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ON THE GEOMETRIC MEAN THEOREM

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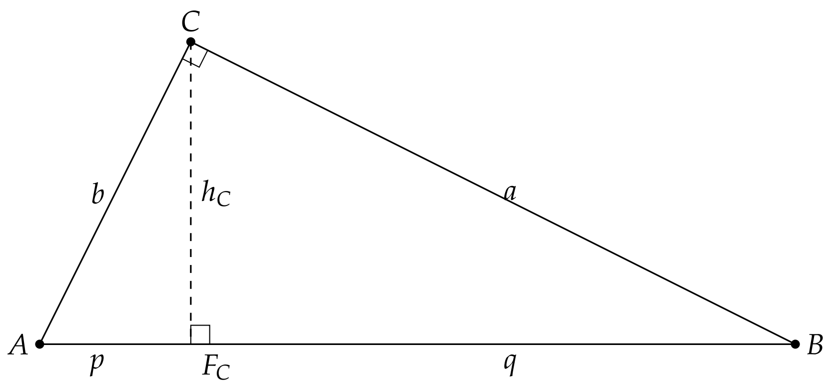
ABSTRACT. We consider a modified Pythagorean identity which yields an analogue of the geometric mean theorem for obtuse triangles. Several consequences of this identity are also examined, including: the associated orthic triangle being isosceles, the nine-point center lying on a side, and the Kosnita point coinciding with a vertex.

1. INTRODUCTION

Given a right triangle with an altitude h_C as shown below:



the geometric mean theorem is the statement that the length of the altitude from the $90°$ vertex C is the geometric mean of the two segments AF_C and BF_C it creates on the hypotenuse, namely

$$h_C = \sqrt{pq} = \sqrt{AF_C \times BF_C} \tag{1.1}$$

It was proved in [1] that equation (1.1) is equivalent to the Pythagorean identity:

$$c^2 = a^2 + b^2 \tag{1.2}$$

However, if we view the segments p and q above as the distances from A and B to the foot of the altitude from vertex C, then the equivalence of (1.1) and (1.2) holds because the altitude h_C is *internal*. In the case of obtuse triangles where two altitudes are *external*, for example, as shown below:

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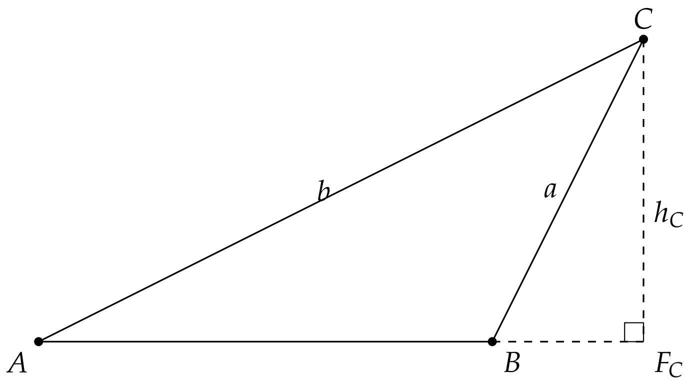


we prove an analogous geometric mean theorem, namely

$$h_C = \sqrt{AF_C \times BF_C} \tag{1.3}$$

which is equivalent to a modified Pythagorean identity:

$$(a^2 - b^2)^2 = (ac)^2 + (cb)^2 \tag{1.4}$$

2. The altitude version adapted for obtuse triangles

Notation. *For a triangle ABC, we will denote the side-lengths by $a = BC$, $b = CA$, and $c = AB$. The interior angles of the triangle will be denoted by $\angle A = \angle CAB$, $\angle B = \angle ABC$, and $\angle C = \angle BCA$. For simplicity we will sometimes drop the angle symbol and just write A for $\angle A$.*

Definition 2.1. *Let ABC be a triangle.*

a) *ABC is a non-right triangle if none of the three angles is $90°$.*
b) *ABC is an obtuse triangle if it contains an angle greater than $90°$.*
c) *ABC is an isosceles triangle if two of its sides have equal lengths.*
d) *ABC is an equilateral triangle if all its three sides have equal lengths.*

Proposition 2.1. *Let ABC be an obtuse triangle. Then the following statements are equivalent:*

(1) $\cos^2 A + \cos^2 B = 1$
(2) $(a^2 - b^2)^2 = (ac)^2 + (cb)^2$

Proof. First suppose that $\cos^2 A + \cos^2 B = 1$. The cosine formula gives

$$\left(\frac{b^2+c^2-a^2}{2bc}\right)^2 + \left(\frac{a^2+c^2-b^2}{2ac}\right)^2 = 1$$

$$(a^2+b^2)c^4 - 2(a^4+b^4)c^2 + (a^2+b^2)(a^2-b^2) = 0$$

$$\frac{2(a^4+b^4) \pm \sqrt{4(a^4+b^4)^2 - 4(a^2+b^2)^2(a^2-b^2)^2}}{2(a^2+b^2)} = c^2$$

$$\frac{2(a^4+b^4) \pm 4a^2b^2}{2(a^2+b^2)} = c^2$$

$$\therefore a^2 + b^2 = c^2$$

$$\text{or } \frac{(a^2-b^2)^2}{a^2+b^2} = c^2$$

Since $\triangle ABC$ is obtuse, we take the second possibility $\frac{(a^2-b^2)^2}{a^2+b^2} = c^2$, which is just equation (1.4) re-arranged. Conversely, suppose that equation (1.4) holds. Since we can't have $a = b$, suppose that $b > a$. An explicit computation of $\cos A$ and $\cos B$ gives:

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{b^2-a^2+\left(\frac{(a^2-b^2)^2}{a^2+b^2}\right)}{2b\left(\frac{b^2-a^2}{\sqrt{a^2+b^2}}\right)} = \frac{b}{\sqrt{a^2+b^2}}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{a^2-b^2+\left(\frac{(a^2-b^2)^2}{a^2+b^2}\right)}{2a\left(\frac{b^2-a^2}{\sqrt{a^2+b^2}}\right)} = -\frac{a}{\sqrt{a^2+b^2}}$$

from which it follows that $\cos^2 A + \cos^2 B = 1$. □

Corollary 2.1. *If equation (1.4) holds in a non-right $\triangle ABC$, then so is the equation $\sin^2 A + \sin^2 B = 1$. Thus, it is not only right triangles that satisfy the trigonometric identities $\cos^2 A + \cos^2 B = 1$ and $\sin^2 A + \sin^2 B = 1$.*

Corollary 2.2. *If a non-right $\triangle ABC$ satisfies equation (1.4), then $A - B = \pm 90°$, so the triangle is necessarily obtuse.*

Indeed:

$$\cos^2 A + \cos^2 B = \sin^2 A + \sin^2 B \implies \cos 2A = -\cos 2B \implies 2A - 2B = \pm 180°,$$

and so $A - B = \pm 90°$.

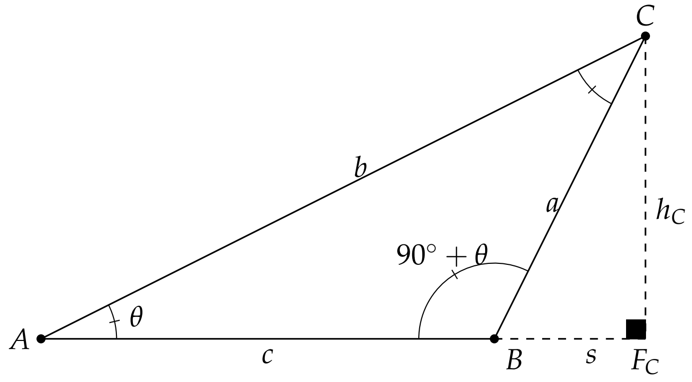
Proposition 2.2. *Let ABC be a non-right triangle. Then the following statements are equivalent:*

(1) $(a^2-b^2)^2 = (ac)^2 + (cb)^2$

(2) $h_C = \sqrt{AF_C \times BF_C}$, *where F_C is the foot of the altitude from vertex C.*

Proof. The second condition is the altitude version of the geometric mean theorem, adapted for obtuse triangles.

First suppose that $(a^2-b^2)^2 = (ac)^2 + (cb)^2$. As a consequence of the preceding corollary, we can denote the interior angles of $\triangle ABC$ as $\angle A = \theta$, $\angle B = 90° + \theta$, $\angle C = 90° - 2\theta$.



In $\triangle ACF_C$, $\angle ACF_C = 90^\circ - \theta$. In $\triangle BCF_C$, $\angle BCF_C = \theta$, $\angle CBF_C = 90^\circ - \theta$. Thus, the two triangles are similar.

$$\begin{aligned} \implies \frac{c+s}{h_C} &= \frac{h_C}{s} \\ h_C^2 &= (c+s) \times s \\ h_C^2 &= AF_C \times BF_C \end{aligned}$$

Conversely, assume that in the diagram above one has $h_C = \sqrt{AF_C \times BF_C}$. From $\triangle BCF_C$ we have $h_C^2 = a^2 - s^2 = b^2 - (c+s)^2$, giving

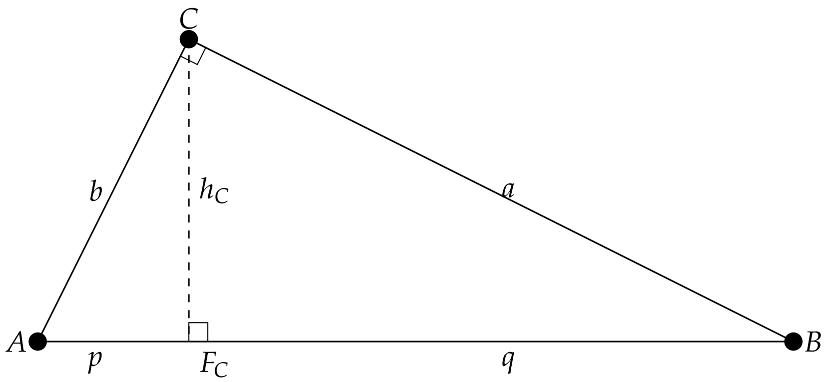
$$s = \frac{b^2 - a^2 - c^2}{2c} \implies c + s = \frac{b^2 - a^2 + c^2}{2c}$$

$$\begin{aligned} h_C^2 &= AF_C \times BF_C \\ \therefore a^2 - \left(\frac{b^2 - a^2 - c^2}{2c}\right)^2 &= \left(\frac{b^2 - a^2 + c^2}{2c}\right)\left(\frac{b^2 - a^2 - c^2}{2c}\right) \\ 4a^2c^2 - \left[(b^2 - a^2)^2 + c^4 - 2c^2(b^2 - a^2)\right] &= (b^2 - a^2)^2 - c^4 \\ \therefore c^2(a^2 + b^2) &= (b^2 - a^2)^2 \end{aligned}$$

□

3. The "leg version" adapted for obtuse triangles

Let R be the radius of the circumscribed circle of ABC. If $\angle C = 90^\circ$, as per the diagram below



then each leg is the geometric mean of the hypotenuse and the part of the leg directly below the hypotenuse:

$$a^2 = qc, \quad b^2 = pc \tag{3.1}$$

Since $c = 2R$ for a right triangle, the relations in (3.1) may be re-written as

$$a^2 = (BF_C)(2R), \quad b^2 = (AF_C)(2R) \tag{3.2}$$

We will now derive analogues of the relations in (3.2) for triangles that satisfy equation (1.4).

Proposition 3.1. *Let R be the radius of the circumscribed circle of a non-right triangle ABC. Then the following two statements are equivalent:*

(1) $a^2 + b^2 = 4R^2$
(2) $(a^2 - b^2)^2 = (ac)^2 + (cb)^2$

Proof. First suppose that $(a^2 - b^2)^2 = (ac)^2 + (cb)^2$. Earlier in the course of proving Proposition (2.1), we obtained $\cos A = \frac{b}{\sqrt{a^2+b^2}}$, when $b > a$. From this we get $\sin A = \frac{a}{\sqrt{a^2+b^2}}$. By the extended law of sines (see [3]), $R = \frac{a}{2\sin A}$. Substituting:

$$R = \frac{a}{2\left(\frac{a}{\sqrt{a^2+b^2}}\right)} = \frac{\sqrt{a^2+b^2}}{2} \implies 4R^2 = a^2 + b^2$$

Now suppose that $a^2 + b^2 = 4R^2$. Again we use the extended law of sines, this time writing $R = \frac{c}{2\sin C}$. Then:

$$\begin{aligned}
a^2 + b^2 &= 4R^2 \\
a^2 + b^2 &= 4\left(\frac{c}{2\sin C}\right)^2 \\
(a^2 + b^2)\sin^2 C &= c^2 \\
(a^2 + b^2)(1 - \cos^2 C) &= (a^2 + b^2 - 2ab\cos C) \\
\cos C\Big((a^2 + b^2)\cos C - 2ab\Big) &= 0 \\
\cos C &= \frac{2ab}{a^2 + b^2} \text{ or } 0 \\
\frac{a^2 + b^2 - c^2}{2ab} &= \frac{2ab}{a^2 + b^2} \\
(a^2 + b^2)^2 - 4a^2b^2 &= c^2(a^2 + b^2) \\
(a^2 + b^2)^2 - c^2(a^2 + b^2) &= 4a^2b^2 \\
\therefore (a^2 - b^2)^2 &= (ac)^2 + (cb)^2
\end{aligned}$$

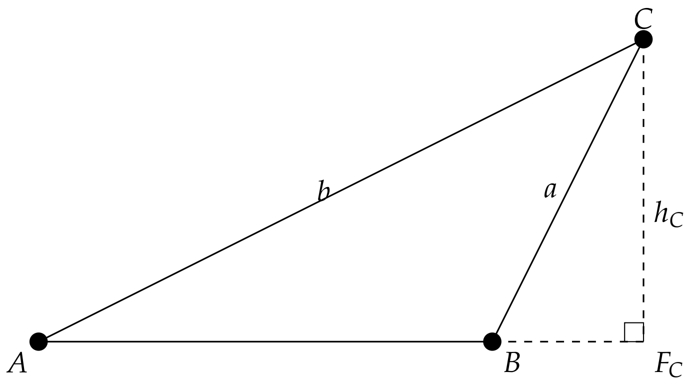
Note that we didn't consider $\cos C = 0$ because we assumed a *non-right* triangle at the beginning. From now on, we will freely utilize the equivalence just established. □

Corollary 3.1. *Let H be the orthocenter of $\triangle ABC$. If equation (1.4) holds, then $AH = b$ and $BH = a$.*

These follow because in any triangle we always have $AH^2 = 4R^2 - a^2$ and $BH^2 = 4R^2 - b^2$.

Proposition 3.2. *Let ABC be a triangle which satisfies equation (1.4). Then we have the following analogues of the "leg version" of the geometric mean theorem:*

(1) $a^2 = (BF_C)(2R)$
(2) $b^2 = (AF_C)(2R)$



Proof. We will use the fact that $a^2 + b^2 = 4R^2$ from Proposition (3.1). Apply Pythagorean theorem to $\triangle BCF_C$:

$$BF_C^2 = a^2 - CF_C^2 = a^2 - \left(\frac{ab}{2R}\right)^2 = \left(\frac{a}{2R}\right)^2 (4R^2 - b^2) = \left(\frac{a}{2R}\right)^2 (a^2) = \left(\frac{a^2}{2R}\right)^2$$

Hence $a^2 = (BF_C)(2R)$. Similarly, we get $b^2 = (AF_C)(2R)$. □

4. Extra consequences

There are several consequences of condition (1.4), as well as several other statements equivalent to it. In this section, we consider a few.

4.1. **Orthic triangle is isosceles.**

Proposition 4.1. *Let ABC be a non-right triangle. Then the following statements are equivalent:*

(1) $(a^2 - b^2)^2 = (ac)^2 + (cb)^2$
(2) $a \cos A + b \cos B = 0$.

Accordingly, if a triangle satisfies equation (1.4), then the associated orthic triangle is automatically isosceles.

Proof. First suppose that $(a^2 - b^2)^2 = (ac)^2 + (cb)^2$ and let $b > a$. From Proposition (2.1) we calculated $\cos A = \frac{b}{\sqrt{a^2+b^2}}$ and $\cos B = -\frac{a}{\sqrt{a^2+b^2}}$. Thus, $a \cos A + b \cos B = 0$. For the converse we suppose $a \cos A + b \cos B = 0$ and then use the cosine law:

$$\begin{aligned} a\left(\frac{b^2 + c^2 - a^2}{2bc}\right) + b\left(\frac{a^2 + c^2 - b^2}{2ac}\right) &= 0 \\ a^2(b^2 + c^2 - a^2) + b^2(a^2 + c^2 - b^2) &= 0 \\ (a^2 + b^2)c^2 + 2a^2b^2 - a^4 - b^4 &= 0 \\ \therefore (a^2 + b^2)c^2 &= (a^2 - b^2)^2 \end{aligned}$$

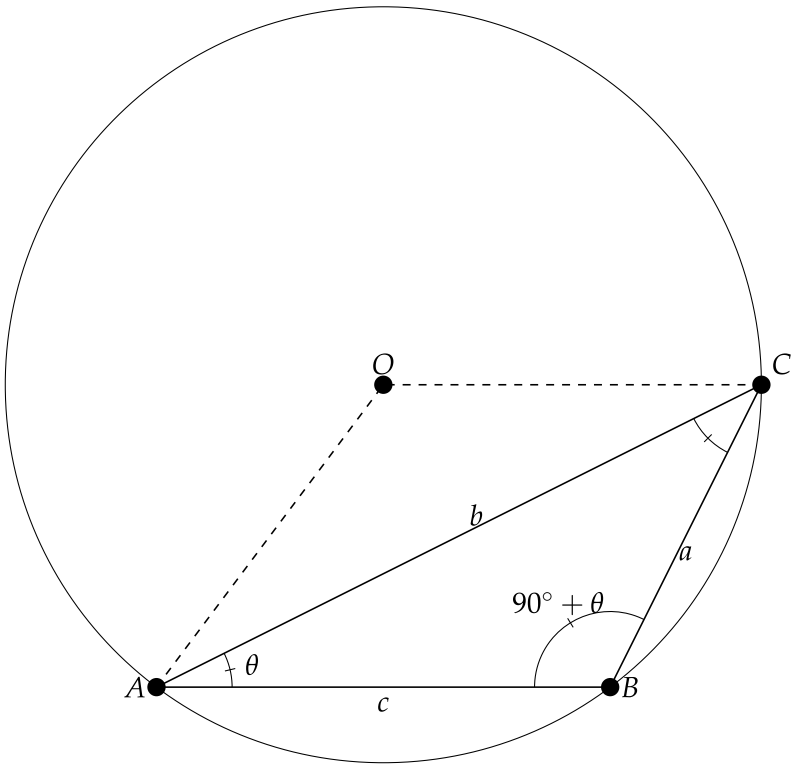
The orthic triangle is isosceles because its side-lengths, relative to the measures of the parent triangle, are given by $|a \cos A|$, $|b \cos B|$, $|c \cos C|$. □

Remark 4.1. *The analogue of the above equivalence for a right triangle with hypotenuse of length c would be $c^2 = a^2 + b^2 \iff a \cos A - b \cos B = 0$. As such, if we allow for degenerate orthic triangles, then the geometric mean theorem is equivalent to the orthic triangle being isosceles, both in the case of right triangles, and in the case of triangles satisfying equation (1.4).*

4.2. **Radius is parallel to a side.**

Proposition 4.2. *Suppose that $\triangle ABC$ satisfies equation (1.4). Then the radius of the circumscribed circle of ABC through vertex C is parallel to side AB.*

Proof. By Corollary (2.2), we have $A - B = \pm 90^\circ$. To be specific, let $B - A = 90^\circ$ and set $A = \theta$. Then $B = 90^\circ + \theta$ and $C = 90^\circ - 2\theta$. Consider the circumcircle shown below:



The angle which the *major arc AC* subtends at the center of the circle is *twice* the angle it subtends at the circumference, and so:

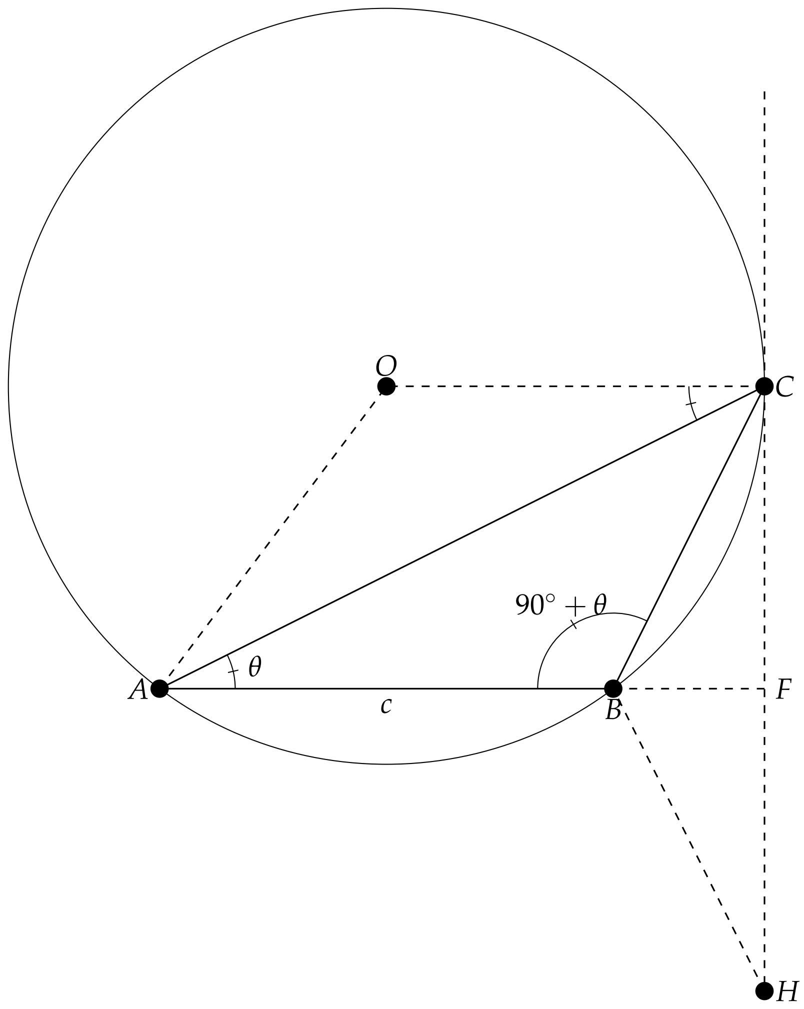
$$\text{reflex } \angle AOC = 2(90^\circ + \theta) \implies \text{obtuse } \angle AOC = 180^\circ - 2\theta$$

Since $\triangle AOC$ is isosceles, we have that $\angle OAC = \angle OCA = \theta$. This shows that radius OC is parallel to side AB. ☐

4.3. **Segment from orthocenter is tangent to circumscribed circle.**

Proposition 4.3. *Suppose that the side-lengths of $\triangle ABC$ satisfy equation (1.4). Then the segment from the orthocenter is tangent to the circumscribed circle of ABC at C.*

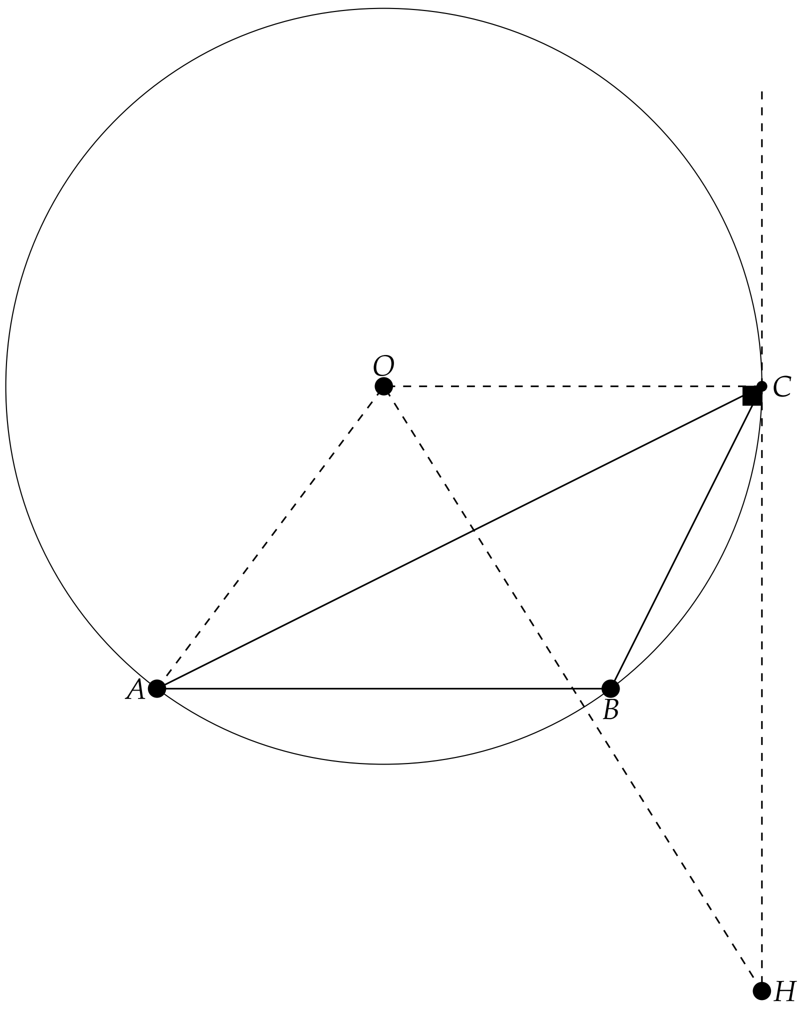
Proof. As usual, we can set the interior angles as $A = \theta$, $B = 90^\circ + \theta$, and $C = 90^\circ - 2\theta$. Let O be the circumcenter, and consider the circumscribed circle of ABC:



Since $\triangle ABC$ is *obtuse*, its orthocenter H is situated outside the triangle as shown above. Join OC and CH. Extend side AB to meet CH at F. Since F now becomes the foot of the altitude from C, we have that $\angle BFH = 90°$. Since OC is parallel to AB, it follows that $\angle OCH = 90°$. Radius is perpendicular to tangent at the point of contact: HC is a tangent to the circumscribed circle at C. □

Proposition 4.4. *Let H be the orthocenter of $\triangle ABC$, and let R be the radius of the circumscribed circle of ABC. If segment CH is tangent to the circumscribed circle at C, then $a^2 + b^2 = 4R^2$.*

Proof. Apply the Pythagorean theorem to the right triangle OCH below:



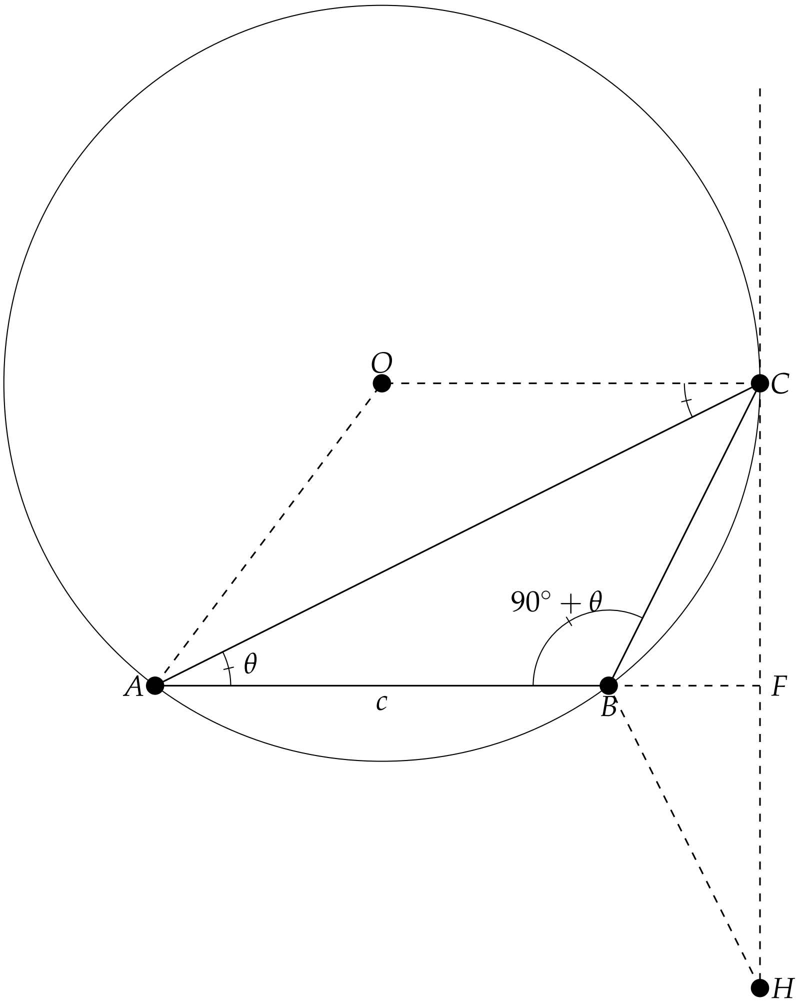
$$OH^2 = OC^2 + CH^2$$
$$9R^2 - a^2 - b^2 - c^2 = R^2 + (4R^2 - c^2)$$
$$\therefore a^2 + b^2 = 4R^2$$

□

4.4. **Orthocenter is a reflection of one vertex over the opposite side.**

Proposition 4.5. *Let O and H be the circumcenter and orthocenter of $\triangle ABC$, and let R be the radius of the circumscribed circle of ABC. If the side-lengths satisfy equation (1.4), then H is a reflection of vertex C over side AB.*

Proof. Suppose that the side-lengths satisfy equation (1.4). We had $a^2 + b^2 = 4R^2$. This in turn implies $BH = a$, as per Corollary (3.1). So $\triangle BCH$ below is isosceles with $BH = BC$:



Extend side AB to meet CH at F. Since F now becomes the foot of the altitude from C, we have that $\angle BFH = 90°$. Altitude BF bisects the base, so $CF = FH$. This proves that H is a reflection of C over side AB. □

Corollary 4.1. *If the side-lengths of $\triangle ABC$ satisfy equation (1.4), then its orthocenter H is a vertex of its reflection triangle. As such, the two triangles are perspective at a vertex of the reflection triangle.*

Proof. The *reflection triangle* (see [4]) is obtained by reflecting each vertex in the opposite side. By Proposition (4.5), the reflection of vertex C over side AB is the orthocenter. Since the orthocenter is always the perspector of the parent triangle and its reflection triangle, it follows in this case that the two triangles are perspective at a vertex of the reflection triangle. □

4.5. **Nine-point center lies on a side.**

Proposition 4.6. *Suppose that the side-lengths of $\triangle ABC$ satisfy equation (1.4). Then the nine-point center lies on side AB.*

Proof. We use the *segment addition postulate* . Note that in any triangle ABC with orthocenter H, circumcenter O, and nine-point center N, we have:

$$AN^2 = \frac{2AH^2 + 2AO^2 - OH^2}{4}$$

We proved that $AH = b$ when $(a^2 - b^2)^2 = (ac)^2 + (cb)^2$. So

$$AN^2 = \frac{2AH^2 + 2AO^2 - OH^2}{4} = \frac{2b^2 + 2R^2 - (9R^2 - a^2 - b^2 - c^2)}{4}$$

After some simplifications, we obtain

$$AN^2 = \frac{(a^2 - 3b^2)^2}{16(a^2 + b^2)} \implies AN = \frac{|(a^2 - 3b^2)|}{4\sqrt{(a^2 + b^2)}}$$

Similarly:

$$BN = \frac{|(b^2 - 3a^2)|}{4\sqrt{(a^2 + b^2)}}$$

Let's examine the absolute values. There are four cases to consider.

First, we can't have $a^2 - 3b^2 > 0$ and $b^2 - 3a^2 > 0$ simultaneously. Otherwise, their sum must be greater than zero as well; but their sum is $-2(a^2 + b^2) < 0$.

Next, suppose that $3b^2 - a^2 > 0$ and $3a^2 - b^2 > 0$. Then the sum is $2(a^2 + b)^2$, and so:

$$AN = \frac{3b^2 - a^2}{4\sqrt{(a^2 + b^2)}}, \ BN = \frac{3a^2 - b^2}{4\sqrt{(a^2 + b^2)}} \implies AN + BN = \frac{\sqrt{a^2 + b^2}}{2}$$

Because $a^2 + b^2 = 4R^2$, this leads to $AN + BN = R$. This is a special case. If the points A, B, N aren't co-linear, then in $\triangle ABN$, the median through N passes through the nine-point circle, and so the length of this median is a radius of the nine-point circle, namely $\frac{R}{2}$. We now have a triangle ABN in which the sum of two sides is R and a median has length $\frac{R}{2}$. This is impossible. Indeed, the side-lengths of $\triangle ABN$ have to be of the form $\frac{R-c}{2}, \frac{R+c}{2}, c$ for sides AN, BN, AB (or sides BN, AN, AB). If we compute the cosine of the angle at N, we obtain

$$\cos N = \frac{\left(\frac{R-c}{2}\right)^2 + \left(\frac{R+c}{2}\right)^2 - c^2}{2 \times \frac{R-c}{2} \times \frac{R+c}{2}} = 1 \implies \angle N = 0$$

The third and fourth cases are the same. For example, $3b^2 - a^2 > 0$ and $b^2 - 3a^2 > 0$. Then take

$$AN = \frac{3b^2 - a^2}{4\sqrt{(a^2 + b^2)}}, \ BN = \frac{b^2 - 3a^2}{4\sqrt{(a^2 + b^2)}}$$

and obtain

$$AN + BN = \frac{4(b^2 - a^2)}{4\sqrt{(a^2 + b^2)}} = c = AB.$$

□

A special case of Proposition (4.6) is the following.

4.6. **Nine-point center coincides with vertex.**

Proposition 4.7. *Let $\triangle ABC$ have side-lengths a, b, c, radius of circumscribed circle R, and nine-point center N. Then the following statements are equivalent:*

(1) $N = B$, the nine-point center is precisely vertex B
(2) $\angle A = \angle C = 30^\circ$, $\angle B = 120^\circ$
(3) $a = c = R$
(4) $a : b : c = 1 : \sqrt{3} : 1$

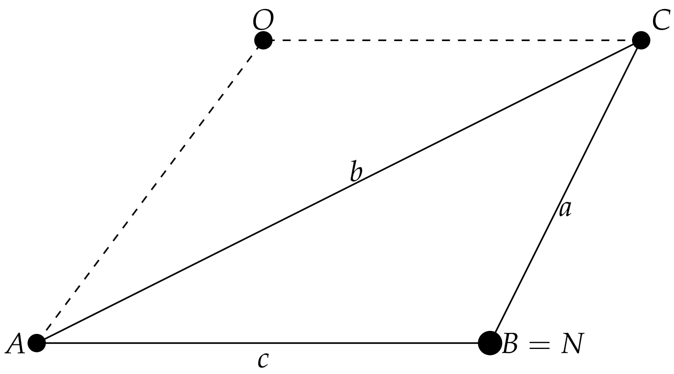
Under the above equivalence, the resulting orthic triangle is equilateral.

Proof. Observe, from (4), that $a = c$ and $b = \sqrt{3}a$. Thus,

$$(b^2 - a^2)^2 = (3a^2 - a^2)^2 = 4a^4 = a^2(a^2 + 3a^2) = c^2(a^2 + b^2),$$

and so if the equivalence holds, then the side-lengths of $\triangle ABC$ is a special case of condition (1.4).

For the proof, we show that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (1). So first suppose that the nine-point center coincides with vertex B. Consider the diagram below:



Let R be the radius of the circumscribed circle of $\triangle ABC$. Since the nine-point circle goes through the midpoint of AB and has radius equal to *half* the radius of the circumscribed circle of the parent triangle, we have that

$$\frac{R}{2} = \frac{c}{2} \implies R = c$$

But then $R = \frac{c}{2\sin C}$, by the extended law of sines. So

$$c = \frac{c}{2\sin C} \implies \sin C = \frac{1}{2} \implies \angle C = 30^\circ, 150^\circ$$

Similarly, the nine-point circle passes through the midpoint of BC, so the radius from B to this midpoint is $\frac{a}{2}$:

$$\frac{R}{2} = \frac{a}{2} \implies R = a \implies \frac{a}{2\sin A} = a \implies \angle A = 30^\circ, 150^\circ$$

The only permissible choice of $\angle A$ and $\angle C$ is $\angle A = \angle C = 30^\circ$. Then $\angle B = 120^\circ$.

For (2) $\implies$ (3), suppose that $\angle A = \angle C = 30°$. Using the extended law of sines again

$$R = \frac{a}{2\sin A} = \frac{a}{2\sin 30°} = a \qquad R = \frac{c}{2\sin C} = \frac{c}{2\sin 30°} = c$$

Thus, $a = c = R$.

For (3) $\implies$ (4) we suppose that $a = c = R$. The extended law of sines gives $\angle A = \angle C = 30°$. Then $\angle B$ has to be $120°$. By the cosine law we get

$$b^2 = a^2 + c^2 - 2ac\cos B = a^2 + a^2 - 2a^2\left(-\frac{1}{2}\right) = 3a^2 \implies b = \sqrt{3}a$$

from which $a : b : c = 1 : \sqrt{3} : 1$.

Finally, we show that (4) $\implies$ (1). Let O and H be the circumcenter and orthocenter, respectively. Since the nine-point center N is the midpoint of the Euler segment OH we have that BN is a median in triangle BOH; thus:

$$BN^2 = \frac{2BO^2 + 2BH^2 - OH^2}{4}$$

Now BO is a radius in the circumscribed circle of $\triangle ABC$, so $BO = R$, and the length of the Euler segment OH satisfies $OH^2 = 9R^2 - a^2 - b^2 - c^2$. The fact that $a : b : c = 1 : \sqrt{3} : 1$ again gives $a = c = R$ and $b = \sqrt{3}a$. Also $BH = \sqrt{4R^2 - b^2} = \sqrt{4a^2 - 3a^2} = a$. Then

$$BN^2 = \frac{2a^2 + 2a^2 - (9a^2 - a^2 - 3a^2 - a^2)}{4} = 0$$

and so B coincides with the nine-point center N.

Usually the orthic triangle has lengths $a\cos A$, $b\cos B$, and $c\cos C$. Under the above equivalence, $\angle A = \angle C = 30°$, and $\angle B = 120°$, and so we get, for the side-lengths of the resulting orthic triangle:

$$a\cos A = \frac{\sqrt{3}}{2}a,\ |b\cos B| = \left|-\frac{1}{2}b\right| = \frac{1}{2}\sqrt{3}a,\ \ c\cos C = \frac{\sqrt{3}}{2}c = \frac{\sqrt{3}}{2}a$$

The orthic triangle is equilateral as expected. □

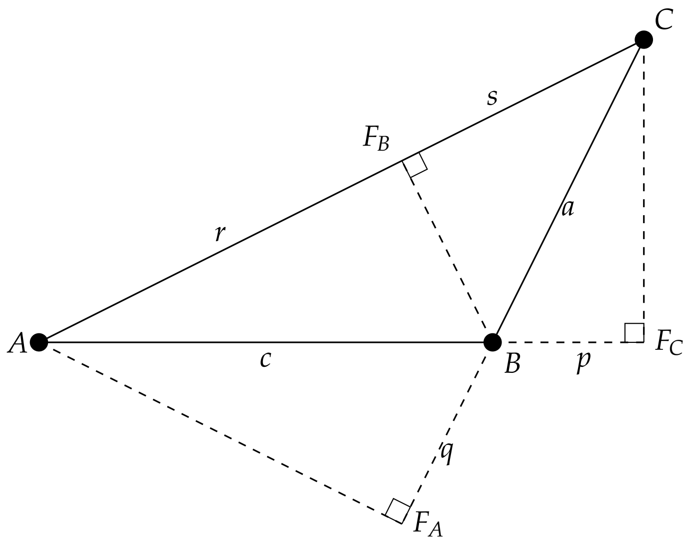
4.7. **Kosnita point coincides with a vertex.**

Proposition 4.8. *Let the side-lengths of $\triangle ABC$ satisfy equation (1.4). Then the Kosnita point of $\triangle ABC$ is precisely C.*

Proof. By Proposition (4.6), the nine-point center lies on side AB. The Kosnita point is the isogonal conjugate of the nine-point center (see [4]). Since the isogonal conjugate of a point on a side AB is the opposite vertex, it follows that the Kosnita point is precisely vertex C. □

4.8. **Equal and unequal products.**

Proposition 4.9. *Let ABC be any triangle with an obtuse angle B, like the one shown below*



Then:

(1) *the product of side AB and its extension to the foot of the altitude from C is equal to the product of side CB and its extension to the foot of altitude from A, namely $cp = aq$*

(2) *we have $q(q+a) = rs$, $rs > aq$, $rs > cp$, if equation (1.4) is satisfied.*

Proof. (1) Let R be the radius of the circumscribed circle of $\triangle ABC$. Note that CF_C is the length of the altitude from vertex C, and so $CF_C = \frac{ab}{2R}$; similarly, $AF_A = \frac{bc}{2R}$ (see [2]). From $\triangle BCF_C$, we have

$$p = \sqrt{a^2 - CF_C^2} = \sqrt{a^2 - \left(\frac{ab}{2R}\right)^2} = \frac{a}{2R}\sqrt{4R^2 - b^2} \implies cp = \frac{ac}{2R}\sqrt{4R^2 - b^2}.$$

From $\triangle ABF_A$ we have

$$q = \sqrt{c^2 - AF_A^2} = \sqrt{c^2 - \left(\frac{bc}{2R}\right)^2} = \frac{c}{2R}\sqrt{4R^2 - b^2} \implies aq = \frac{ac}{2R}\sqrt{4R^2 - b^2}.$$

This shows that $aq = cp$.

(2) Consider $\triangle ABF_B$. Since BF_B is the length of the altitude from vertex B, we have $BF_B = \frac{ac}{2R}$. Thus:

$$r^2 = c^2 - \left(\frac{ac}{2R}\right)^2 = \left(\frac{c}{2R}\right)^2 (4R^2 - a^2) = \left(\frac{c}{2R}\right)^2 b^2 \implies r = \frac{bc}{2R} = AF_A$$

Also:

$$s^2 = a^2 - BF_B^2 = a^2 - \left(\frac{ac}{2R}\right)^2 = \left(\frac{a}{2R}\right)^2 (4R^2 - c^2) = \left(\frac{a}{2R}\right)^2 \left(a^2 + b^2 - \frac{(a^2 - b^2)^2}{a^2 + b^2}\right),$$

giving $s = \frac{a}{R}(CF_C)$. In turn we get $rs = \frac{a}{R}(AF_A \times CF_C)$. Next, we simplify the previous expression for q when $a^2 + b^2 = 4R^2$:

$$q = \frac{c}{2R}\sqrt{4R^2 - b^2} = \frac{c}{2R}\sqrt{a^2} = \frac{ac}{2R} = BF_B.$$

As an aside, we get that the quadrilateral AF_ABF_B is a *kite*. Now consider $q^2 + aq$:

$$\begin{aligned} q^2 + aq &= \frac{a^2c^2}{4R^2} + \frac{a^2c}{2R} \\ &= \frac{a^2}{2R}\left(\frac{c^2}{2R} + c\right) \\ &= \frac{a^2}{2R}\left(\frac{(a^2-b^2)^2}{a^2+b^2} \times \frac{1}{\sqrt{a^2+b^2}} + \frac{b^2-a^2}{\sqrt{a^2+b^2}}\right) \\ &= \frac{a^2}{2R}\left(\frac{b^2-a^2}{\sqrt{a^2+b^2}}\right)\left(\frac{b^2-a^2+a^2+b^2}{a^2+b^2}\right) \\ &= \frac{a^2}{2R}\left(\frac{b^2-a^2}{2R}\right)\frac{2b^2}{4R^2} \\ &= \frac{a^2}{2R}(c)\left(\frac{b^2}{2R^2}\right) \\ &= \frac{a}{R}\left(\frac{ab}{2R}\right)\left(\frac{bc}{2R}\right) \\ &= \frac{a}{R}\left(CF_C \times AF_A\right) \end{aligned}$$

and so it follows that $q^2 + aq = rs$. If we now re-write as $q^2 = rs - aq$, then $q^2 > 0 \implies rs > aq$ and $aq = cp$ then gives $rs > cp$.

□

5. Summary

Let a, b, c be the side-lengths of a *non-right* triangle ABC. Let R be the radius of its circumscribed circle, and F_A, F_B, F_C the feet of the altitudes from A B, C. Then the following statements are equivalent:

(1) $(a^2 - b^2)^2 = (ac)^2 + (cb)^2$
(2) $a \cos A + b \cos B = 0$
(3) $\cos^2 A + \cos^2 B = 1$
(4) $\sin^2 A + \sin^2 B = 1$
(5) $CF_C^2 = (AF_C)(BF_C)$
(6) $a^2 = (BF_C)(2R)$
(7) $b^2 = (AF_C)(2R)$
(8) $a^2 + b^2 = 4R^2$
(9) $A - B = \pm 90°$.

Statement (5) is the altitude version of the geometric mean theorem, while statements (6) and (7) are the leg versions, all adapted for obtuse triangles.

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