Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.11, (2022), Issue 1, pp.90-99

SEVERAL ENHANCEMENTS OF EULER'S INEQUALITY IN TETRAHEDRON

ZHANG YUN

ABSTRACT. In this article, we will revisit Euler's inequality in space, and then give several inequalities in the tetrahedron. These inequalities are an enhancement of the Euler's inequality in space. There is a certain theoretical significance.

1. INTRODUCTION AND MOTIVATIONS

Let R be the radius of the circumscribed sphere of a tetrahedron and let r be the radius of the inscribed sphere of the tetrahedron. Then Euler's famous inequality in space states that[1, 2]

$$R \geq 3r. \tag{1.1}$$

Here we shall find some new expressions, involving the area, the median, the altitude, the radius of its exscribed sphere of such a tetrahedron.

2. MAIN RESULTS

Given a tetrahedron $A_1A_2A_3A_4$, let R and r denote the radius of its circumscribed sphere and inscribed sphere. Let $h_i, m_i (i = 1, 2, 3, 4)$ denote the altitude and the median passing through the vertex A_i, respectively. Let s_i denote the area of the face opposite the vertex A_i and v denote the volume of the tetrahedron $A_1A_2A_3A_4$.

Theorem 1.

$$1 \leq \frac{3}{4}\sum_{i=1}^{4}\frac{s_i}{\sum_{k=1}^{4} s_k - s_i} \leq \left(\frac{R}{3r}\right)^{\frac{2}{3}}\sqrt{4\left(\frac{R}{3r}\right)^4 - 3}$$

Proof. By reference [2]

$$\sum_{i=1}^{n}\frac{y_i^2}{x_i} \geq \frac{\left(\sum_{i=1}^{n} y_i\right)^2}{\sum_{i=1}^{n} x_i}\ (x_i, y_i \in R_+, i = 1, 2, \cdots, n) \tag{2.1}$$

2010 *Mathematics Subject Classification.* 97G40; 51M04; 51M25.
Key words and phrases. Euler's Inequality in Space, Tetrahedron, Euclidean Geometries, Geometric Inequalities.

we have

$$\frac{3}{4}\sum_{i=1}^{4}\frac{s_i}{\sum_{k=1}^{4}s_k-s_i}=\frac{3}{4}\sum_{i=1}^{4}\frac{s_i^2}{s_i\left(\sum_{k=1}^{4}s_k-s_i\right)}\ge\frac{3}{4}\times\frac{\left(\sum_{i=1}^{4}s_i\right)^2}{\sum_{i=1}^{4}s_i\left(\sum_{k=1}^{4}s_k-s_i\right)}$$
$$=\frac{3}{4}\times\frac{\sum_{i=1}^{4}{s_i}^2+2\sum_{1\le i<j\le 4}s_is_j}{2\sum_{1\le i<j\le 4}s_is_j}\ge\frac{3}{4}\times\frac{\frac{2}{3}\sum_{1\le i<j\le 4}s_is_j+2\sum_{1\le i<j\le 4}s_is_j}{2\sum_{1\le i<j\le 4}s_is_j}=1 \tag{2.2}$$

On the other hand

$$\sum_{i=1}^{4}\frac{s_i}{\sum_{k=1}^{4}s_k-s_i}\le\frac{1}{3\sqrt[3]{\prod_{k=1}^{4}s_k}}\sum_{i=1}^{4}s_i\sqrt[3]{s_i}\le\frac{1}{3\sqrt[3]{\prod_{k=1}^{4}s_k}}\sqrt{\left(\sum_{i=1}^{4}s_i^2\right)\left(\sum_{i=1}^{4}\sqrt[3]{s_i^2}\right)} \tag{2.3}$$

Using the power mean inequality

$$\left(\frac{\sum_{i=1}^{n}x_i^{\alpha}}{n}\right)^{\frac{1}{\alpha}}\ge\left(\frac{\sum_{i=1}^{n}x_i^{\beta}}{n}\right)^{\frac{1}{\beta}}\ (x_i,\alpha,\beta\in R_+,i=1,2,\cdots,n,\alpha\ge\beta) \tag{2.4}$$

We get

$$\sum_{i=1}^{4}\sqrt[3]{s_i^2}\le 4\left(\frac{\sum_{i=1}^{4}s_i}{4}\right)^{\frac{2}{3}}=2^{\frac{2}{3}}\left(\sum_{i=1}^{4}s_i\right)^{\frac{2}{3}} \tag{2.5}$$

By reference [1],

$$24\sqrt{3}r^2\le\sum_{i=1}^{4}s_i\le\frac{8\sqrt{3}}{3}R^2 \tag{2.6}$$

$$v\le\frac{2^{\frac{3}{2}}}{3^{\frac{7}{4}}}\left(\prod_{i=1}^{4}s_i\right)^{\frac{3}{8}},v=\frac{1}{3}r\sum_{i=1}^{4}s_i \tag{2.7}$$

$$\sum_{i=1}^{4}\frac{1}{s_i}\le\frac{2\sqrt{3}}{9r^2},\prod_{k=1}^{4}h_k\ge 256r^4 \tag{2.8}$$

And then,

$$2^4\times 3^6r^8\le\prod_{i=1}^{4}s_i\le\frac{16R^8}{9} \tag{2.9}$$

$$\sum_{i=1}^{4}\sqrt[3]{s_i^2}\le 2^{\frac{2}{3}}\left(\frac{8\sqrt{3}R^2}{3}\right)^{\frac{2}{3}}=2^{\frac{8}{3}}\times 3^{-\frac{1}{3}}R^{\frac{4}{3}} \tag{2.10}$$

$$\sum_{i=1}^{4}{s_i}^2=\left(\sum_{i=1}^{4}s_i\right)^2-2\sum_{1\le i<j\le 4}s_is_j\le\left(\sum_{i=1}^{4}s_i\right)^2-12\left(\prod_{i=1}^{4}s_i\right)^{\frac{1}{2}}$$
$$\le\left(\frac{8\sqrt{3}R^2}{3}\right)^2-12\left(2^4\times 3^6r^8\right)^{\frac{1}{2}}=\frac{2^6R^4-2^4\times 3^5r^4}{3} \tag{2.11}$$

$$\sum_{i=1}^{4}\frac{s_i}{\sum_{k=1}^{4}s_k-s_i}\leq\frac{1}{3}\times\left(2^4\times3^6r^8\right)^{-\frac{1}{3}}\times\left(\frac{2^6R^4-2^4\times3^5r^4}{3}\right)^{\frac{1}{2}}\times \left(2^{\frac{8}{3}}\times3^{-\frac{1}{3}}R^{\frac{4}{3}}\right)^{\frac{1}{2}}=\frac{4}{3}\left(\frac{R}{3r}\right)^{\frac{2}{3}}\sqrt{4\left(\frac{R}{3r}\right)^4-3} \tag{2.12}$$

$$\frac{3}{4}\sum_{i=1}^{4}\frac{s_i}{\sum_{k=1}^{4}s_k-s_i}\leq\left(\frac{R}{3r}\right)^{\frac{2}{3}}\sqrt{4\left(\frac{R}{3r}\right)^4-3} \tag{2.13}$$

Thus Theorem 1 has been proved. □

We have noticed

$$1\leq\left(\frac{R}{3r}\right)^{\frac{2}{3}}\sqrt{4\left(\frac{R}{3r}\right)^4-3}\Leftrightarrow 1\leq\left(\frac{R}{3r}\right)^{\frac{4}{3}}\left[4\left(\frac{R}{3r}\right)^4-3\right] \Leftrightarrow 4\left[\left(\frac{R}{3r}\right)^{\frac{4}{3}}\right]^4-3\left(\frac{R}{3r}\right)^{\frac{4}{3}}-1\geq0 \tag{2.14}$$

For convenience, put $\left(\frac{R}{3r}\right)^{\frac{4}{3}}=x$, then $x>0$, and

$$4\left[\left(\frac{R}{3r}\right)^{\frac{4}{3}}\right]^4-3\left(\frac{R}{3r}\right)^{\frac{4}{3}}-1\geq0\Leftrightarrow4x^4-3x-1\geq0\Leftrightarrow\left(3x^4-3x\right)+\left(x^4-1\right)\geq0$$
$$\Leftrightarrow(x-1)\left[3x\left(x^2+x+1\right)+(x+1)\left(x^2+1\right)\right]\geq0\Leftrightarrow x\geq1\Leftrightarrow\left(\frac{R}{3r}\right)^{\frac{4}{3}}\geq1\Leftrightarrow R\geq3r \tag{2.15}$$

This is Euler's inequality. It shows that Theorem 1 is stronger than $R\geq3r$.
For a triangle $A_1A_2A_3$, the same as the above the tetrahedron $A_1A_2A_3A_4$, we get

Corollary 1. *In the triangle $A_1A_2A_3$, we have*

$$1\leq\frac{2}{3}\sum_{i=1}^{3}\frac{a_i}{\sum_{k=1}^{3}a_k-a_i}\leq\frac{1}{3}\sqrt{8\left(\frac{R}{2r}\right)+1}$$

It is easy to prove that Corollary 1 is stronger than Euler's inequality $R\geq2r$.

Theorem 2.

$$\left(\frac{R}{3r}\right)^{-2}\leq\frac{9\sqrt{3}}{32}\sum_{i=1}^{4}\frac{h_i^2}{\sum_{k=1}^{4}s_k-s_i}\leq\left(\frac{R}{3r}\right)^{\frac{2}{3}}\sqrt{97\left(\frac{R}{3r}\right)^{16}-96}$$

Proof. According to the proof of Theorem 1, we get

$$
\begin{aligned}
&\sum_{i=1}^{4} \frac{h_i^2}{\sum_{k=1}^{4} s_k - s_i} \geq \frac{\left(\sum_{i=1}^{4} h_i\right)^2}{\sum_{i=1}^{4}\left(\sum_{k=1}^{4} s_k - s_i\right)} = \frac{\left(\sum_{i=1}^{4} h_i\right)^2}{3\sum_{i=1}^{4} s_i} = \frac{1}{3\sum_{i=1}^{4} s_i}\left(\sum_{k=1}^{4} \frac{3v}{s_k}\right)^2 \\
&= \frac{(3v)^2}{3\sum_{i=1}^{4} s_i}\left(\sum_{k=1}^{4} \frac{1}{s_k}\right)^2 = \frac{\left(r\sum_{i=1}^{4} s_i\right)^2}{3\sum_{i=1}^{4} s_i}\left(\sum_{i=1}^{4} \frac{1}{s_i}\right)^2 \geq \frac{r^2}{3}\left(\sum_{i=1}^{4} s_i\right)\left(\frac{16}{\sum_{i=1}^{4} s_i}\right)^2 \\
&= \frac{256r^2}{3\sum_{i=1}^{4} s_i} \geq \frac{256r^2}{3} \times \frac{3}{8\sqrt{3}R^2} = \frac{32\sqrt{3}}{27}\left(\frac{R}{3r}\right)^{-2}
\end{aligned}
\tag{2.16}
$$

Then

$$
\frac{9\sqrt{3}}{32}\sum_{i=1}^{4} \frac{h_i^2}{\sum_{k=1}^{4} s_k - s_i} \geq \frac{9\sqrt{3}}{32} \times \frac{32\sqrt{3}}{27}\left(\frac{R}{3r}\right)^{-2} = \left(\frac{R}{3r}\right)^{-2}
\tag{2.17}
$$

So the left side has been proved. We see the right side again.

$$
\begin{aligned}
\sum_{i=1}^{4} \frac{h_i^2}{\sum_{k=1}^{4} s_k - s_i} &\leq \sum_{i=1}^{4} \frac{h_i^2}{3\sqrt[3]{\frac{\prod_{k=1}^{4} s_k}{s_i}}} = \frac{1}{3}\sum_{i=1}^{4} \sqrt[3]{\frac{s_i}{\prod_{k=1}^{4} s_k}} h_i^2 \\
&= \frac{1}{3\sqrt[3]{\prod_{k=1}^{4} s_k}}\sum_{i=1}^{4} h_i^2 \sqrt[3]{s_i} \leq \frac{1}{3\sqrt[3]{\prod_{k=1}^{4} s_k}}\sqrt{\left(\sum_{i=1}^{4} h_i^4\right)\left(\sum_{j=1}^{4} \sqrt[3]{s_j^2}\right)}
\end{aligned}
\tag{2.18}
$$

From the proof process of Theorem 1, we obtain

$$
\sqrt[3]{\prod_{k=1}^{4} s_k} \geq 2^{\frac{4}{3}} \times 3^2 r^{\frac{8}{3}}
\tag{2.19}
$$

$$
\sum_{i=1}^{4} \sqrt[3]{{s_i}^2} \leq 2^{\frac{8}{3}} \times 3^{-\frac{1}{3}} R^{\frac{4}{3}}
\tag{2.20}
$$

$$\begin{aligned}
\sum_{i=1}^{4} h_i^2 &= \left(\sum_{i=1}^{4} h_i\right)^2 - 2\sum_{1\le i<j\le 4} h_i h_j = \left(\sum_{i=1}^{4} \frac{3v}{s_i}\right)^2 - 2\sum_{1\le i<j\le 4} h_i h_j \\
&= (3v)^2\left(\sum_{i=1}^{4}\frac{1}{s_i}\right)^2 - 2\sum_{1\le i<j\le 4} h_i h_j \\
&\le \left(r\sum_{i=1}^{4} s_i\right)^2\left(\frac{2\sqrt{3}}{9r^2}\right)^2 - 2\times 6\sqrt[6]{h_1^3h_2^3h_3^3h_4^3} \\
&= r^2\left(\sum_{i=1}^{4} s_i\right)^2 \times \frac{2^2\times 3}{3^4r^4} - 12\left(\prod_{k=1}^{4} h_k\right)^{\frac{1}{2}} \\
&\le r^2\left(\frac{8\sqrt{3}R^2}{3}\right)^2 \times \frac{4}{3^3r^4} - 12\left[\frac{(3v)^4}{\prod_{i=1}^{4} s_i}\right]^{\frac{1}{2}} \\
&= \frac{2^8R^4}{3^4r^2} - 3\times 2^2\left(r\sum_{i=1}^{4} s_i\right)^2 \times \frac{1}{\left(\prod_{k=1}^{4} s_k\right)^{\frac{1}{2}}} \\
&\le \frac{2^8R^4}{3^4r^2} - 3\times 2^2r^2\times\left(24\sqrt{3}r^2\right)^2\times\frac{1}{\left(\frac{16R^8}{9}\right)^{\frac{1}{2}}} = \frac{2^6\left(4R^8-3^9r^8\right)}{3^4R^4r^2}
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
\sum_{i=1}^{4} h_i^4 &= \left(\sum_{i=1}^{4} h_i^2\right)^2 - 2\sum_{1\le i<j\le 4} h_i^2h_j^2 \le \left[\frac{2^6\left(4R^8-3^9r^8\right)}{3^4R^4r^2}\right]^2 - 2\times 6\prod_{i=1}^{4} h_i \\
&= \frac{2^{12}\left(4R^8-3^9r^8\right)^2}{3^8R^8r^4} - 12\prod_{i=1}^{4}\frac{3v}{s_i} = \frac{2^{12}\left(4R^8-3^9r^8\right)^2}{3^8R^8r^4} - 12\left(r\sum_{i=1}^{4} s_i\right)^4\times\frac{1}{\prod_{k=1}^{4} s_k} \\
&\le \frac{2^{12}\left(4R^8-3^9r^8\right)^2}{3^8R^8r^4} - 12r^4\times\left(24\sqrt{3}r^2\right)^4\times\frac{1}{\frac{16R^8}{9}} \\
&= \frac{2^{12}\left(4R^8-3^9r^8\right)^2}{3^8R^8r^4} - 12r^4\times 3^4\times 2^{12}\times 3^2r^8\times\frac{9}{16R^8} \\
&= \frac{2^{12}\left(4R^8-3^9r^8\right)^2}{3^8R^8r^4} - \frac{2^{10}\times 3^9r^{12}}{R^8} = \frac{2^{10}\left[64R^{16}-96R^8(3r)^8+33(3r)^{16}\right]}{3^8R^8r^4} \\
&\le \frac{2^{10}\left[64R^{16}-96R^8(3r)^8+33R^{16}\right]}{3^8R^8r^4} = \frac{2^{10}\left[97R^{16}-96R^8(3r)^8\right]}{3^8R^8r^4} \\
&\le \frac{2^{10}\left[97R^{16}-96(3r)^8(3r)^8\right]}{3^8R^8r^4} = \frac{2^{10}\left[97R^{16}-96(3r)^{16}\right]}{3^8R^8r^4} \\
&\le \frac{2^{10}\left[97R^{16}-96(3r)^{16}\right]}{3^8(3r)^8r^4} = \frac{2^{10}\left[97R^{16}-96(3r)^{16}\right]}{3^4(3r)^{12}}
\end{aligned} \tag{2.22}$$

And then

$$\sqrt{\left(\sum_{i=1}^{4} h_i^4\right)\left(\sum_{k=1}^{4}\sqrt[3]{s_k{}^2}\right)} \le \sqrt{2^{\frac{8}{3}} \times 3^{-\frac{1}{3}} R^{\frac{4}{3}} \times \frac{2^{10}\left[97R^{16}-96(3r)^{16}\right]}{3^4(3r)^{12}}} = \frac{2^{\frac{4}{3}} \times 2^5 R^{\frac{2}{3}}}{3^2(3r)^6} \times \frac{1}{3^{\frac{1}{6}}}\sqrt{97R^{16}-96(3r)^{16}} \tag{2.23}$$

$$\frac{1}{3\sqrt[3]{\prod_{k=1}^{4} s_k}}\sqrt{\left(\sum_{k=1}^{4} h_k^4\right)\left(\sum_{i=1}^{4}\sqrt[3]{s_i^2}\right)} \le \frac{1}{3} \times \frac{1}{2^{\frac{4}{3}} \times 3^2 r^{\frac{8}{3}}} \times \frac{2^{\frac{4}{3}} \times 2^5 R^{\frac{2}{3}}}{3^2(3r)^6 \times 3^{\frac{1}{6}}}\sqrt{97R^{16}-96(3r)^{16}} = \frac{2^5 R^{\frac{2}{3}}}{3^5 r^2 \times r^{\frac{2}{3}}(3r)^6 \times 3^{\frac{1}{6}}}\sqrt{97R^{16}-96(3r)^{16}} = \frac{32\sqrt{3}}{27}\left(\frac{R}{3r}\right)^{\frac{2}{3}}\sqrt{97\left(\frac{R}{3r}\right)^{16}-96} \tag{2.24}$$

Thus

$$\frac{9\sqrt{3}}{32}\sum_{i=1}^{4}\frac{h_i^2}{\sum_i^4 s_k - s_i} \le \frac{9\sqrt{3}}{32} \times \frac{32\sqrt{3}}{27}\left(\frac{R}{3r}\right)^{\frac{2}{3}}\sqrt{97\left(\frac{R}{3r}\right)^{16}-96} = \left(\frac{R}{3r}\right)^{\frac{2}{3}}\sqrt{97\left(\frac{R}{3r}\right)^{16}-96} \tag{2.25}$$

This completes the proof of Theorem 2. □

From $\left(\frac{R}{3r}\right)^{-2} \le \left(\frac{R}{3r}\right)^{\frac{2}{3}}\sqrt{97\left(\frac{R}{3r}\right)^{16}-96}$, we get

$$1 \le \left(\frac{R}{3r}\right)^{\frac{8}{3}}\sqrt{97\left(\frac{R}{3r}\right)^{16}-96} \Leftrightarrow 1 \le \left(\frac{R}{3r}\right)^{\frac{16}{3}}\left[97\left(\frac{R}{3r}\right)^{16}-96\right] \tag{2.26}$$

$$97\left[\left(\frac{R}{3r}\right)^{\frac{16}{3}}\right]^4 - 96\left(\frac{R}{3r}\right)^{\frac{16}{3}} - 1 \ge 0 \tag{2.27}$$

Put $\left(\frac{R}{3r}\right)^{\frac{16}{3}} = x$, then $x > 0$, and

$$97\left[\left(\frac{R}{3r}\right)^{\frac{16}{3}}\right]^4 - 96\left(\frac{R}{3r}\right)^{\frac{16}{3}} - 1 \ge 0 \Leftrightarrow 97x^4 - 96x - 1 \ge 0 \Leftrightarrow (x-1)\left[96x\left(x^2+x+1\right)+\left(x^2+1\right)(x+1)\right] \ge 0 \Leftrightarrow x \ge 1 \Leftrightarrow \left(\frac{R}{3r}\right)^{\frac{16}{3}} \ge 1 \Leftrightarrow R \ge 3r \tag{2.28}$$

This is Euler's inequality. This implies that Euler's inequality $R \ge 3r$ can be obtained by the proof of Theorem 2.

Corollary 2. *In the triangle $A_1A_2A_3$, we can get*

$$\left(\frac{R}{2r}\right)^{-2} \leq \frac{4\sqrt{3}}{9}\sum_{i=1}^{3}\frac{h_i}{\sum_{k=1}^{3}a_k - a_i} \leq \frac{2}{3}\sqrt{\frac{1}{4}\left(\frac{R}{2r}\right)^{-2}+2}$$

It shows that Euler's inequality $R \geq 2r$ can be obtained by Corollary 2.

Theorem 3.

$$\left(\frac{R}{3r}\right)^{-2} \leq \frac{9\sqrt{3}}{32}\sum_{i=1}^{4}\frac{m_i^2}{\sum_{k=1}^{4}s_k - s_i} \leq \left(\frac{R}{3r}\right)^{\frac{2}{3}}\sqrt{4\left(\frac{R}{3r}\right)^{4}-3}$$

Proof.

$$\sum_{i=1}^{4}\frac{m_i^2}{\sum_{k=1}^{4}s_k - s_i} \geq \frac{\left(\sum_{i=1}^{4}m_i\right)^2}{3\sum_{i=1}^{4}s_i} \geq \frac{1}{3}\times\frac{1}{\frac{8\sqrt{3}R^2}{3}}\left(\sum_{i=1}^{4}m_i\right)^2 = \frac{1}{8\sqrt{3}R^2}\left(\sum_{i=1}^{4}m_i\right)^2 \tag{2.29}$$

By reference[1], we obtain

$$16r \leq \sum_{i=1}^{4}m_i \leq \frac{16R}{3} \tag{2.30}$$

$$64r^2 \leq \sum_{i=1}^{4}m_i^2 \leq \frac{64R^2}{9} \tag{2.31}$$

$$\prod_{i=1}^{4}h_i \geq 256r^4 \tag{2.32}$$

And so

$$\frac{1}{8\sqrt{3}R^2}\left(\sum_{i=1}^{4}m_i\right)^2 \geq \frac{1}{8\sqrt{3}R^2}\times(16r)^2 = \frac{32\sqrt{3}}{27}\left(\frac{R}{3r}\right)^{-2} \tag{2.33}$$

Hence

$$\frac{9\sqrt{3}}{32}\sum_{i=1}^{4}\frac{{m_i}^2}{\sum_{k=1}^{4}s_k - s_i} \geq \frac{9\sqrt{3}}{32}\times\frac{32\sqrt{3}}{27}\left(\frac{R}{3r}\right)^{-2} = \left(\frac{R}{3r}\right)^{-2} \tag{2.34}$$

On the other hand,

$$\begin{aligned}\sum_{i=1}^{4}\frac{m_i^2}{\sum_{k=1}^{4}s_k - s_i} &\leq \frac{1}{3\sqrt[3]{\prod_{k=1}^{4}s_k}}\times\sum_{i=1}^{4}\sqrt[3]{s_i}m_i^2 \leq \frac{1}{3\sqrt[3]{\prod_{k=1}^{4}s_k}}\sqrt{\left(\sum_{i=1}^{4}\sqrt[3]{s_i^2}\right)\left(\sum_{i=1}^{4}m_i^4\right)}\\ &\leq \frac{1}{3\sqrt[3]{\prod_{k=1}^{4}s_k}}\sqrt{2^{\frac{2}{3}}\left(\sum_{i=1}^{4}s_i\right)^{\frac{2}{3}}\left(\sum_{k=1}^{4}m_k^4\right)} = \frac{2^{\frac{1}{3}}}{3}\left(\frac{\sum_{i=1}^{4}s_i}{\prod_{k=1}^{4}s_k}\right)^{\frac{1}{3}}\sqrt{\sum_{i=1}^{4}m_i^4}\\ &\leq \frac{2^{\frac{1}{3}}}{3}\left(\frac{8\sqrt{3}R^2}{2^4\times 3^7r^8}\right)^{\frac{1}{3}}\sqrt{\sum_{i=1}^{4}m_i^4} = 2^{-\frac{5}{2}}\times\frac{1}{r^2}\left(\frac{R}{3r}\right)^{\frac{2}{3}}\sqrt{\sum_{i=1}^{4}m_i^4}\end{aligned} \tag{2.35}$$

We consider

$$m_i \geq h_i, \prod_{i=1}^{4}h_i \geq 256r^4 \tag{2.36}$$

Then

$$\prod_{i=1}^{4} m_i \geq \prod_{i=1}^{4} h_i \geq 256r^4 \tag{2.37}$$

$$\begin{aligned}\sum_{i=1}^{4} m_i^4 &= \left(\sum_{i=1}^{4} m_i^2\right)^2 - 2\sum_{1\leq i<j\leq 4} m_i^2 m_j^2 \leq \left(\sum_{i=1}^{4} m_i^2\right)^2 - 12\prod_{i=1}^{4} m_i \\ &\leq \left(\sum_{i=1}^{4} m_i^2\right)^2 - 12\times 256r^4 \leq \left(\frac{64R^2}{9}\right)^2 - 3\times 2^{10}r^4 = \frac{2^{10}\left[4R^4 - 3(3r)^4\right]}{3^4}\end{aligned} \tag{2.38}$$

$$\sqrt{\sum_{i=1}^{4} m_i^4} \leq \frac{2^5\sqrt{4R^4 - 3(3r)^4}}{9} \tag{2.39}$$

$$\begin{aligned}2^{-\frac{5}{2}} \times \frac{1}{r^2}\left(\frac{R}{3r}\right)^{\frac{2}{3}} \sqrt{\sum_{i=1}^{4} m_i^4} &\leq 2^{-\frac{5}{2}} \times \frac{1}{r^2}\left(\frac{R}{3r}\right)^{\frac{2}{3}} \times \frac{2^5\sqrt{4R^4 - 3(3r)^4}}{9} \\ &= \frac{32\sqrt{3}}{27}\left(\frac{R}{3r}\right)^{\frac{2}{3}} \sqrt{4\left(\frac{R}{3r}\right)^4 - 3}\end{aligned} \tag{2.40}$$

So

$$\sum_{i=1}^{4} \frac{m_i^2}{\sum_{k=1}^{4} s_k - s_i} \leq \frac{32\sqrt{3}}{27}\left(\frac{R}{3r}\right)^{\frac{2}{3}} \sqrt{4\left(\frac{R}{3r}\right)^4 - 3} \tag{2.41}$$

And

$$\frac{9\sqrt{3}}{32}\sum_{i=1}^{4} \frac{m_i^2}{\sum_{k=1}^{4} s_k - s_i} \leq \frac{9\sqrt{3}}{32} \times \frac{32\sqrt{3}}{27}\left(\frac{R}{3r}\right)^{\frac{2}{3}} \sqrt{4\left(\frac{R}{3r}\right)^4 - 3} = \left(\frac{R}{3r}\right)^{\frac{2}{3}} \sqrt{4\left(\frac{R}{3r}\right)^4 - 3} \tag{2.42}$$

Thus Theorem 3 has been proved.

□

From $\left(\frac{R}{3r}\right)^{-2} \leq \left(\frac{R}{3r}\right)^{\frac{2}{3}} \sqrt{4\left(\frac{R}{3r}\right)^4 - 3}$, we get

$$\begin{aligned}1 \leq \left(\frac{R}{3r}\right)^{\frac{8}{3}} \sqrt{4\left(\frac{R}{3r}\right)^4 - 3} &\Leftrightarrow 1 \leq \left(\frac{R}{3r}\right)^{\frac{16}{3}} \left[4\left(\frac{R}{3r}\right)^4 - 3\right] \\ &\Leftrightarrow 4\left[\left(\frac{R}{3r}\right)^{\frac{4}{3}}\right]^7 - 3\left[\left(\frac{R}{3r}\right)^{\frac{4}{3}}\right]^4 - 1 \geq 0\end{aligned} \tag{2.43}$$

Put $\left(\frac{R}{3r}\right)^{\frac{4}{3}} = x$, then $x > 0$. So

$$\begin{aligned}
&4\left[\left(\frac{R}{3r}\right)^{\frac{4}{3}}\right]^7 - 3\left[\left(\frac{R}{3r}\right)^{\frac{4}{3}}\right]^4 - 1 \geq 0 \Leftrightarrow 4x^7 - 3x^4 - 1 \geq 0 \\
&\Leftrightarrow (x-1)\left[3x^4\left(x^2+x+1\right) + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\right] \geq 0 \\
&\Leftrightarrow x \geq 1 \Leftrightarrow \left(\frac{R}{3r}\right)^{\frac{4}{3}} \geq 1 \Leftrightarrow R \geq 3r
\end{aligned} \tag{2.44}$$

This is Euler's inequality. It shows that Euler's inequality $R \geq 3r$ is separated by $\frac{9\sqrt{3}}{32}\sum_{i=1}^{4}\frac{m_i^2}{\sum_{k=1}^{4}s_k - s_i}$.

Corollary 3. *In the triangle $A_1A_2A_3$, we get*

$$\left(\frac{R}{2r}\right)^{-2} \leq \frac{4\sqrt{3}}{9}\sum_{i=1}^{3}\frac{m_i}{\sum_{k=1}^{3}a_k - a_i} \leq \frac{1}{3}\sqrt{8\left(\frac{R}{2r}\right)^2 + 1} \tag{2.45}$$

It is easy to prove that Euler's inequality $R \geq 2r$ is separated by $\frac{4\sqrt{3}}{9}\sum_{i=1}^{3}\frac{m_i}{\sum_{k=1}^{3}a_k - a_i}$.

Theorem 4.

$$\frac{9\sqrt{3}}{8}\sum_{i=1}^{4}\frac{{r_i}^2}{\sum_{k=1}^{4}s_k - s_i} \geq \left(\frac{R}{3r}\right)^{-2}$$

Proof. By $r_i = \frac{3v}{\sum_{k=1}^{4}s_k - 2s_i}$, we have

$$\begin{aligned}
\sum_{i=1}^{4} r_i &\geq 4\sqrt[4]{\frac{(3v)^4}{\prod_{i=1}^{4}\left(\sum_{k=1}^{4}s_k - 2s_i\right)}} = 4 \times 3v \times \frac{1}{\sqrt[4]{\prod_{i=1}^{4}\left(\sum_{k=1}^{4}s_k - 2s_i\right)}} \\
&\geq 4 \times r\left(\sum_{i=1}^{4}s_i\right) \times \frac{1}{\sqrt[4]{\left[\frac{\sum_{i=1}^{4}\left(\sum_{k=1}^{4}s_k - 2s_i\right)}{4}\right]^4}} = 4r\left(\sum_{i=1}^{4}s_i\right)\frac{2}{\sum_{i=1}^{4}s_i} = 8r
\end{aligned} \tag{2.46}$$

And so

$$\begin{aligned}
\sum_{i=1}^{4}\frac{r_i^2}{\sum_{k=1}^{4}s_k - s_i} &\geq \frac{\left(\sum_{i=1}^{4}r_i\right)^2}{\sum_{i=1}^{4}\left(\sum_{k=1}^{4}s_k - s_i\right)} = \frac{\left(\sum_{i=1}^{4}r_i\right)^2}{3\sum_{i=1}^{4}s_i} \\
&\geq \frac{1}{3\sum_{i=1}^{4}s_i} \times (8r)^2 \geq \frac{64r^2}{3} \times \frac{3}{8\sqrt{3}R^2} \\
&= \frac{8}{9\sqrt{3}}\left(\frac{R}{3r}\right)^{-2}
\end{aligned} \tag{2.47}$$

$$\frac{9\sqrt{3}}{8}\sum_{i=1}^{4}\frac{r_i^2}{\sum_{k=1}^{4}s_k - s_i} \geq \frac{9\sqrt{3}}{8} \times \frac{8}{9\sqrt{3}}\left(\frac{R}{3r}\right)^{-2} = \left(\frac{R}{3r}\right)^{-2} \tag{2.48}$$

□

Like Corollary 1, Corollary 2 and Corollary 3, we have

Corollary 4. *In the triangle $A_1A_2A_3$, we get*

$$\left(\frac{R}{2r}\right)^{-2} \leq \frac{4\sqrt{3}}{9}\sum_{i=1}^{3}\frac{r_i}{\sum_{k=1}^{3}a_k - a_i} \leq \frac{1}{9}\sqrt{192\left(\frac{R}{2r}\right)^2 - 111}$$

In fact that

$$\begin{aligned}\left(\frac{R}{2r}\right)^{-2} \leq \frac{1}{9}\sqrt{192\left(\frac{R}{2r}\right)^2 - 111} &\Leftrightarrow 1 \leq \frac{1}{9}\left(\frac{R}{2r}\right)^2\sqrt{192\left(\frac{R}{2r}\right)^2 - 111} \\ &\Leftrightarrow 81 \leq (\frac{R}{2r})^4\left[192\left(\frac{R}{2r}\right)^2 - 111\right] \\ &\Leftrightarrow 192\left[\left(\frac{R}{2r}\right)^2\right]^3 - 111\left[\left(\frac{R}{2r}\right)^2\right]^2 - 81 \geq 0 \qquad (2.49) \\ &\Leftrightarrow \left[\left(\frac{R}{2r}\right)^2 - 1\right]\left[192\left(\frac{R}{2r}\right)^4 + 81\left(\frac{R}{2r}\right)^2 + 81\right] \geq 0 \\ &\Leftrightarrow \left(\frac{R}{2r}\right)^2 \geq 1 \Leftrightarrow R \geq 2r\end{aligned}$$

This is Euler's inequality. It shows that Euler's inequality $R \geq 2r$ is separated by $\frac{4\sqrt{3}}{9}\sum_{i=1}^{3}\frac{r_i}{\sum_{k=1}^{3}a_k - a_i}$.

Comparing Theorem 1, Theorem 2, Theorem 3 and Theorem 4, we have the following:
Conjecture
There exists a function f that satisfies

$$\frac{9\sqrt{3}}{8}\sum_{i=1}^{4}\frac{r_i^2}{\sum_{k=1}^{4}s_k - s_i} \leq f\left(\frac{R}{3r}\right)$$

Acknowledgments. The author thanks Dr. Yu Shi (Chinese Academy of Sciences) for the help in editing the article with LaTeX.

References

[1] Kuang Jichang. *Applied Inequalities*. Jinan: Shandong science and technology press. 2004, 1, pp. 251-253,China.
[2] Zhang Yun. *A Simple Proof of Euler's Inequality in Space*. Math Spectrum, 2003/2004, Volume 36, Number 3, pp.51.
[3] Zhang Yun. *An Inequality Connected with a Special Point of a Triangle*. Math Spectrum, 2007/2008, Volume 40, Number 1, pp.3-8.
[4] Zhang Yun. *Problems 11783*. The American Mathematical Monthly, 2014, Volume 121, Issue 6.
[5] Zhang Yun. *Problems 11972*. The American Mathematical Monthly, 2017, Volume 124, Issue 4.

Bo-Ai branch school of Xi'An senior high school
Caotan 6th road, Weiyang district, Xi'An, Shaan xi, China, 710000 .
Email address: yunzhangmath@126.com