1}{3}z + \frac{b}{9a} \tag{2.95}$$

and

$$r(z) = \frac{1}{9a}\left[2\left(3ac-b^2\right)z+(9ad-bc)\right]. \quad (2.96)$$

Let us define the numbers:

$$\Delta_1 := \begin{vmatrix} 3a & b \\ b & c \end{vmatrix} \quad (2.97)$$

$$= 3ac-b^2 \quad (2.98)$$

and

$$\Delta_2 := \begin{vmatrix} 3a & b \\ c & 3d \end{vmatrix} \quad (2.99)$$

$$= 9ad-bc. \quad (2.100)$$

We have:

$$r(z) = \frac{1}{9a}\left(2\Delta_1 z+\Delta_2\right). \quad (2.101)$$

Thus, we have:

$$\begin{aligned} f'(z_1)f'(z_2)f'(z_3) &= 27ar(w_1)r(w_2) \\ &= 27a\cdot\frac{1}{81a^2}\left(2\Delta_1 w_1+\Delta_2\right)\left(2\Delta_1 w_2+\Delta_2\right) \\ &= \frac{1}{3a}\left(2\Delta_1 w_1+\Delta_2\right)\left(2\Delta_1 w_2+\Delta_2\right). \quad (2.102) \end{aligned}$$

If $\Delta_1 = 0$, then $f'(z_1)f'(z_2)f'(z_3) = \Delta_2^2/(3a)$.
If $\Delta_1 \neq 0$, then

$$\begin{aligned} f'(z_1)f'(z_2)f'(z_3) &= \frac{4\Delta_1^2}{9a^2}\cdot 3a\left(w_1+\frac{\Delta_2}{2\Delta_1}\right)\left(w_2+\frac{\Delta_2}{2\Delta_1}\right) \\ &= \frac{4\Delta_1^2}{9a^2}\cdot f'\left(-\frac{\Delta_2}{2\Delta_1}\right) \\ &= \frac{4\Delta_1^2}{9a^2}\left[3a\left(-\frac{\Delta_2}{2\Delta_1}\right)^2+2b\left(-\frac{\Delta_2}{2\Delta_1}\right)+c\right]. \\ &= \frac{1}{9a^2}\left[3a\Delta_2^2-4b\Delta_1\Delta_2+4c\Delta_1^2\right]. \end{aligned}$$

Thus, we can see that in all cases, we have:

$$f'(z_1)f'(z_2)f'(z_3) = \frac{1}{9a^2}\left[3a\Delta_2^2-4b\Delta_1\Delta_2+4c\Delta_1^2\right]. \quad (2.99)$$

Using formulas (2.91), (2.92), and (2.99), we obtain:

$$\begin{aligned} E &= -\frac{1}{a}\cdot\frac{f'(z_1)f'(z_2)f'(z_3)}{f^2((z_1+z_2+z_3)/3)} \\ &= -\frac{1}{a}\cdot\frac{1}{9a^2}\cdot\left[3a\Delta_2^2-4b\Delta_1\Delta_2+4c\Delta_1^2\right]\cdot\frac{1}{f^2(-b/(3a))} \\ &= -\frac{3a\Delta_2^2-4b\Delta_1\Delta_2+4c\Delta_1^2}{9a^3f^2(-b/(3a))}. \quad (2.98) \end{aligned}$$

We can now present the answer to our question.

Theorem 2.2. *Let $f(z) = az^3 + bz^2 + cz + d$ be a polynomial of degree with complex coefficients. Then the following statements are equivalent:*

(1) The roots z_1, z_2, and z_3 form a non-equilateral non-degenerate triangle ABC whose Euler and Siebeck-Marden lines coincide or are perpendicular.

(2) The roots z_1, z_2, and z_3 form a non-equilateral non-degenerate triangle ABC, such that

$$m(\sphericalangle MAB) + m(\sphericalangle MBC) + m(\sphericalangle MCA) \quad = \quad 180^\circ, \tag{2.99}$$

where G is the centroid of the triangle ABC.

(3) The roots z_1, z_2, and z_3 form a non-equilateral non-degenerate isosceles triangle ABC.

(4) $f(-b/(3a)) \neq 0$, and the number:

$$E \quad = \quad -\frac{3a\Delta_2^2 - 4b\Delta_1\Delta_2 + 4c\Delta_1^2}{9a^3 f^2(-b/(3a))} \tag{2.100}$$

is real, negative and different from -27, where $\Delta_1 = 3ac - b^2$ and $\Delta_2 = 9ad - bc$.

Moreover, if anyone of the above four equivalent conditions holds, then the Euler and Siebeck-Marden lines coincide if and only $E > -27$. If $E < -27$, then the Euler and Siebeck-Marden lines are perpendicular.

Proof. The proof is basically done in the previous propositions. Condition $f(-b/(3a)) \neq 0$ ensures the fact that the three roots z_1, z_2, and z_3 of the polynomial f cannot be all equal, since if they were all three equal, then they will be equal to their average $(z_1 + z_2 + z_3)/3 = -b/(3a)$, and thus $-b/(3a)$ will be a root of f. This will imply $f(-b/(3a)) = 0$, contradiction.

The number E is in fact equal to the number:

$$\frac{(z_1 - z_2)^2(z_2 - z_3)^2(z_3 - z_1)^2}{(\bar{z} - z_1)^2(\bar{z} - z_2)^2(\bar{z} - z_3)^2}, \tag{2.101}$$

where $\bar{z} = (z_1 + z_2 + z_3)/3 = -b/(3a)$. Since this number is real and negative, we conclude that

$$F \quad = \quad \frac{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}{(\bar{z} - z_1)(\bar{z} - z_2)(\bar{z} - z_3)} \tag{2.102}$$

cannot be real. This implies that the roots z_1, z_2, and z_3 are not collinear, since otherwise the complex numbers $z_1 - z_2$, $z_2 - z_3$, $z_3 - z_1$, $\bar{z} - z_1$, $\bar{z} - z_2$, and $\bar{z} - z_3$ will represent collinear vectors, which will imply that the number F is real, which ia a contradiction. □

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