Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.11, (2022), Issue 1, pp.51-73

ON EXTERIOR HOFSTADTER ELEMENTS

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ABSTRACT. The Hofstadter triangles with their elements (transversals, points, and sectrices) were investigated first by Hofstadter and then were studied and analyzed by many researchers. Based on these known elements, new ones closely related to Hofstadter's elements called "*twins*" have been defined, analyzed and studied very recently. All of them lie basically in the interior of a given triangle. In this article we analyze and study in an analogous way what we call "**exterior** *Hofstadter elements*" in contrast to the known ones which we call "**interior** *Hofstadter elements*". The "*exterior twins*" of the former elements will be studied in a forthcoming work soon. This is because their analysis and study are much more complicated.

1. INTRODUCTION

Let a scalene triangle ΔABC, with sides BC, CA, AB whose lengths are $|BC| = a$, $|CA| = b$, $|AB| = c$ and lie opposite the angles α, β, γ in radians, respectively. Without loss of generality we may assume that $\alpha < \gamma < \beta$. Let the two straight semi-lines of the ordered pair $(\mathcal{E}_{\mathrm{BC1}}, \mathcal{E}_{\mathrm{CB2}})$ have as origins (poles) the vertices B and C, pass through C and B, respectively, rotate about B counterclockwise and about C clockwise at constant rates proportional to β and γ, and for any $x \in [0,1]$ the angles swept by them are βx and γx, respectively. (*Note:* The two letters in the subscript denote, for $x = 0$, the side of the triangle, which lies on the (set) intersection of the members of the pair, the first letter denotes the corresponding origin (pole), and the numbers 1 and 2 denote counterclockwise and clockwise rotation.) Let the two straight semi-lines of the pair $(\mathcal{E}_{\mathrm{BC1}}, \mathcal{E}_{\mathrm{CB2}})$ intersect at the point A_x, a vertex of the (interior) Hofstadter triangle for $x \in [0,1]$. The straight line AA_x which intersects BC at the point $A_{x,in}$ is called (interior) $x-$transversal. Similarly, using a cyclic notation, consider two more pairs of straight semi-lines $(\mathcal{E}_{\mathrm{CA1}}, \mathcal{E}_{\mathrm{AC2}})$ and $(\mathcal{E}_{\mathrm{AB1}}, \mathcal{E}_{\mathrm{BA2}})$, their corresponding points of intersection B_x and C_x (the other two vertices of the (interior) Hofstadter triangle), and the two straight lines BB_x, CC_x, which intersect the sides of the triangle at $B_{x,in}$ and $C_{x,in}$, respectively. The straight line segments $AA_{x,in}$ $BB_{x,in}$, $CC_{x,in}$, are the three (interior) $x-$transversals, and the triangle $\Delta A_{x,in}B_{x,in}C_{x,in}$ will be called "*the first (interior) companion to Hofstadter triangle*". As x increases continuously in the interval $[0,1]$ the three $x-$transversals are concurrent at a point called "*interior Hofstadter point H_x*" (see, e.g., [4]). One more characteristic of the arcs drawn by the

2010 *Mathematics Subject Classification.* Primary 51-02; Secondary......
Key words and phrases. interior and exterior Hofstadter triangles, transversals, points, sectrices.

points A_x, B_x, C_x, as x increases from 0 to 1, is that they pass through some well-known points of the triangle, like the vertices of the "*first equilateral Morley triangle*" for $x = \frac{1}{3}$ [6], the incenter for $x = \frac{1}{2}$, etc. The locus of the "*(interior) Hofstadter point* H_x" is called "$x-$*sectrix*" a term borrowed from the "*Maclaurin sectrices*" [5]. The Hofstadter triangles and their elements (transversals, points and sectrices) may be called "*interior Hofstadter elements*".

For the interested reader we mention that in [2] we introduced the "*(interior) twin Hofstadter elements*" as follows: Let A'_x, B'_x, C'_x be projections of the vertices A, B, C onto the sides BC, CA, AB, respectively. Then, the aforementioned elements are the triangles $\Delta A'_x B'_x C'_x$, the transversals $AA'_{x,in}$, $BB'_{x,in}$, $CC'_{x,in}$, the points of concurrency of the three transversals H'_x, and the sectrix is the locus of H'_x as x increases continuously in the interval $[0,1]$. In analogy to a previous terminology the triangle $\Delta A'_{x,in} B'_{x,in} C'_{x,in}$ will be called "*the second (interior) companion to Hofstader triangle*". (Note that $H'_0 \equiv H_0$.)

2. Background material

To define what we shall call "*exterior Hofstadter elements*", consider that the three pairs of straight semi-lines rotate in the same way as before but this time they sweep the exterior angles of the triangle ΔABC. Suppose that the two straight semi-lines of the pair $(\mathcal{E}_{\mathrm{BC1}}, \mathcal{E}_{\mathrm{CB2}})$ start from the positions BA and CA and sweep the exterior angles of ΔABC, $\pi - \beta$ and $\pi - \gamma$, at constant rates different from the previous ones, so that for any $y \in [0,1]$, they have swept angles equal to $(\pi - \beta)y$ and $(\pi - \gamma)y$, respectively. The point of their intersection is denoted by A_y, $y \in [0,1]$, is a vertex of the "*exterior Hofstadter triangle*". The extended straight line AA_y intersects BC at $A_{y,ex}$ and straight line segment $AA_{y,ex}$ is the "*exterior $y-$transversal*". Consider the other two pairs of straight semi-lines and define cyclicly, the points B_y, C_y, their projections onto the sides CA and AB by B'_y and C'_y, as well as the intersections of BB_y and CC_y with the sides CA and AB by $B_{y,ex}$ and $C_{y,ex}$, respectively. The triangle $\Delta A_y B_y C_y$ is the "*exterior Hofstadter triangle*" and the straight line segments $AA_{y,ex}$, $BB_{y,ex}$, $CC_{y,ex}$ are the "*exterior Hofstadter $y-$transversals*". As will be proved in the sequel, the exterior Hofstadter $y-$transversals are concurrent at a point, the "*exterior Hofstadter point* H_y". As y increases continuously in the interval $[0,1]$, the locus of the exterior Hofstadter points H_y is called "*exterior $y-$sectrix*". It is worth pointing out that the term "*exterior*" comes from the fact that the triangles $\Delta A_y B_y C_y$ lie in the exterior of the triangle ΔABC while the classical Hofstadter triangles lie in the interior of ΔABC.

First, we observe that the sum of the angles the two semi-lines of the pair $(\mathcal{E}_{\mathrm{BC1}}, \mathcal{E}_{\mathrm{CB2}})$ formed with BC, for a certain $y \in [0,1]$, equals $(\beta + (\pi - \beta)y) + (\gamma + (\pi - \gamma)y) = \pi - \alpha + (\pi + \alpha)y$. Note that for $y = 0$ this angle is $\pi - \alpha$ while for $y = 1$ it is 2π. This simply means that there exists a $y \in (0,1)$ for which the two semi-lines are parallel. For this to happen there must be $(\beta + (\pi - \beta)y) + (\gamma + (\pi - \gamma)y) = \pi$; hence, $y = \frac{\alpha}{\pi+\alpha}$. So we have to distinguish and examine three cases: (a) $y \in [\, 0, \frac{\alpha}{\pi+\alpha}\,)$, (b) $y = \frac{\alpha}{\pi+\alpha}$, (c) $y \in (\, \frac{\alpha}{\pi+\alpha}, 1\,]$. Obviously, in case (a) the two semi-lines intersect at a point A_y which lies in the same half-plane with A with respect to (*wrt*) the straight line BC while in case (c) the same semi-lines intersect at a point A_y lying in the other half-plane *wrt* BC that does not contain A. In case (b) the semi-lines are parallel; also, parallel to them will be the

semi-lines AA_y with $A_{y \to \left(\frac{\alpha}{\pi+\alpha}\right)^-}$ and $A_{y \to \left(\frac{\alpha}{\pi+\alpha}\right)^+}$ (two points at infinity in the Projective Geometry sense).
Note that something similar happens when the other two pairs of straight semi-lines $(\mathcal{E}_{\text{CA1}}, \mathcal{E}_{\text{AC2}})$ and $(\mathcal{E}_{\text{AB1}}, \mathcal{E}_{\text{BA2}})$ are considered. The former pair of semi-lines are parallel for $y = \frac{\gamma}{\pi+\gamma}$ and the latter for $y = \frac{\beta}{\pi+\beta}$; at the same time BB_y will be parallel to the former pair, with $B_{y \to \left(\frac{\gamma}{\pi+\gamma}\right)^-}$ $\left(\text{and } B_{y \to \left(\frac{\gamma}{\pi+\gamma}\right)^+}\right)$, while CC_y will be parallel to the latter pair, with $B_{y \to \left(\frac{\beta}{\pi+\beta}\right)^-}$ $\left(\text{and } B_{y \to \left(\frac{\beta}{\pi+\beta}\right)^+}\right)$ in the sense already explained. Since the function $\frac{z}{\pi+z}$ is strictly increasing for $z \in (0, +\infty)$ it will be

$$0 < \frac{\alpha}{\pi+\alpha} < \frac{\gamma}{\pi+\gamma} < \frac{\beta}{\pi+\beta} < \frac{1}{2}, \tag{2.1}$$

meaning that as y increases in $[0, 1]$, first the semi-lines of the pair $(\mathcal{E}_{\text{BC1}}, \mathcal{E}_{\text{CB2}})$ become parallel, then those of the pair $(\mathcal{E}_{\text{CA1}}, \mathcal{E}_{\text{AC2}})$ and, finally, the ones of the pair $(\mathcal{E}_{\text{AB1}}, \mathcal{E}_{\text{BA2}})$. Although in the three particular cases $y \in \left\{\frac{\alpha}{\pi+\alpha}, \frac{\gamma}{\pi+\gamma}, \frac{\beta}{\pi+\beta}\right\}$ some of the elements of the "*exterior Hofstadter triangles*" are not defined, there are others as, e.g., their transversals and points that are well-defined as limiting cases. Some others will be examined and studied more analytically in the Appendix.
In the following we shall exhaustively examine only the pair $(\mathcal{E}_{\text{BC1}}, \mathcal{E}_{\text{CB2}})$ of straight semi-lines for all $y \in [0, 1] \setminus \left\{\frac{\alpha}{\pi+\alpha}, \frac{\gamma}{\pi+\gamma}, \frac{\beta}{\pi+\beta}\right\}$. The two cases $y = 0$, $y = 1$ will be examined as limiting cases. The corresponding elements, properties and results for the other two pairs of straight semi-lines will be obtained cyclicly. For the investigation that follows we need some basic statements. The first is Ceva's Theorem [8] which will be given without proof and the second is also a theorem concerning the trilinear coordinates of the point of concurrency of three line segments that are drawn from the three vertices of any triangle and end up at the opposite sides. Finally, we will present Desargues's Theorem [1] (see also [7]) to which we will only refer in the text briefly.

Theorem 2.1. *(Ceva's Theorem): Let a triangle ΔABC and three straight line segments from its vertices A, B, C to the points D, E, F of the opposite sides, respectively. Sufficient and necessary condition for the three line segments AD, BE, CF to be concurrent is that the following equality holds*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1. \tag{2.2}$$

Remark 2.2. *One should bear in mind that if in (2.2) the straight line segments involved are taken as vectors as, e.g.,*

$$\frac{\vec{BD}}{\vec{DC}} \cdot \frac{\vec{CE}}{\vec{EA}} \cdot \frac{\vec{AF}}{\vec{FB}} = +1^1,$$

the end result is always $+1$ no matter whether the point in question lies inside or outside the triangle.

[1] In the following we will avoid using the vector symbol "$\vec{\cdot}$" unless it is absolutely necessary. The distinction between length and vector of a line segment will become clear from the context.

Definition 2.1. *Under the notation of Theorem 2.1 the trilinear coordinates of the point of concurrency O are the directed distances of O from the sides BC, CA, AB, respectively, or entities proportional to them with a positive ratio. Let the three coordinates be h_a, h_b, h_c, respectively, then the set of trilinear coordinates is written symbolically as the ratio notation*

$$h_a : h_b : h_c \;=\; v : w : z, \tag{2.3}$$

where v, w, z are entities positively proportional to h_a, h_b, h_c, respectively. (Note: In a certain case we may omit the left hand side of the equality above.)

Remark 2.3. *It has to be said that whenever the point O and a certain vertex of the triangle ΔABC, e.g., A lie in the same half-plane wrt the extended straight line BC, then the coordinate h_a will be positive otherwise it will be negative. (**Attention:** In general, although we may multiply all three trilinear coordinates of a certain point by any positive number, since then the point remains unchanged, we **cannot** do the same with a negative number. Also, if the point in question is a vertex of the triangle two of its coordinates will be zero while the third one must be given by its **real** length; see, e.g., (3.15) later on.)*

Theorem 2.4. *Under the notation and the assumptions of Theorem 2.1 and Definition 2.1, the trilinear coordinates of the point of concurrency O, strictly inside the triangle, are given by*

$$h_a : h_b : h_c = \frac{(OBC)}{a} : \frac{(OCA)}{b} : \frac{(OAB)}{c}. \tag{2.4}$$

Theorem 2.5. *(Desargues's Theorem): Given two triangles ΔABC and $\Delta A'B'C'$. The two propositions below are equivalent, in the Projective Geometry sense.*
a) The straight lines AA', BB', CC' pass through a common point.
b) The intersections of the pairs of straight lines

$$A'' = (BC, B'C'), \quad B'' = (CA, C'A'), \quad C'' = (AB, A'B')$$

lie on a straight line.
(See also http://users.math.uoc.gr/pamfilos/gGallery/problems/Desargues.html, where an elementary Euclidean Geometry proof based on Menelaus's Theorem can be found.)

Corollary 2.1. *The intersections of the pairs of straight lines*

$$A_{x,1} = (BC, B_{x,in}C_{x,in}), \quad B_{x,1} = (CA, C_{x,in}A_{x,in}), \quad C_{x,1} = (AB, A_{x,in}B_{x,in})$$

lie on a straight line.

Proof: For the two triangles ΔABC and $\Delta A_{x,in}B_{x,in}C_{x,in}$ their transversals $AA_{x,in}$, $BB_{x,in}$, $CC_{x,in}$ are concurrent meaning that part (a) of the Desargues Theorem is valid. Consequently, so is part (b) and the proof is completed. □

Before we go on further we should said that the Hofstadter triangles appear in pairs, except in trivial cases, in such a way that the elements of one of the members of the pair can provide directly the elements of the other. So, in Section 3 we will study and analyze analytically any of the two members of the pair ignoring the existence of the other and in Section 4 we will examine very briefly this pair-wise property. As we will see the elements of the members of each pair are interrelated.

3. Exterior Hofstadter triangles and their elements

As has already been mentioned, in the following we will exhaustively examine only one case while the relevant results for the other two will be either omitted, if they are of no further use, or will be given by cyclic permutations.

3.1. **Exterior Hofstadter triangles.** For $y \in [0,1]$ we examine first the general case $y \in (0,1)\setminus\left\{\frac{\alpha}{\pi+\alpha}, \frac{\gamma}{\pi+\gamma}, \frac{\beta}{\pi+\beta}\right\}$ while the limiting cases $y = 0$ (or $\lim_{y\to 0^+}$), $y = 1$ (or $\lim_{y\to 1^-}$) are examined right afterwards. In the following, the notation, properties and statements introduced in Section 2 are adopted and the three altitudes of the triangle ΔABC, let them be AD, BE and CF, are also considered.
We begin with the determination of the points A_y and A'_y.
Observe that from the right triangles $\Delta BA'_yA_y$ and $\Delta A'_yCA_y$ we obtain

$$\begin{aligned} BA'_y &= A_yA'_y \cot\left(\beta + (\pi-\beta)y\right) = -A_yA'_y \cot\left((\pi-\beta)(1-y)\right), \\ A'_yC &= A_yA'_y \cot\left(\gamma + (\pi-\gamma)y\right) = -A_yA'_y \cot\left((\pi-\gamma)(1-y)\right), \end{aligned} \tag{3.1}$$

respectively, because $(\beta + (\pi-\beta)y) + ((\pi-\beta)(1-y)) = (\gamma + (\pi-\gamma)y) + ((\pi-\gamma)(1-y)) = \pi$. Adding the above equalities we obtain

$$\begin{aligned} a &= BA'_y + A'_yC = -A_yA'_y\left(\cot\left((\pi-\beta)(1-y)\right) + \cot\left((\pi-\gamma)(1-y)\right)\right) \\ &= -A_yA'_y \cdot \frac{\sin((\pi+\alpha)(1-y))}{\sin((\pi-\beta)(1-y))\sin((\pi-\gamma)(1-y))}, \end{aligned} \tag{3.2}$$

from which

$$A_yA'_y = -a \cdot \frac{\sin\left((\pi-\beta)(1-y)\right)\sin\left((\pi-\gamma)(1-y)\right)}{\sin\left((\pi+\alpha)(1-y)\right)}. \tag{3.3}$$

Using the expression from (3.3) into the two expressions in (3.1) we obtain

$$BA'_y = a \cdot \frac{\sin((\pi-\gamma)(1-y))\cos((\pi-\beta)(1-y))}{\sin((\pi+\alpha)(1-y))}, \quad A'_yC = a \cdot \frac{\sin((\pi-\beta)(1-y))\cos((\pi-\gamma)(1-y))}{\sin((\pi+\alpha)(1-y))}. \tag{3.4}$$

Cyclicly, we can obtain

$$B_yB'_y = -b \cdot \frac{\sin((\pi-\gamma)(1-y))\sin((\pi-\alpha)(1-y))}{\sin((\pi+\beta)(1-y))}, \quad C_yC'_y = -c \cdot \frac{\sin((\pi-\alpha)(1-y))\sin((\pi-\beta)(1-y))}{\sin((\pi+\gamma)(1-y))}, \tag{3.5}$$

$$\begin{aligned} CB'_y &= b \cdot \frac{\sin((\pi-\alpha)(1-y))\cos((\pi-\gamma)(1-y))}{\sin((\pi+\beta)(1-y))}, \quad B'_yA = b \cdot \frac{\sin((\pi-\gamma)(1-y))\cos((\pi-\alpha)(1-y))}{\sin((\pi+\beta)(1-y))}, \\ AC'_y &= c \cdot \frac{\sin((\pi-\beta)(1-y))\cos((\pi-\alpha)(1-y))}{\sin(((\pi+\gamma)(1-y))}, \quad C'_yB = c \cdot \frac{\sin((\pi-\alpha)(1-y))\cos((\pi-\beta)(1-y))}{\sin((\pi+\gamma)(1-y))}. \end{aligned} \tag{3.6}$$

We continue with the limiting case $y = 0$ or, equivalently, $y \to 0^+$. We use (3.4) and (3.3) to determine the points $A'_{y=0}$ and $A_{y=0}$. From the first of (3.4) we have

$$\begin{aligned} BA'_{y=0} &= \lim_{y\to 0^+} BA'_y = a \cdot \lim_{y\to 0^+} \frac{\sin((\pi-\gamma)(1-y))\cos((\pi-\beta)(1-y))}{\sin((\pi+\alpha)(1-y))} = a \cdot \frac{\sin\gamma\cos(\pi-\beta)}{\sin(\pi+\alpha)} \\ &= a \cdot \frac{\sin\gamma\cos\beta}{\sin\alpha} = c\cos\beta. \end{aligned} \tag{3.7}$$

From the second of (3.4) it can also be found that

$$\begin{aligned} A'_{y=0}C &= \lim_{y\to 0^+} A'_yC = a \cdot \lim_{y\to 0^+} \frac{\sin((\pi-\beta)(1-y))\cos((\pi-\gamma)(1-y))}{\sin((\pi+\alpha)(1-y))} = a \cdot \frac{\sin\beta\cos\gamma}{\sin\alpha} \\ &= b\cos\gamma. \end{aligned} \tag{3.8}$$

Cyclicly, we can obtain that

$$CB'_{y=0} = a\cos\gamma, \quad B'_{y=0}A = c\cos\alpha, \quad AC'_{y=0} = b\cos\alpha, \quad C'_{y=0}B = a\cos\beta. \tag{3.9}$$

Observe that the expressions (3.7) and (3.8) give the projections of the sides BA and AC onto the side BC, respectively, meaning that $A'_{y=0}$ is the projection of the vertex A onto the side BC and, therefore, $A_{y=0} \equiv A$ and $A'_{y=0} \equiv D$. In a similar way we can interpret the expressions in (3.9).
On the other hand, for the limiting case $A_{y=0}A'_{y=0} = \lim_{y\to 0^+} A_yA'_y$ we may find that

$$\begin{aligned} A_{y=0}A'_{y=0} &= \lim_{y\to 0^+} A_yA'_y = \lim_{y\to 0^+} -a\cdot \frac{\sin((\pi-\beta)(1-y))\sin((\pi-\gamma)(1-y))}{\sin(((\pi+\alpha)(1-y))} = a\cdot\frac{\sin\beta\sin\gamma}{\sin\alpha} \\ &= c\sin\beta, \end{aligned} \tag{3.10}$$

which is the altitude AD. Cyclicly, we obtain

$$B_{y=0}B'_{y=0} = \lim_{y\to 0^+} B_yB'_y = a\sin\gamma, \quad C_{y=0}C'_{y=0} = \lim_{y\to 0^+} C_yC'_y = b\sin\alpha. \tag{3.11}$$

Similar interpretations to the points $A_{y=0}$ and $A'_{y=0}$ can be given for the points $B_{y=0}$, $B'_{y=0}$ and $C_{y=0}$, $C'_{y=0}$ as well as to $A_{y=0}A'_{y=0}$ above for the expressions in (3.11).
Now, for the limiting case $BA'_{y=1} = \lim_{y\to 1^-} BA'_y$ we can successively obtain that

$$\begin{aligned} BA'_{y=1} = \lim_{y\to 1^-} BA'_y &= a\cdot \lim_{y\to 1^-} \frac{\sin((\pi-\gamma)(1-y))\cos((\pi-\beta)(1-y))}{\sin(((\pi+\alpha)(1-y))} \\ &= a\cdot \frac{\lim_{y\to 1^-}\frac{\sin(((\pi-\gamma)(1-y)))}{(\pi-\gamma)(1-y)}\cdot(\pi-\gamma)(1-y)\cdot\lim_{y\to 1^-}\cos((\pi-\beta)(1-y))}{\lim_{y\to 1^-}\frac{\sin((\pi+\alpha)(1-y))}{(\pi+\alpha)(1-y)}\cdot(\pi+\alpha)(1-y)} \\ &\overset{(*)}{=} a\cdot\frac{1\cdot(\pi-\gamma)\cdot 1}{1\cdot(\pi+\alpha)} = a\frac{(\pi-\gamma)}{(\pi+\alpha)}. \end{aligned} \tag{3.12}$$

In should be explained that to obtain the right side of $\overset{(*)}{=}$, first we simplified by the factor $1-y$, since as $y \to 1^-$, $1-y>0$, and then we gave the three limits that are equal to the numbers indicated in it. In exactly the same way it can be found that

$$A'_{y=1}C = \lim_{y\to 1^-} A'_yC = a\frac{(\pi-\beta)}{(\pi+\alpha)}. \tag{3.13}$$

Similarly, we find from (3.12) and (3.13) the other four entities below

$$CB'_{y=1} = b\frac{(\pi-\alpha)}{(\pi+\beta)},\ B'_{y=1}A = b\frac{(\pi-\gamma)}{(\pi+\beta)},\ AC'_{y=1} = c\frac{(\pi-\beta)}{(\pi+\gamma)},\ C'_{y=1}B = c\frac{(\pi-\alpha)}{(\pi+\gamma)}. \tag{3.14}$$

Remark 3.1. *As is readily seen from (3.4), neither of the two expressions for BA'_y and A'_yC is finite, for $y = \frac{\alpha}{\pi+\alpha}$, although their sum equals BC. This has as a consequence that none of the explicit quantities that are associated with the above value of y as, e.g., $AA'_{y=\frac{\alpha}{\pi+\alpha}}$ is finite. The same observation can be made in the particular cases $y = \frac{\gamma}{\pi+\gamma}$ and $y = \frac{\beta}{\pi+\beta}$ with $CC'_{y=\frac{\gamma}{\pi+\gamma}}$ and $BB'_{y=\frac{\beta}{\pi+\beta}}$, respectively, and with any similar elements associated with them. However, as we will see in the sequel many limiting results can be found based on these three particular values of y some of which can guarantee continuity.*

We go on with the determination of the trilinear coordinates of the vertices of the "*exterior Hofstadter triangles*" and state and prove a relevant proposition.

Theorem 3.2. *For any $y \in [0,1] \backslash \left\{ \frac{\alpha}{\pi+\alpha}, \frac{\gamma}{\pi+\gamma}, \frac{\beta}{\pi+\beta}, \right\}$, the vertices of the "exterior Hofstadter triangles" $\Delta A_y B_y C_y$ have trilinear coordinates as follows:*

$$\begin{aligned} \left|A_{y=0}A'_{y=0}\right| : \left|A_{y=0}A_{y=0,\mathrm{CA}}\right| : \left|A_{y=0}A_{y=0,\mathrm{AB}}\right| &= 2R\sin\beta\sin\gamma : 0 : 0, \\ \left|B_{y=0}B_{y=0,\mathrm{BC}}\right| : \left|B_{y=0}B'_{y=0}\right| : \left|B_{y=0}B_{y=0,\mathrm{AB}}\right| &= 0 : 2R\sin\alpha\sin\gamma : 0, \\ \left|C_{y=0}C_{y=0,\mathrm{BC}}\right| : \left|C_{y=0}C_{y=0,\mathrm{CA}}\right| : \left|C_{y=0}C'_{y=0}\right| &= 0 : 0 : 2R\sin\alpha\sin\beta. \end{aligned} \tag{3.15}$$

$$\begin{aligned} \left|A_yA'_y\right| : -\left|A_yA_{y,\mathrm{CA}}\right| : -\left|A_yA_{y,\mathrm{AB}}\right| &= 1 : -\tfrac{\sin((\pi-\gamma)y)}{\sin((\pi-\gamma)(1-y))} : -\tfrac{\sin((\pi-\beta)y)}{\sin((\pi-\beta)(1-y))} && \textit{for } y \in \left(0, \tfrac{\alpha}{\pi+\alpha}\right), \\ -\left|A_yA'_y\right| : \left|A_yA_{y,\mathrm{CA}}\right| : \left|A_yA_{y,\mathrm{AB}}\right| &= -1 : \tfrac{\sin((\pi-\gamma)y)}{\sin((\pi-\gamma)(1-y))} : \tfrac{\sin((\pi-\beta)y)}{\sin((\pi-\beta)(1-y))} && \textit{for } y \in \left(\tfrac{\alpha}{\pi+\alpha}, 1\right), \\ -\left|B_yB_{y,\mathrm{BC}}\right| : \left|B_yB'_y\right| : -\left|B_yB_{y,\mathrm{AB}}\right| &= -\tfrac{\sin((\pi-\gamma)y)}{\sin((\pi-\gamma)(1-y))} : 1 : -\tfrac{\sin((\pi-\alpha)y)}{\sin((\pi-\alpha)(1-y))} && \textit{for } y \in \left(0, \tfrac{\beta}{\pi+\beta}\right), \\ \left|B_yB_{y,\mathrm{BC}}\right| : -\left|B_yB'_y\right| : \left|B_yB_{y,\mathrm{AB}}\right| &= \tfrac{\sin((\pi-\gamma)y)}{\sin((\pi-\gamma)(1-y))} : -1 : \tfrac{\sin((\pi-\alpha)y)}{\sin((\pi-\alpha)(1-y))} && \textit{for } y \in \left(\tfrac{\beta}{\pi+\beta}, 1\right), \\ -\left|C_yC_{y,\mathrm{BC}}\right| : -\left|C_yC_{y,\mathrm{CA}}\right| : \left|C_yC'_y\right| &= -\tfrac{\sin((\pi-\beta)y)}{\sin((\pi-\beta)(1-y))} : -\tfrac{\sin((\pi-\alpha)y)}{\sin(\pi-\alpha)(1-y))} : 1 && \textit{for } y \in \left(0, \tfrac{\gamma}{\pi+\gamma}\right), \\ \left|C_yC_{y,\mathrm{BC}}\right| : \left|C_yC_{y,\mathrm{CA}}\right| : -\left|C_yC'_y\right| &= \tfrac{\sin((\pi-\gamma)y)}{\sin((\pi-\gamma)(1-y))} : \tfrac{\sin((\pi-\alpha)y)}{\sin(\pi-\alpha)(1-y))} : -1 && \textit{for } y \in \left(\tfrac{\gamma}{\pi+\gamma}, 1\right). \end{aligned} \tag{3.16}$$

$$\begin{aligned} \left|A_{y=1}A'_{y=1}\right| : \left|A_{y=1}A_{y=1,\mathrm{CA}}\right| : \left|A_{y=1}A_{y=1,\mathrm{AB}}\right| &= 0 : \tfrac{\pi-\beta}{\sin\beta} : \tfrac{\pi-\gamma}{\sin\gamma}, \\ \left|B_{y=1}B_{y=1,\mathrm{BC}} : B_{y=1}B'_{y=1}\right| : \left|B_{y=1}B_{y=1,\mathrm{AB}}\right| &= \tfrac{\pi-\alpha}{\sin\alpha} : 0 : \tfrac{\pi-\gamma}{\sin\gamma}, \\ \left|C_{y=1}C_{y=1,\mathrm{BC}}\right| : \left|C_{y=1}C_{y=1,\mathrm{CA}}\right| : \left|C_{y=1}C'_{y=1}\right| &= \tfrac{\pi-\alpha}{\sin\alpha} : \tfrac{\pi-\beta}{\sin\beta} : 0. \end{aligned} \tag{3.17}$$

Proof: We begin with the general case where $y \in (0,1) \backslash \left\{ \frac{\alpha}{\pi+\alpha}, \frac{\gamma}{\pi+\gamma}, \frac{\beta}{\pi+\beta} \right\}$ and specifically with $\alpha \in \left(0, \frac{\alpha}{\pi+\alpha}\right)$. The distance of A_y from the side BC is given by $A_yA'_y$ in (3.3). Let $A_{y,\mathrm{CA}}$ and $A_{y,\mathrm{AB}}$ be the projections of A_y onto the sides CA and AB, respectively. From the right triangles $\Delta A_yA_{y,\mathrm{CA}}C$ and $\Delta A_yA'_yC$, we have successively,

$$\left|A_yA_{y,\mathrm{CA}}\right| = \left|A_yC\right| \sin\left((\pi-\gamma)y\right) = \frac{\left|A_yA'_y\right| \sin\left((\pi-\gamma)y\right)}{\sin(\gamma+(\pi-\gamma)y)} = \frac{\left|A_yA'_y\right| \sin\left((\pi-\gamma)y\right)}{\sin((\pi-\gamma)(1-y))}.$$

Similarly, from the two right triangles and $\Delta A_yA'_yA_{y,\mathrm{AB}}$ and $\Delta A_yA'_yB$ we find that

$$\left|A_yA_{y,\mathrm{AB}}\right| = \frac{\left|A_yA'_y\right| \sin\left((\pi-\beta)y\right)}{\sin\left((\pi-\beta)(1-y)\right)}.$$

In view of Remark 2.3 regarding the position of A_y and the signs of its distances from the corresponding sides we obtain the results in the first of relations (3.16). This is because A_y and $A_{y,\mathrm{CA}}$ lie in different half-planes *wrt* the extended straight line CA and something similar holds for A_y and $A_{y,\mathrm{AB}}$ *wrt* to the extended straight line AB. On the other hand, when $y \in \left(\frac{\alpha}{\pi+\alpha}, 1\right)$, A_y lies within the angle $\langle BAC$ but A_y and A lie in different half-planes *wrt* to the extended straight line BC and so the signs of the trilinear coordinates of A_y are the opposite of the previous ones. So, for $y \in \left(0, \frac{\alpha}{\pi+\alpha}\right)$,

$$\left|A_yA_{y,\mathrm{CA}}\right| = \frac{\left|A_yA'_y\right| \sin((\pi-\gamma)y)}{\sin((\pi-\gamma)(1-y))}, \quad \left|A_yA_{y,\mathrm{AB}}\right| = \frac{\left|A_yA'_y\right| \sin((\pi-\beta)y)}{\sin((\pi-\beta)(1-y))}, \tag{3.18}$$

while for $y \in \left(\frac{\alpha}{\pi+\alpha}, 1\right)$, the distance $\left|A_yA'_y\right|$ is negative while the other two distances given in the expressions (3.18) are positive. It is interesting to note that except for the sign difference the expressions for the trilinear coordinates of A_y in the first two ratio equalities (3.16) are the same.

From our analysis in the present proof not only the trilinear coordinates of the vertex A_y of the exterior Hofstadter triangle are obtained but also their cyclic ones for B_y and C_y. All of them are given in (3.16).

For the case $y = 0$ we have already found in (3.10) and in a note following it that $A_{y=0}A'_{y=0} \equiv AD = c \sin \beta$ (or $b \sin \gamma = 2R \sin \beta \sin \gamma$) > 0, where R is the circumcircle radius of the triangle ΔABC. Clearly, the distances of $A_{y=0} \equiv A$ from CA and AB are zeros. Cyclicly, we readily obtain the distances of the points $B_{y=0} \equiv B$ and $C_{y=0} \equiv C$ from the same sides as before.
The trilinear coordinates, in case $A_{y=0}A'_{y=0}$ is involved, can be found as a limiting case allowing $y \to \frac{\alpha}{\pi+\alpha}$. However, dividing through by $A_{y=0}A'_{y=0} > 0$ and taking limits we readily find that the other two trilinear coordinates are equal to zero as expected. Similar conclusions can be drawn for the other two cases. All of the corresponding coordinates are given in (3.15).

For the case $y = 1$ we obviously have $A_{y=1}A'_{y=1} = B_{y=1}B'_{y=1} = C_{y=1}C'_{y=1} = 0$. Alternatively, this is also yielded from (3.3) and its cyclic expressions as is shown in the sequel. Note that going from the expression on the left of $\stackrel{(*)}{=}$ below to the one on the right we applied first De L'Hôpital's Rule and then we took the appropriate limits.

$$\begin{aligned}
A_{y=1}A'_{y=1} &= \lim_{y\to 1^-} A_yA'_y = -a \cdot \lim_{y\to 1^-} \frac{\sin((\pi-\beta)(1-y))\sin((\pi-\gamma)(1-y))}{\sin((\pi+\alpha)(1-y))} \\
&= -a \cdot \lim_{y\to 1^-} \frac{\sin((\pi-\beta)(1-y))\sin((\pi-\gamma)(1-y))}{\sin((\pi+\alpha)(1-y))} \\
&\stackrel{(*)}{=} -a \cdot \frac{(\beta-\pi)\lim_{y\to 1^-}\cos((\pi-\beta)(1-y))\lim_{y\to 1^-}\sin((\pi-\gamma)(1-y))}{(\pi+\alpha)\lim_{y\to 1^-}\cos((\pi-\alpha)+(\pi+\alpha)y)} \\
&\quad -a \cdot \frac{\lim_{y\to 1^-}\sin((\pi-\beta)(1-y))(\gamma-\pi)\lim_{y\to 1^-}\cos((\pi-\gamma)(1-y))}{(\pi+\alpha)\lim_{y\to 1^-}\cos((\pi-\alpha)+(\pi+\alpha)y)} \\
&= -a \cdot \frac{(\beta-\pi)\cos 0 \sin 0 + (\gamma-\pi)\sin 0 \cos 0}{(\pi+\alpha)\cos(2\pi)} = -a \cdot \frac{(\beta-\pi)\cdot 1\cdot 0 + (\gamma-\pi)\cdot 0 \cdot 1}{(\pi+\alpha)\cdot 1} = 0.
\end{aligned} \tag{3.19}$$

Then, using the expressions found in (3.12)-(3.14) we readily obtain the trilinear coordinates of the points $A_{y=1} \equiv A'_{y=1}$, $B_{y=1} \equiv B'_{y=1}$, $C_{y=1} \equiv C'_{y=1}$ presented in (3.17). For example, from the right triangles $\Delta A'_{y=1}CA_{y=1,\mathrm{CA}}$ and $\Delta A'_{y=1}AB_{y=1,\mathrm{AB}}$, using (3.12) and (3.13), we obtain $A_{y=1}A_{y=1,\mathrm{CA}} = A'_{y=1}A_{y=1,\mathrm{CA}} = A'_1C\sin\gamma = a \cdot \frac{(\pi-\beta)\sin\gamma}{\pi+\alpha}$; so from the other right triangle we have $A_{y=1}A_{y=1,\mathrm{AB}} = a \cdot \frac{(\pi-\gamma)\sin\beta}{\pi+\alpha}$. Then, dividing through by $a \cdot \frac{\sin\beta\sin\gamma}{\pi+\alpha}$ we obtain $\frac{\pi-\beta}{\sin\beta}$ and $\frac{\pi-\gamma}{\sin\gamma}$ for the last two coordinates. The expressions just obtained and their cyclic ones are presented in (3.17). □

The "*exterior Hofstadter triangle* $\Delta A_yB_yC_y$" is depicted in Figure 1. The angles of the original triangle ΔABC are $\alpha = \frac{\pi}{4} < \gamma = \frac{\pi}{3} < \beta = \frac{5\pi}{12}$, and y was chosen to be $y = \frac{1}{10} \in \left(0, \frac{1}{5}\right) \equiv \left(0, \frac{\alpha}{\pi+\alpha}\right)$.

Remark 3.3. *For $y = \frac{1}{2}$ the "exterior Hofstadter triangle" $\Delta A_{y=\frac{1}{2}}B_{y=\frac{1}{2}}C_{y=\frac{1}{2}}$ has vertices the excenters of the triangle ΔABC. The trilinear coordinates of these vertices are readily obtained*

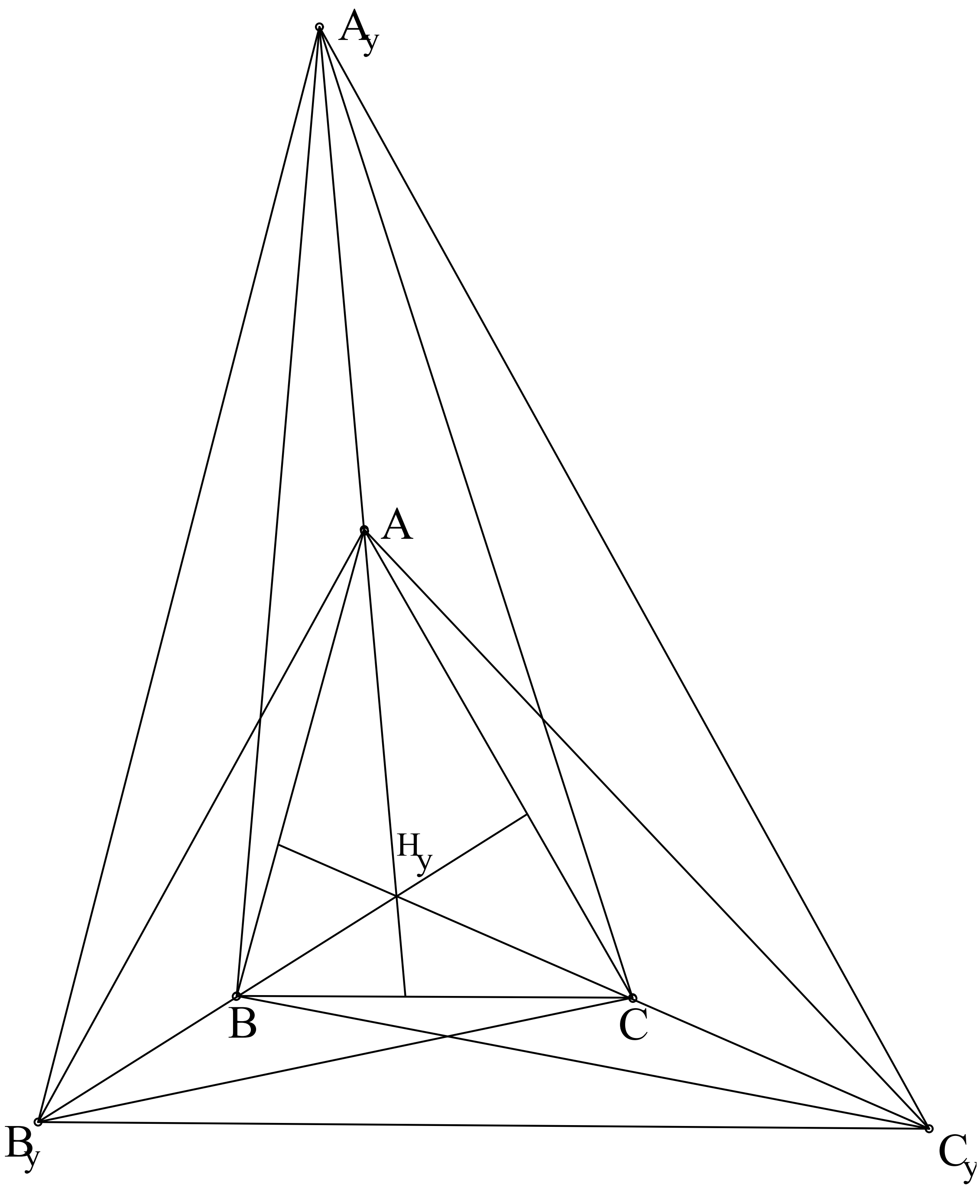

Figure 1. *Exterior Hofstadter triangle $\Delta A_y B_y C_y$, with its transversals AA_y, BB_y, CC_y, and Hofstadter point.* $\left(\alpha = \frac{\pi}{4} < \gamma = \frac{\pi}{3} < \beta = \frac{5\pi}{12},\ y = \frac{1}{10} \in \left(0, \frac{1}{5}\right) \equiv \left(0, \frac{\alpha}{\pi+\alpha}\right)\right)$

from (3.16) by setting $y=\frac{1}{2}$. From (3.16) we obtain that

$$\begin{array}{rcl} -\left|A_{y=\frac{1}{2}}A'_{y=\frac{1}{2}}\right| : \left|A_{y=\frac{1}{2}}A_{y=\frac{1}{2},\mathrm{CA}}\right| : \left|A_{y=\frac{1}{2}}A_{y=\frac{1}{2},\mathrm{AB}}\right| &=& -1:1:1,\\ \left|B_{y=\frac{1}{2}}B_{y=\frac{1}{2},\mathrm{BC}}\right| : -\left|B_{y=\frac{1}{2}}B'_{y=\frac{1}{2}}\right| : \left|B_{y=\frac{1}{2}}B_{y=\frac{1}{2},\mathrm{AB}}\right| &=& 1:-1:1,\\ \left|C_{y=\frac{1}{2}}C_{y=\frac{1}{2},\mathrm{BC}}\right| : \left|C_{y=\frac{1}{2}}C_{y=\frac{1}{2},\mathrm{CA}}\right| : -\left|C_{y=\frac{1}{2}}C'_{y=\frac{1}{2}}\right| &=& 1:1:-1. \end{array} \tag{3.20}$$

As an application we will show below that the "*exterior Hofstadter triangle*" corresponding to $y=\frac{2}{3}$ is connected with one of the well-known 18 "*equilateral Morley triangles*" (see, e.g., [3]). More specifically:

Theorem 3.4. *The "exterior Hofstadter triangle" whose vertices are the points of intersection of the trisectors of the exterior angles of a given triangle ΔABC corresponding to $y=\frac{2}{3}$ has trilinear components of its vertices that are the opposite in sign of the corresponding ones of the "third equilateral Morley triangle". More specifically, the trilinear coordinates of the triangle in question are given by*

$$\begin{array}{rcl} -\left|A_{y=\frac{2}{3}}A'_{y=\frac{2}{3}}\right| : \left|A_{y=\frac{2}{3}}A_{y=\frac{2}{3},\mathrm{CA}}\right| : \left|A_{y=\frac{2}{3}}A_{y=\frac{2}{3},\mathrm{AB}}\right| &=& -1:2\cos\left(\frac{\pi-\gamma}{3}\right):2\cos\left(\frac{\pi-\beta}{3}\right),\\ \left|B_{y=\frac{2}{3}}B_{y=\frac{2}{3},\mathrm{BC}}\right| : -\left|B_{y=\frac{2}{3}}B'_{y=\frac{2}{3}}\right| : \left|B_{y=\frac{2}{3}}B_{y=\frac{2}{3},\mathrm{AB}}\right| &=& 2\cos\left(\frac{\pi-\gamma}{3}\right):-1:2\cos\left(\frac{\pi-\alpha}{3}\right),\\ \left|C_{y=\frac{2}{3}}C_{y=\frac{2}{3},\mathrm{BC}}\right| : \left|C_{y=\frac{2}{3}}C_{y=\frac{2}{3},\mathrm{CA}}\right| : -\left|C_{y=\frac{2}{3}}C'_{y=\frac{2}{3}}\right| &=& 2\cos\left(\frac{\pi-\beta}{3}\right):2\cos\left(\frac{\pi-\alpha}{3}\right):-1. \end{array} \tag{3.21}$$

Proof: Let $A_{y=\frac{2}{3}}$, $B_{y=\frac{2}{3}}$, $C_{y=\frac{2}{3}}$ be the vertices of the triangle $\Delta A_{y=\frac{2}{3}}B_{y=\frac{2}{3}}C_{y=\frac{2}{3}}$ formed by the points of intersection of the exterior trisectors of the triangle ΔABC corresponding to $y=\frac{2}{3}$. To determine the trilinear coordinates of its vertices, say that of $A_{y=\frac{2}{3}}$, observe that this point lies under the side BC and so $\langle A_{y=\frac{2}{3}}BC=\frac{\pi-\beta}{3}$. Let then $-\left|A_{y=\frac{2}{3}}A'_{y=\frac{2}{3}}\right| : \left|A_{y=\frac{2}{3}}A_{y=\frac{2}{3},\mathrm{CA}}\right| : \left|A_{y=\frac{2}{3}}A_{y=\frac{2}{3},\mathrm{AB}}\right|$ be the trilinear coordinates of $A_{y=\frac{1}{3}}$. From the two right triangles $\Delta A_{y=\frac{2}{3}}A_{y=\frac{2}{3},\mathrm{CA}}$ and $A_{y=\frac{2}{3}}A'_{y=\frac{2}{3}}C$ we can obtain successively that

$$\begin{array}{rcl} \left|A_{y=\frac{2}{3}}A_{y=\frac{2}{3},\mathrm{CA}}\right| &=& \left|A_{y=\frac{2}{3}}C\right|\sin\left(\frac{2(\pi-\gamma)}{3}\right)=\dfrac{\left|A_{y=\frac{2}{3}}A'_{y=\frac{2}{3}}\right|\sin\left(\frac{2(\pi-\gamma)}{3}\right)}{\sin\left(\frac{\pi-\gamma}{3}\right)}\\ &=& 2\left|A_{y=\frac{2}{3}}A'_{y=\frac{2}{3}}\right|\cos\left(\frac{\pi-\gamma}{3}\right), \end{array}$$

where we have expanded the sine in the numerator as sine of twice the argument $\frac{\pi-\gamma}{3}$ and then simplified by $\sin\left(\frac{\pi-\gamma}{3}\right)$. Working in the same way we can find that

$$\left|A_{y=\frac{2}{3}}A_{y=\frac{2}{3},\mathrm{AB}}\right|=2\left|A_{y=\frac{2}{3}}A'_{y=\frac{2}{3}}\right|\cos\left(\frac{\pi-\beta}{3}\right).$$

Therefore the trilinear coordinates of $A_{y=\frac{2}{3}}$ are

$$-\left|A_{y=\frac{2}{3}}A'_{y=\frac{2}{3}}\right| : 2\left|A_{y=\frac{2}{3}}A'_{y=\frac{2}{3}}\right|\cos\left(\frac{\pi-\gamma}{3}\right) : 2\left|A_{y=\frac{2}{3}}A'_{y=\frac{2}{3}}\right|\cos\left(\frac{\pi-\beta}{3}\right)$$

and dividing through by $\left|A_{y=\frac{2}{3}}A'_{y=\frac{1}{3}}\right|$ we end up with

$$-\left|A_{y=\frac{2}{3}}A'_{y=\frac{2}{3}}\right| : \left|A_{y=\frac{2}{3}}A_{y=\frac{2}{3},\mathrm{CA}}\right| : \left|A_{y=\frac{2}{3}}A_{y=\frac{2}{3},\mathrm{AB}}\right| = -1 : 2\cos\left(\frac{\pi-\gamma}{3}\right) : 2\cos\left(\frac{\pi-\beta}{3}\right). \tag{3.22}$$

Cyclicly, we can find directly the trilinear coordinates of the vertices $B_{y=\frac{2}{3}}$ and $C_{y=\frac{2}{3}}$ of the triangle $\Delta A_{y=\frac{2}{3}}B_{y=\frac{2}{3}}C_{y=\frac{2}{3}}$; all of them are given in (3.21).
Now, in [3] it is given that the trilinear coordinates of the vertices of the "*third equilateral Morley triangle*" $\Delta A_y B_y C_y$, say for the vertex A_y, are the following

$$1 : 2\cos\left(\frac{\gamma-4\pi}{3}\right) : 2\cos\left(\frac{\beta-4\pi}{3}\right). \tag{3.23}$$

These coordinates compared with the ones in (3.22) are all opposite in sign because the arguments of the corresponding cosines sum up to $-\pi$, e.g., $\frac{\pi-\gamma}{3}+\frac{\gamma-4\pi}{3}=-\pi$ and the proof is complete. □

3.2. **Exterior Hofstadter transversals and their concurrency.** Let AA_y intersect BC at $A_{y,ex}$, BB_y intersect CA at $B_{y,ex}$, CC_y intersect AB at $C_{y,ex}$. It is understood that $y \in [0,1]\setminus\left\{\frac{\alpha}{\pi+\alpha}, \frac{\gamma}{\pi+\gamma}, \frac{\beta}{\pi+\beta}\right\}$. As always, we consider the case where y lies in the union of the four open intervals above excluding the two extreme cases $y=0$ and $y=1$ which will be examined afterwards.
To determine the three points $A_{y,ex}$, $B_{y,ex}$, $C_{y,ex}$, next the transversals $AA_{y,ex}$, $BB_{y,ex}$, $CC_{y,ex}$ of the exterior Hofstadter triangle and then the three ratios $\frac{BA_{y,ex}}{A_{y,ex}C}$, $\frac{CB_{y,ex}}{B_{y,ex}A}$, $\frac{AC_{y,ex}}{C_{y,ex}B}$, we will determine the first ratio only while the other two will be given by cyclic permutations. Note that actually we have to consider y in each of the four open intervals $\left(0,\frac{\alpha}{\pi+\alpha}\right)$, $\left(\frac{\alpha}{\pi+\alpha},\frac{\gamma}{\pi+\gamma}\right)$, $\left(\frac{\gamma}{\pi+\gamma},\frac{\beta}{\pi+\beta}\right)$, $\left(\frac{\beta}{\pi+\beta},1\right)$ separately. However, since the points $A_{y,ex}$, $B_{y,ex}$, $C_{y,ex}$ lie strictly inside the corresponding sides of the triangle the ratio, e.g., $\frac{BA_{y,ex}}{A_{y,ex}C}$ will be always positive no matter whether $y \in \left(0,\frac{\alpha}{\pi+\alpha}\right)$ or $y \in \left(\frac{\alpha}{\pi+\alpha},1\right)$ and this is despite that the corresponding expressions in (3.16) are different in sign. This will be made clear below in the final expressions (3.24) and (3.25). So, we determine the ratios in only one of the four intervals and especially in the first one.
The triangles $\Delta BA_{y,ex}A_y$ and $\Delta A_{y,ex}CA_y$ share the same altitude and then their areas will be proportional to the their bases $BA_{y,ex}$ and $A_{y,ex}C$. The same property will be satisfied by the areas of the triangles $\Delta BA_{y,ex}A$ and $\Delta A_{y,ex}CA$. Hence, we can obtain successively

$$\begin{aligned}\frac{BA_{y,ex}}{A_{y,ex}C} = \frac{\left|BA_{y,ex}\right|}{\left|A_{y,ex}C\right|} &= \frac{(BA_{y,ex}A_y)}{(A_{y,ex}CA_y)} = \frac{(BA_{y,ex}A)}{(A_{y,ex}CA)} \overset{(*)}{=} \frac{(BA_{y,ex}A_y)-(BA_{y,ex}A)}{(A_{y,ex}CA_y)-(A_{y,ex}CA)} = \frac{(ABA_y)}{(ACA_y)} \\ &\overset{(**)}{=} \frac{c\left|A_yA_{y,AB}\right|}{b\left|A_yA_{y,CA}\right|} \overset{(***)}{=} \frac{c\left|A_yB\right|\sin((\pi-\beta)y)}{b\left|A_yC\right|\sin((\pi-\gamma)y)} \overset{(****)}{=} \frac{c\,\sin((\pi-\gamma)(1-y))\sin((\pi-\beta)y)}{b\,\sin((\pi-\beta)(1-y))\sin((\pi-\gamma)y)}.\end{aligned} \tag{3.24}$$

Firstly, equality $\overset{(*)}{=}$ holds because of the previous equality of equal ratios. Secondly, equality $\overset{(**)}{=}$ holds since the terms of the right fraction express twice the areas of the two triangles ABA_y and ACA_y. Thirdly, going via $\overset{(***)}{=}$ the expressions for $A_yA_{y,AB}$ and

$A_yA_{y,CA}$ were replaced by their equivalent expressions in (3.18). Fourthly, equality $\stackrel{(****)}{=}$ holds because the ratio of the two sides of the triangle ΔA_yBC equals the ratio of the sines of the opposite angles. Similarly, we can determine the cyclic expressions to (3.24) which are given as follows

$$\frac{CB_{y,ex}}{B_{y,ex}A} = \frac{a\ \sin((\pi-\alpha)(1-y))\sin((\pi-\gamma)y)}{c\ \sin((\pi-\gamma)(1-y))\sin((\pi-\alpha)y)}, \quad \frac{AC_{y,ex}}{C_{y,ex}B} = \frac{b\ \sin((\pi-\beta)(1-y))\sin((\pi-\alpha)y)}{a\ \sin((\pi-\alpha)(1-y))\sin((\pi-\beta)y)}. \tag{3.25}$$

From (3.24) and (3.25) we can obtain that

$$\begin{gathered}\frac{BA_{y,ex}}{A_{y,ex}C}\cdot\frac{CB_{y,ex}}{B_{y,ex}A}\cdot\frac{AC_{y,ex}}{C_{y,ex}B} = \\ \frac{c\ \sin((\pi-\gamma)(1-y))\sin((\pi-\beta)y)}{b\ \sin((\pi-\beta)(1-y))\sin((\pi-\gamma)y)}\cdot\frac{a\ \sin((\pi-\alpha)(1-y))\sin((\pi-\gamma)y)}{c\ \sin((\pi-\gamma)(1-y))\sin((\pi-\alpha)y)}\cdot\frac{b\ \sin((\pi-\beta)(1-y))\sin((\pi-\alpha)y)}{a\ \sin((\pi-\alpha)(1-y))\sin((\pi-\beta)y)} = 1.\end{gathered} \tag{3.26}$$

By virtue of Ceva's Theorem 2.1 we conclude that the three transversals $AA_{y,ex}$, $BB_{y,ex}$, $CC_{y,ex}$ are concurrent at a point H_y that constitutes the "*exterior Hofstadter point* H_y" corresponding to the y considered.

For $y = 0$, from the two extreme ratios of (3.24) we have

$$\begin{aligned}\frac{BA_{y=0,ex}}{A_{y=0,ex}C} &= \lim_{y\to0^+}\frac{c\ \sin((\pi-\gamma)(1-y))\sin((\pi-\beta)y)}{b\ \sin((\pi-\beta)(1-y))\sin((\pi-\gamma)y)} \\ &= \lim_{y\to0^+}\frac{c\cdot\sin((\pi-\gamma)(1-y))\cdot\frac{\sin((\pi-\beta)y)}{(\pi-\beta)y}\cdot(\pi-\beta)y}{b\cdot\sin((\pi-\beta)(1-y))\cdot\frac{\sin((\pi-\gamma)y)}{(\pi-\gamma)y}\cdot(\pi-\gamma)y} \\ &= \frac{c\cdot\lim_{y\to0^+}\sin((\pi-\gamma)(1-y))\cdot\lim_{y\to0^+}\frac{\sin((\pi-\beta)y)}{(\pi-\beta)y}\cdot(\pi-\beta)}{b\cdot\lim_{y\to0^+}\sin((\pi-\beta)(1-y))\cdot\lim_{y\to0^+}\frac{\sin((\pi-\gamma)y)}{(\pi-\gamma)y}\cdot(\pi-\gamma)} \\ &= \frac{c\cdot\sin(\pi-\gamma)\cdot1\cdot(\pi-\beta)}{b\cdot\sin(\pi-\beta)\cdot1\cdot(\pi-\gamma)} \\ &= \frac{c(\pi-\beta)\sin\gamma}{b(\pi-\gamma)\sin\beta} \\ &= \frac{(\pi-\beta)\sin^2\gamma}{(\pi-\gamma)\sin^2\beta}.\end{aligned} \tag{3.27}$$

Cyclicly, we can obtain for the ratios $\frac{CB_{y=0,ex}}{B_{y=0,ex}A}$ and $\frac{AC_{y=0,ex}}{C_{y=0,ex}B}$ that

$$\frac{CB_{y=0,ex}}{B_{y=0,ex}A} = \frac{(\pi-\gamma)\sin^2\alpha}{(\pi-\alpha)\sin^2\gamma}, \quad \frac{AC_{y=0,ex}}{C_{y=0,ex}B} = \frac{(\pi-\alpha)\sin^2\beta}{(\pi-\beta)\sin^2\alpha} \tag{3.28}$$

and, finally, from (3.27) and (3.28) we obtain

$$\frac{BA_{y=0,ex}}{A_{y=0,ex}C}\cdot\frac{CB_{y=0,ex}}{B_{y=0,ex}A}\cdot\frac{AC_{y=0,ex}}{C_{y=0,ex}B} = 1. \tag{3.29}$$

Therefore, by Ceva's Theorem 2.1 the three *exterior transversals* for $y = 0$ are concurrent at the "*exterior Hofstadter point* $H_{y=0}$".

For $y = 1$, we work in an analogous way as in the previous case. So, we find that

$$\begin{aligned}
\frac{BA_{y=1,ex}}{A_{y=1,ex}C} &= \lim_{y\to 1^-} \frac{c\sin((\pi-\gamma)(1-y))\sin((\pi-\beta)y)}{b\sin((\pi-\beta)(1-y))\sin((\pi-\gamma)y)} \\
&= \lim_{y\to 1^-} \frac{c\cdot\frac{\sin((\pi-\gamma)(1-y))}{(\pi-\gamma)(1-y)}\cdot(\pi-\gamma)(1-y)\cdot\sin((\pi-\beta)y)}{b\cdot\frac{\sin((\pi-\beta)(1-y))}{(\pi-\beta)(1-y)\cdot\sin((\pi-\beta)y)}\cdot(\pi-\gamma)(1-y)\cdot\sin((\pi-\gamma)y)} \\
&= \frac{c\cdot\lim_{y\to 1^-}\frac{\sin((\pi-\gamma)(1-y))}{(\pi-\gamma)(1-y)}\cdot(\pi-\gamma)(1-y)\cdot\lim_{y\to 1^-}\sin((\pi-\beta)y)}{b\cdot\lim_{y\to 1^-}\frac{\sin((\pi-\beta)(1-y))}{(\pi-\beta)(1-y)}\cdot(\pi-\beta)(1-y)\cdot\lim_{y\to 1^-}\sin((\pi-\gamma)y)} \\
&= \frac{c\cdot 1\cdot(\pi-\gamma)\cdot\sin(\pi-\beta)}{b\cdot 1\cdot(\pi-\beta)\cdot\sin(\pi-\gamma)} \\
&= \frac{(\pi-\gamma)}{(\pi-\beta)}.
\end{aligned} \tag{3.30}$$

By cyclic permutations we can obtain

$$\frac{CB_{y=1,ex}}{B_{y=1,ex}A} = \frac{(\pi-\alpha)}{(\pi-\gamma)}, \quad \frac{AC_{y=1,ex}}{C_{y=1,ex}B} = \frac{(\pi-\beta)}{(\pi-\alpha)}$$

and, therefore,

$$\frac{BA_{y=1,ex}}{A_{y=1,ex}C}\cdot\frac{CB_{y=1,ex}}{B_{y=1,ex}A}\cdot\frac{AC_{y=1,ex}}{C_{y=1,ex}B} = \frac{(\pi-\gamma)}{(\pi-\beta)}\cdot\frac{(\pi-\alpha)}{(\pi-\gamma)}\cdot\frac{(\pi-\beta)}{(\pi-\alpha)} = 1.$$

Consequently, for $y = 1$ the *exterior Hofstadter transversals* are concurrent and their point of concurrency is the "*exterior Hofstadter point* $H_{y=1}$".

Remark 3.5. *Although the trilinear coordinates of the three points $A_{y,ex}$, $B_{y,ex}$, $C_{y,ex}$ are not needed explicitly in the sequel, they can be determined very easily. For example, for the point $A_{y,ex}$ we know the ratio of the straight line segments $BA_{y,ex}$ and $A_{y,ex}C$ from (3.24) and also their sum $BA_{y,ex} + A_{y,ex}C = a$. Hence, $A_{y,ex}$ is the point that separates internally BC into the given ratio. Then, it can easily be found that*

$$BA_{y,ex} = ac\cdot\frac{\sin((\pi-\gamma)(1-y))\sin((\pi-\beta)y)}{S_{BC}}, \quad A_{y,ex}C = ab\cdot\frac{\sin((\pi-\beta)(1-y))\sin((\pi-\gamma)y)}{S_{BC}},$$
$$S_{\mathrm{BC}} = b\cdot\sin((\pi-\beta)(1-y))\sin((\pi-\gamma)y) + c\cdot\sin((\pi-\gamma)(1-y))\sin((\pi-\beta)y).$$

From the expressions for $BA_{y,ex}$ and $A_{y,ex}C$ the trilinear coordinates of $A_{y,ex}$ can be readily determined. Then, the other two points $B_{y,ex}$ and $C_{y,ex}$ can be determined cyclicly. The limiting cases for $y = 0$ and $y = 1$ can be found from (3.27) and (3.30) as well as from their corresponding cyclic expressions.

Remark 3.6. *It is worth to point out that as $y \to \left(\frac{\alpha}{\pi+\alpha}\right)^-$, the triangle $BCA_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-}$ tends to the "infinite half-strip" with base BC and sides straight semi-lines, with origins B and C, parallel to each other and forming with BC angles $\beta + (\pi-\beta)\frac{\alpha}{\pi+\alpha}$ and $\gamma + (\pi-\gamma)\frac{\alpha}{\pi+\alpha}$, respectively. On the other hand, if $y \to \left(\frac{\alpha}{\pi+\alpha}\right)^+$, the triangle $BCA_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^+}$ tends to the "complementary" to the previous "infinite half-strip" with base BC and sides straight semi-lines with origins B and C, parallel to each other and forming with BC angles $(\pi-\beta) - (\pi-\beta)\frac{\alpha}{\pi+\alpha} = \frac{(\pi-\beta)\pi}{\pi+\alpha}$ and $(\pi-\gamma) - (\pi-\gamma)\frac{\alpha}{\pi+\alpha} = \frac{(\pi-\gamma)\pi}{\pi+\alpha}$, respectively. These observations will enable us to define not only the ratios in (3.24) and (3.25), despite the fact that in the limit the triangles involved become infinite half-strips, but also to show that the limiting transversals of the "exterior triangles" **do** exist. So **do** their corresponding limiting "exterior Hofstadter points".*

Remark 3.7. *The straight line segments (y-transversals) $AA_{y,ex}$, $BB_{y,ex}$, $CC_{y,ex}$, for the y's examined, joining the vertices of the two triangles ΔABC and $\Delta A_{y,ex}B_{y,ex}C_{y,ex}$ are concurrent. Consequently, Desargues's Theorem 2.5 for these triangles applies. (Note: The triangle $\Delta A_{y,ex}B_{y,ex}C_{y,ex}$ is called "the first (exterior) companion to Hofstadter triangle".)*

3.3. **Exterior Hofstadter points.** To determine the trilinear coordinates of the "*exterior Hofstadter points H_y*" for all $y \in [0,1]\backslash\left\{\frac{\alpha}{\pi+\alpha}, \frac{\gamma}{\pi+\gamma}, \frac{\beta}{\pi+\beta}\right\}$ we follow the classical way, omitting for the time being the extreme points of the interval considered.
Each pair of triangles $\Delta ABA_{y,ex}$ and $\Delta AA_{y,ex}C$, as well as $\Delta H_yBA_{y,ex}$ and $\Delta H_yA_{y,ex}C$, share the same altitudes. Hence, their areas will be proportional to their bases. Having this in mind, denoting by $H_{y,\mathrm{BC}}$, $H_{y,\mathrm{CA}}$, $H_{y,\mathrm{AB}}$ the trilinear coordinates of the *exterior Hofstadter point H_y* and applying properties of equal ratios we can successively obtain

$$\frac{BA_{y,ex}}{A_{y,ex}C} = \frac{(ABA_{y,ex})}{(AA_{y,ex}C)} = \frac{(H_yBA_{y,ex})}{(H_yA_{y,ex}C)} = \frac{(ABA_{y,ex})-(H_yBA_{y,ex})}{(ABA_{y,ex})-(AA_{y,ex}C)} = \frac{(H_yAB)}{(H_yCA)} = \frac{c\cdot H_{y,\mathrm{AB}}}{b\cdot H_{y,\mathrm{CA}}}. \tag{3.31}$$

Equating the result found for the ratio $\frac{BA_{y,ex}}{A_{y,ex}C}$ in (3.24) and in the extreme right side of (3.31) above we obtain after some simple manipulations that

$$\frac{H_{y,\mathrm{CA}}}{\frac{\sin((\pi-\beta)(1-y))}{\sin((\pi-\beta)y)}} = \frac{H_{y,\mathrm{AB}}}{\frac{\sin((\pi-\gamma)(1-y))}{\sin((\pi-\gamma)y)}}.$$

From the above expression we can obtain the trilinear coordinates of $H_{y\in(0,1)\backslash\left\{\frac{\alpha}{\pi+\alpha},\frac{\gamma}{\pi+\gamma},\frac{\beta}{\pi+\beta}\right\}}$ which are as follows

$$\frac{\sin((\pi-\alpha)(1-y))}{\sin((\pi-\alpha)y)} : \frac{\sin((\pi-\beta)(1-y))}{\sin((\pi-\beta)y)} : \frac{\sin((\pi-\gamma)(1-y))}{\sin((\pi-\gamma)y)}. \tag{3.32}$$

From (3.32) we can readily determine the trilinear coordinates of the limiting "*exterior Hofstadter points $H_{y=0}$ and $H_{y=1}$*". To determine $H_{y=0}$, first we write the expressions in (3.32) as

$$\frac{\sin((\pi-\alpha)(1-y))}{\frac{\sin((\pi-\alpha)y)}{((\pi-\alpha)y)}((\pi-\alpha)y)} : \frac{\sin((\pi-\beta)(1-y))}{\frac{\sin((\pi-\beta)y)}{((\pi-\beta)y)}((\pi-\beta)y)} : \frac{\sin((\pi-\gamma)(1-y))}{\frac{\sin((\pi-\gamma)y)}{((\pi-\gamma)y)}((\pi-\gamma)y)}, \tag{3.33}$$

next we simplify by y, and then we take limits as $y \to 0^+$. As a result we have that the trilinear coordinates of $H_{y=0}$ are as follows

$$\frac{\sin\alpha}{\pi-\alpha} : \frac{\sin\beta}{\pi-\beta} : \frac{\sin\gamma}{\pi-\gamma}. \tag{3.34}$$

For the trilinear coordinates of $H_{y=1}$ we work in a similar way but this time first we write the trilinear coordinates (3.32) as

$$\begin{aligned} \frac{\frac{\sin((\pi-\alpha)(1-y))}{((\pi-\alpha)(1-y))}((\pi-\alpha)(1-y))}{\sin((\pi-\alpha)y)} &: \frac{\frac{\sin((\pi-\beta)(1-y))}{((\pi-\beta)(1-y))}((\pi-\beta)(1-y))}{\sin((\pi-\beta)y)} \\ &: \frac{\frac{\sin((\pi-\gamma)(1-y))}{((\pi-\gamma)(1-y))}((\pi-\gamma)(1-y))}{\sin((\pi-\gamma)y)}, \end{aligned} \tag{3.35}$$

next we simplify by $1-y$, and then take limits as $y \to 1^-$ to find

$$\frac{\pi-\alpha}{\sin\alpha} : \frac{\pi-\beta}{\sin\beta} : \frac{\pi-\gamma}{\sin\gamma}. \tag{3.36}$$

In view of Remark 3.6 let us see what happens to the "*exterior Hofstadter transversals*" and to the "*exterior Hofstadter points H_y*" when $y \to \left(\frac{\alpha}{\pi+\alpha}\right)^-$. From (3.24) considering the two extreme expressions, we get

$$\lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-} \frac{BA_{y,ex}}{A_{y,ex}C} = \lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-} \frac{c\,\sin((\pi-\gamma)(1-y))\sin((\pi-\beta)y)}{b\,\sin((\pi-\beta)(1-y))\sin((\pi-\gamma)y)} = \frac{c\,\sin\left(\frac{((\pi-\gamma)\pi)}{(\pi+\alpha)}\right)\sin\left(\frac{((\pi-\beta)\alpha)}{(\pi+\alpha)}\right)}{b\,\sin\left(\frac{((\pi-\beta)\pi)}{(\pi+\alpha)}\right)\sin\left(\frac{((\pi-\gamma)\alpha)}{(\pi+\alpha)}\right)},$$

while from (3.25) we get the following two equalities

$$\lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-} \frac{CB_{y,ex}}{B_{y,ex}A} = \lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-} \frac{a\,\sin((\pi-\alpha)(1-y))\sin(\pi-\gamma)y}{c\,\sin((\pi-\gamma)(1-y))\sin((\pi-\alpha)y} = \frac{a\,\sin\left(\frac{((\pi-\alpha)\pi)}{(\pi+\alpha)}\right)\sin\left(\frac{((\pi-\gamma)\alpha)}{(\pi+\alpha)}\right)}{c\,\sin\left(\frac{((\pi-\gamma)\pi)}{(\pi+\alpha)}\right)\sin\left(\frac{((\pi-\alpha)\alpha)}{(\pi+\alpha)}\right)},$$

$$\lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-} \frac{AC_{y,ex}}{C_{y,ex}B} = \lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-} \frac{b\,\sin((\pi-\beta)(1-y))\sin((\pi-\alpha)y)}{a\,\sin((\pi-\alpha)(1-y))\sin((\pi-\beta)y)} = \frac{b\,\sin\left(\frac{((\pi-\beta)\pi)}{(\pi+\alpha)}\right)\sin\left(\frac{((\pi-\alpha)\alpha)}{(\pi+\alpha)}\right)}{a\,\sin\left(\frac{((\pi-\alpha)\pi)}{(\pi+\alpha)}\right)\sin\left(\frac{((\pi-\beta)\alpha)}{(\pi+\alpha)}\right)}.$$

Using the above limiting results for $y \to \left(\frac{\alpha}{\pi+\alpha}\right)^-$ we have for the corresponding limiting "*exterior Hofstadter transversals*" that

$$\lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-} \left(\frac{BA_{y,ex}}{A_{y,ex}C}\cdot\frac{CB_{y,ex}}{B_{y,ex}A}\cdot\frac{AC_{y,ex}}{C_{y,ex}B}\right) = 1,$$

and by Ceva's Theorem 2.1 these limiting transversals are concurrent at the limiting "*exterior Hofstadter point $H_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-}$*" whose coordinates are found by considering the trilinear coordinates in (3.32) and taking limits for $y \to \left(\frac{\alpha}{\pi+\alpha}\right)^-$ we readily find that

$$\frac{\sin\left(\frac{((\pi-\alpha)\pi)}{(\pi+\alpha)}\right)}{\sin\left(\frac{((\pi-\alpha)\alpha)}{(\pi+\alpha)}\right)} : \frac{\sin\left(\frac{((\pi-\beta)\pi)}{(\pi+\alpha)}\right)}{\sin\left(\frac{((\pi-\beta)\alpha)}{(\pi+\alpha)}\right)} : \frac{\sin\left(\frac{((\pi-\gamma)\pi)}{(\pi+\alpha)}\right)}{\sin\left(\frac{((\pi-\gamma)\alpha)}{(\pi+\alpha)}\right)}. \tag{3.37}$$

Obviously, we obtain the same limits and the same trilinear coordinates for the "*exterior Hofstadter point $H_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^+}$*" meaning that there is **no** "*discontinuity*" in the exterior Hofstadter point as we go from $y \to \left(\frac{\alpha}{\pi+\alpha}\right)^-$ to $y \to \left(\frac{\alpha}{\pi+\alpha}\right)^+$ despite the fact that at this point there is a discontinuity for the vertex $A_{y=\frac{\alpha}{\pi+\alpha}}$.
It is understood that similar expressions for the concurrency of the three limiting "*exterior Hofstadter transversals for $y \to \left(\frac{\gamma}{\pi+\gamma}\right)^-$ and $y \to \left(\frac{\beta}{\pi+\beta}\right)^-$*" at the limiting "*exterior Hofstadter points $H_{y\to\left(\frac{\gamma}{\pi+\gamma}\right)^-}$, $H_{y\to\left(\frac{\beta}{\pi+\beta}\right)^-}$*", respectively, can be found. Here we omit these elements since they are trivially obtained.
Concluding the preceding analysis, besides the limiting "*exterior Hofstadter point*" when $y \to \left(\frac{\alpha}{\pi+\alpha}\right)$, the issue, as to what happens to the various limiting values may be raised. However, based on Remarks 3.5, 3.6, 3.7, as well as on the formulas in them, we can readily find out that nothing changes as regards these limiting values. Therefore, in the preceding study we may simply consider that $y \to \frac{\alpha}{\pi+\alpha}$. The same applies to the other two particular points $y = \frac{\gamma}{\pi+\gamma}, \frac{\beta}{\pi+\beta}$. This observation suggests that the trilinear coordinates of the three particular points that were excluded from our analysis can be incorporated into the trilinear coordinates in (3.32) and also in all the associated expressions in the aforementioned Remarks except in the trilinear coordinates of the vertices of the "*exterior Hofstadter triangle*" which are studied in the Appendix.

Figure 2 is similar to Figure 1 except that $y = \frac{\alpha}{\pi+\alpha}$ and so $A_{y \to \frac{\alpha}{\pi+\alpha}}$ tends to the point at infinity in the Projective Geometry sense. In the figure it was assumed that $y \to \left(\frac{\alpha}{\pi+\alpha}\right)^-$. If it is assumed that $y \to \left(\frac{\alpha}{\pi+\alpha}\right)^+$ then the arrow would be in the other half-plane of A *wrt* BC and in the opposite direction. In both cases the "*exterior Hofstadter point* $H_{y=\frac{\alpha}{\pi+\alpha}}$" would be the same as is depicted in the figure.

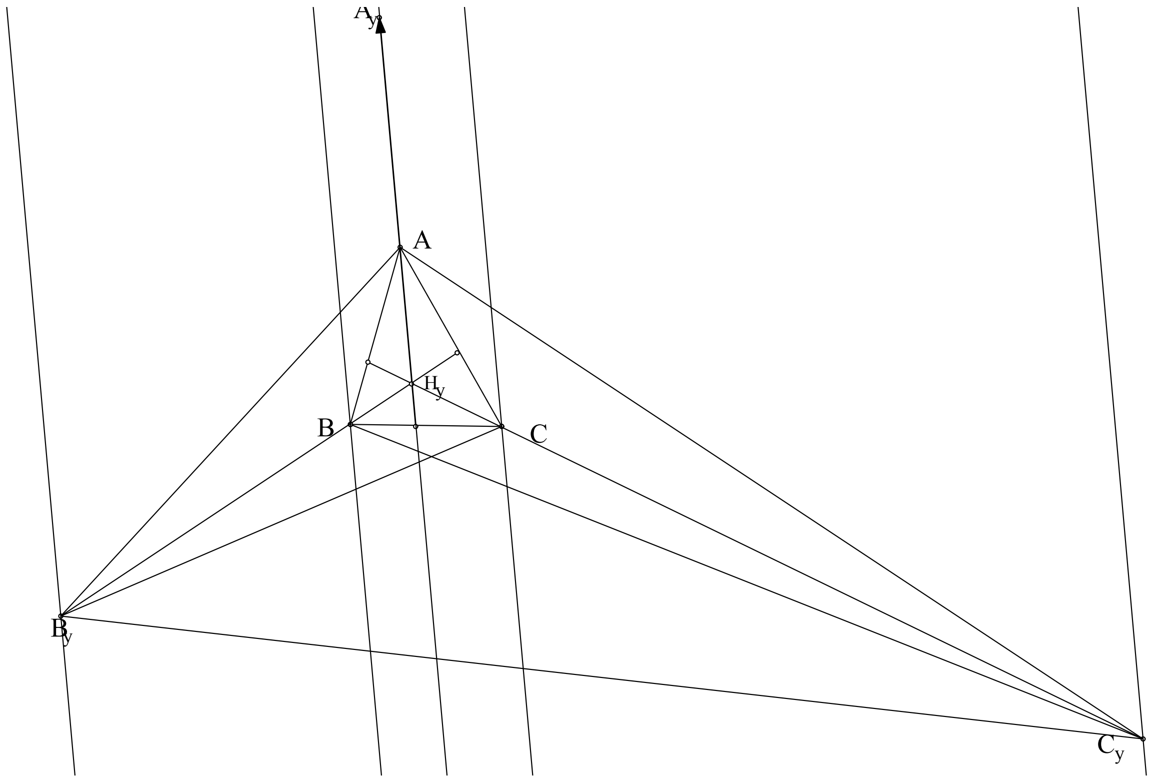

Figure 2. *Degenerate exterior Hofstadter triangle* $\Delta A_yB_yC_y$, *with its transversals* AA_y, BB_y, CC_y, *and Hofstadter point* H_y *in case* $y \to \left(\frac{\alpha}{\pi+\alpha}\right)^-$. $\left(\alpha = \frac{\pi}{4} < \gamma = \frac{\pi}{3} < \beta = \frac{5\pi}{12},\ y = \frac{1}{5} = \frac{\alpha}{\pi+\alpha}\right)$

All of the results of the "*exterior Hofstadter points* H_y" can be summarized in the following statement.

Theorem 3.8. *For any* $y \in [0,1]$, *the "exterior Hofstadter transversals"* $AA_{y,ex}$, $BB_{y,ex}$, $CC_{y,ex}$ *are concurrent at the "exterior Hofstadter point* H_y*" whose trilinear coordinates are given by the expressions*

$$\begin{array}{c} \frac{\sin\alpha}{\pi-\alpha} : \frac{\sin\beta}{\pi-\beta} : \frac{\sin\gamma}{\pi-\gamma} \ \textit{for}\ \ y = 0, \\ \frac{\sin((\pi-\alpha)(1-y))}{\sin((\pi-\alpha)y)} : \frac{\sin((\pi-\beta)(1-y))}{\sin((\pi-\beta)y)} : \frac{\sin((\pi-\gamma)(1-y))}{\sin((\pi-\gamma)y)} \ \textit{for}\ \ y \in (0,1), \\ \frac{\pi-\alpha}{\sin\alpha} : \frac{\pi-\beta}{\sin\beta} : \frac{\pi-\gamma}{\sin\gamma} \ \textit{for}\ \ y = 1. \end{array} \quad (3.38)$$

Moreover, continuity of the locus of $H_{y\in[0,1]}$, *called "exterior Hofstadter sectrix" is then guaranteed.*

Proof: In (3.38) above, the first line is nothing but the relations in (3.34), the second line refers to any $y \in (0,1)$ incorporating the three particular values $y = \frac{\alpha}{\pi+\alpha}, \frac{\gamma}{\pi+\gamma}, \frac{\beta}{\pi+\beta}$ in relations (3.32), and the third line is the relations in (3.36). □

3.4. **Exterior Hofstadter sectrices.** As has already been mentioned the "*exterior Hofstadter secrtix*" is the locus of the "*exterior Hofstadter points H_y*" as y increases continuously in the interval $[0,1]$. It is a continuous arc that is drawn by the middle ratio expressions in (3.8). It begins at the point $H_{y=0}$, with its trilinear coordinates being given by the first ratios in (3.8) and ends at the point $H_{y=1}$ with trilinear coordinates the third ratios in (3.8). Note that the latter ratio expressions constitute the inverses of the former ones.

By the way, if we are interested in the "*y-sectrices*" of the vertices of the "*exterior Hofstadter triangle $\Delta A_y B_y C_y$*", i.e., their loci, these are given by the trilinear coordinates of the points in (3.15)-(3.17) as y increases continuously in the interval $[0,1] \setminus \left\{\frac{\alpha}{\pi+\alpha}, \frac{\gamma}{\pi+\gamma}, \frac{\beta}{\pi+\beta}\right\}$. It is obvious that each of these three loci consists of two branches that will be studied analytically in the Appendix. For example, the first branch of the locus of vertex A_y of the triangle $\Delta A_y B_y C_y$ has as starting point $A_{y=0}$, given by the first ratios in (3.15), and is described by the first ratios in (3.16) for $y \in \left(0, \frac{\alpha}{\pi+\alpha}\right)$, while the second branch is described by the second ratios in (3.16) for $y \in \left(\frac{\alpha}{\pi+\alpha}, 1\right)$ and has as ending point $A_{y=1}$, given by the first trilinear coordinates in (3.17).

However, since the three particular points $y = \frac{\alpha}{\pi+\beta}, \frac{\alpha}{\pi+\beta}, \frac{\alpha}{\pi+\beta}$ have been excluded from the corresponding study in Section 3.1, this omission will be completed in the Appendix.

4. The pair-wise property of the exterior Hofstadter triangles and their elements

If we look very carefully into what we have found so far we see that it is not necessary to analyze and study the "*exterior Hofstadter triangles and their elements*" for all the values of $y \in [0,1]$. This becomes apparent because the formulas found, except in some extreme rare cases, are functions of both y and $y' = 1 - y$. Since $y + y' = 1$ it is implied that the determination of a certain "*exterior Hofstadter triangle*" or any element associated with it for $y \in \left[0, \frac{1}{2}\right)$ can directly yield another "*exterior Hofstadter triangle*" and its relevant elements provided we interchange the roles of of y and $y' = 1 - y \in \left(\frac{1}{2}, 1\right]$. For $y = \frac{1}{2} = y'$ the aforementioned two triangles coincide and produce as the exterior Hofstadter triangle the triangle of the excenters of the original triangle ΔABC, where the trilinear coordinates of its vertices were given in Remark 3.3. The transversals $A_{y=\frac{1}{2}=y'}A$, $B_{y=\frac{1}{2}=y'}B$, $C_{y=\frac{1}{2}=y'}C$ are the interior bisectors of the original triangle ΔABC and its Hofstadter point $H_{y=\frac{1}{2}=y'}$ is the incenter of the triangle ΔABC with trilinear coordinates $1:1:1$.

Also, the graph of a certain exterior Hofstadter triangle $\Delta A_y B_y C_y$ for $y \in (0, \frac{1}{2})$ produces almost automatically the Hofstadter triangle $\Delta A_{y'} B_{y'} C_{y'}$, with $y' = 1 - y$. In Figure 1 one may see the latter triangle $\Delta A_{y'} B_{y'} C_{y'}$ which corresponds to $y' = 1 - y = 1 - \frac{1}{10} = \frac{9}{10}$. In the same figure one may see the transversals and the Hofstadter point of the exterior Hofstadter triangle $\Delta A_{y=\frac{9}{10}} B_{y=\frac{9}{10}} C_{y=\frac{9}{10}}$.

In Figure 3 we have exactly the same original triangle ΔABC and its exterior one $\Delta A_{y=\frac{1}{10}} A_{y=\frac{1}{10}} A_{y=\frac{1}{10}}$ as in Figure 1. To the latter there corresponds a "*exterior pair-wise triangle*", namely

$\Delta A_{y'=\frac{9}{10}}B_{y'=\frac{9}{10}}C_{y'=\frac{9}{10}}$, with $y' = 1 - y = 1 - \frac{1}{10} = \frac{9}{10} \in \left(\frac{1}{2}, 1\right)$. The *"exterior pair-wise transversals"* of the latter triangle are $AA_{y'}$, $BB_{y'}$, $CC_{y'}$ and are concurrent at the *"exterior pair-wise Hofstadter point $H_{y'}$"*. Note that the angles

$$\langle BCD_1 = \langle ACD_2 = \frac{9(\pi-\gamma)}{10},\ \langle CAE_1 = \langle BAE_2 = \frac{9(\pi-\alpha)}{10},\ \langle ABF_1 = \langle CBF_2 = \frac{9(\pi-\beta)}{10}$$

. Hence, the following pairs of angles will be equal

$$\langle ABA_y = \langle CBC_y = \frac{\pi-\beta}{10},\ \langle BCB_y = \langle CAC_y = \frac{\pi-\gamma}{10},\ \langle CAC_y = \langle BAB_y = \frac{\pi-\alpha}{10}.$$

The last two series of equalities make the fivefold of points $(B_yF_1A_{y'}CD_1)$, $(C_yD_2A_{y'}BF_2)$, $(C_yD_1B_{y'}AE_1)$, $(A_yE_2B_{y'}CD_2)$, $(A_yE_1C_{y'}BF_1)$, $(B_yF_2C_{y'}AE_2)$ lie on straight lines.

In what follows we will refer to some extreme cases where the **pair-wise property** holds and others where it seems that it does not.

Looking in relationships (3.38) we see that the trilinear expressions of the exterior Hofstadter points for $y = 0$ and $y' = 1 - y = 1$ have ratios such that the former are the inverse of the latter and vice versa.

Something similar to the above does not seem to happen if we look at the trilinear coordinates of the vertices A_y of the exterior Hofstadter triangle $\Delta A_yB_yC_y$ for $y = 0$ and $y' = 1 - y = 1$ which are given in (3.15) and (3.17), respectively. However, this is not the case as we can see below.

For $y \to 0^+$ let the trilinear coordinates of $A_{y\to 0^+}$ be proportional to

$$\sin((\pi-\gamma)(1-y))\sin((\pi-\beta)(1-y)) : -\sin((\pi-\gamma)y)\sin((\pi-\beta)(1-y)) :$$
$$-\sin((\pi-\beta)y)\sin((\pi-\gamma)(1-y)).$$

Taking limits as $y \to 0^+$ we end up with

$$\sin\gamma\sin\beta : 0 : 0$$

which are proportional to

$$2R\sin\gamma\sin\beta : 0 : 0$$

that is the ratios in (3.15). On the other hand, for $y \to 1^-$ the trilinear coordinates of $A_{y\to 1^-}$ are proportional to

$$-\sin((\pi-\gamma)(1-y))\sin((\pi-\beta)(1-y)) : \sin((\pi-\gamma)y)\sin((\pi-\beta)(1-y)) :$$
$$\sin((\pi-\beta)y)\sin((\pi-\gamma)(1-y)).$$

Multiplying through by $\frac{(1-y)}{\sin((\pi-\gamma)(1-y))\sin((\pi-\beta)(1-y))}$ and making some rearrangements we obtain

$$-(1-y) : \frac{\sin((\pi-\gamma)y)}{\frac{\sin((\pi-\gamma)(1-y))}{(\pi-\gamma)(1-y)}(\pi-\gamma)} : \frac{\sin((\pi-\beta)y)}{\frac{\sin((\pi-\beta)(1-y))}{(\pi-\beta)(1-y)}(\pi-\beta)}.$$

Now, taking limits as $y \to 1^-$ we end up with

$$0 : \frac{\sin(\pi-\gamma)}{\pi-\gamma} : \frac{\sin(\pi-\beta)}{\pi-\beta} \quad \text{or even} \quad 0 : \frac{\pi-\beta}{\sin\beta} : \frac{\pi-\gamma}{\sin\gamma}$$

that is the ratios found in (3.17).

It is worth mentioning that this pair-wise property holds also in the case of the interior Hofstadter triangles and their elements.

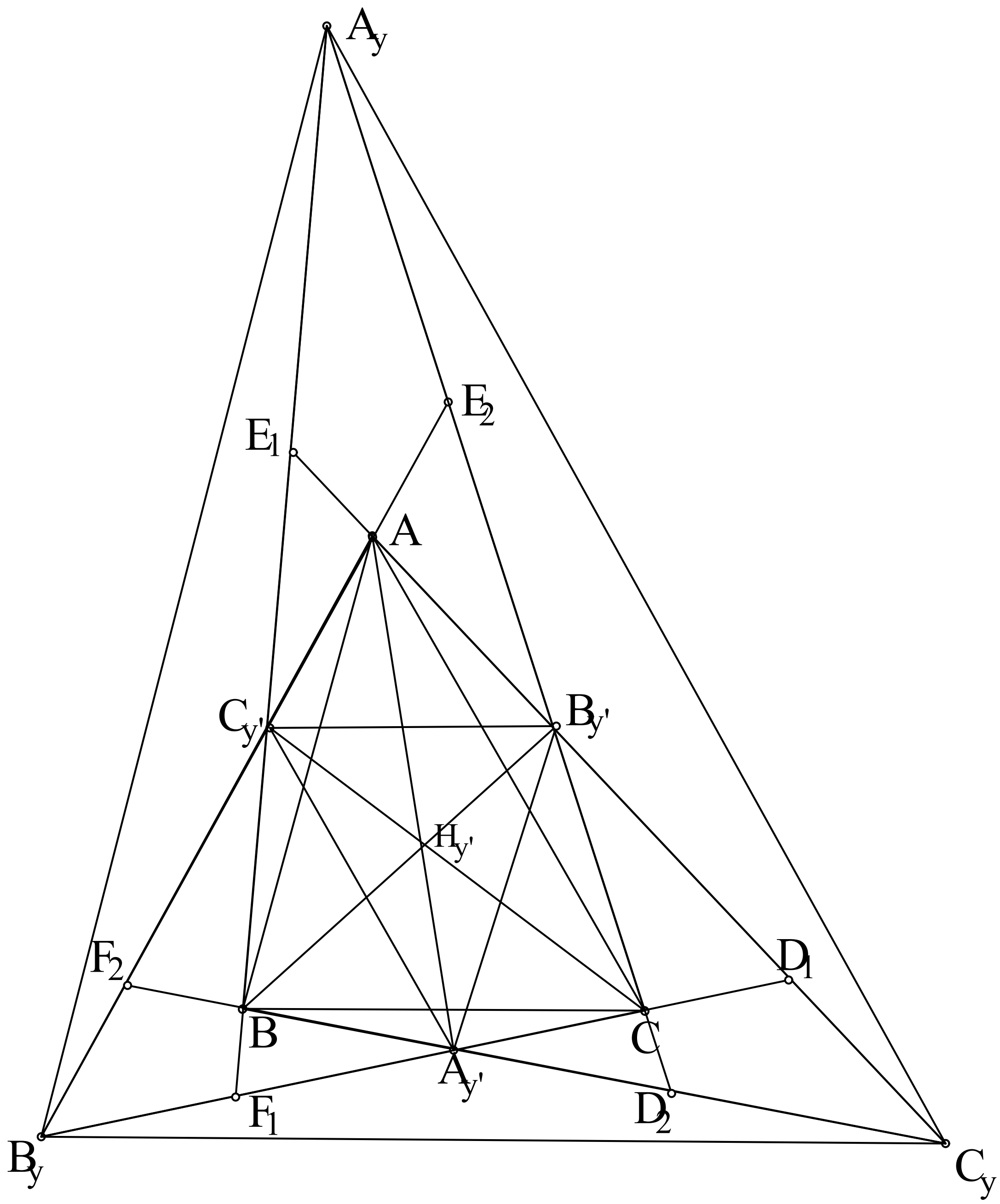

Figure 3. *Exterior pair-wise Hofstadter triangle* $\Delta A_{y'}B_{y'}C_{y'}$, *with its transversals* $AA_{y'}$, $BB_{y'}$, $CC_{y'}$, *and Hofstadter point* $H_{y'}$. $\left(\alpha = \frac{\pi}{4} < \gamma = \frac{\pi}{3} < \beta = \frac{5\pi}{12},\ y' = \frac{9}{10} = 1 - \frac{1}{10} = y\right)$

5. Concluding remarks and discussion

In the present work together with the Appendix that follows we have covered the case of the *"(exterior) Hofstadter triangles and their elements"*. The corresponding *"interior ones"* were covered in [4] and [2] and some more were also mentioned or introduced in this work. Furthermore, in the present work we incorporated Desargues's Theorem 2.5 which usually goes together with the three concurrent extended straight line segments $AA_{y,ex}$, $BB_{y,ex}$, $CC_{y,ex}$, joining the corresponding vertices of the pair of triangles ΔABC and $A_{y,ex}B_{y,ex}C_{y,ex}$. (*Note:* It is pointed out that the Desargues Theorem applies also to the corresponding interior transversals and their respective interior pairs of triangles as in the case of the exterior ones.)
Two more issues are given below.

(1) There was not any mention in the investigation of an isosceles or of an equilateral triangle since the results obtained in these cases are rather trivial and thus they have been omitted.
(2) Suppose that the three pairs of the originally considered straight semi-lines go on rotating about their poles and sweep first the vertically opposite angles of the interior angles of ΔABC, α, β, γ, and then the vertically opposite angles of the exterior angles $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$. Suppose also that the rotations are such that the former angles are swept as the respective interior ones were, i.e., at constant rates so that for any $x \in [0,1]$ the angles swept are αx, βx, γx, respectively, and then the latter angles are swept so that for any $y \in [0,1]$ the angles swept are $(\pi-\alpha)y$, $(\pi-\beta)y$, $(\pi-\gamma)y$ for any $y \in [0,1]$, respectively. Evidently, the graphs drawn will copy the "*interior* and the "*exterior*" "*Hofstadter elements*". Consequently, there will be a periodical phenomenon of period π which may be repeated indefinitely.

Acknowledgement: The author wishes to express his sincere thanks to his friend Professor Paris Pamfilos of the Mathematics and Applied Mathematics Department of the University of Crete who provided him with his new version of EucliDraw and explained to the author a number of issues in connection with the drawing of the figures.

Appendix

Appendix A. Particular cases

For our analysis to be complete we have to find out what happens to the vertices of the "*exterior Hofstadter triangle* $\Delta A_yB_yC_y$" and also to its "*twin triangle* $\Delta A'_yB'_yC'_y$", although we have not studied the latter in this work, when y tends to any of the three particular points $\frac{\alpha}{\pi+\alpha}$, $\frac{\gamma}{\pi+\gamma}$, $\frac{\beta}{\pi+\beta}$. For this it suffices to study only the case $y \to \frac{\alpha}{\pi+\alpha}$ since the conclusions for the other two will be obtained by a cyclic permutation. In addition, it is easy to see that for $y=\frac{\alpha}{\pi+\alpha}$, the points $B_{y=\frac{\alpha}{\pi+\alpha}}$ and $C_{y=\frac{\alpha}{\pi+\alpha}}$ and whatever is associated with them are well-defined. It is reminded that the straight semi-lines $BB_{y=\frac{\alpha}{\pi+\alpha}}$, $CC_{y=\frac{\alpha}{\pi+\alpha}}$ with origins B and C, respectively, are parallel to each other. So, for the time being we do not have to deal with them. Hence, what we want to find first is where each of the two points $A_{y=\frac{\alpha}{\pi+\alpha}}$ and $A'_{y=\frac{\alpha}{\pi+\alpha}}$ lies as the point $A_{y=\frac{\alpha}{\pi+\alpha},ex}$ was already determined in Remark 3.5.

For $y \to \frac{\alpha}{\pi+\alpha}$ we distinguish two cases: (a) $y \to \left(\frac{\alpha}{\pi+\alpha}\right)^-$ and (b) $y \to \left(\frac{\alpha}{\pi+\alpha}\right)^+$.
Case (a): Starting with the relation on the left of (3.4) we successively find that

$$\begin{aligned}
BA'_{\left(\frac{\alpha}{\pi+\alpha}\right)^-} &= \lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-} BA'_y = a\cdot \lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-} \frac{\sin((\pi-\gamma)(1-y))\cos((\pi-\beta)(1-y))}{\sin((\pi+\alpha)(1-y))} \\
&= a\cdot \frac{\sin\left(\frac{(\pi-\gamma)\pi}{\pi+\alpha}\right)\cos\left(\frac{(\pi-\beta)\pi}{\pi+\alpha}\right)}{\lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-}\sin((\pi+\alpha)(1-y))} \\
&\overset{(*)}{=} a\cdot \frac{\sin\left(\frac{(\pi-\gamma)\pi}{\pi+\alpha}\right)\cos\left(\frac{(\pi-\beta)\pi}{\pi+\alpha}\right)}{\sin(\pi^+)} \\
&= a\cdot \frac{\sin\left(\frac{(\pi-\gamma)\pi}{\pi+\alpha}\right)\cos\left(\frac{(\pi-\beta)\pi}{\pi+\alpha}\right)}{0^-} = -\infty.
\end{aligned} \tag{A.1}$$

It should be explained that to the right of $\overset{(*)}{=}$ we set symbolically $\sin(\pi^+)$ to show that the corresponding limit of the argument tends to π from the right and for the same reason we set in the denominator of the next equal expression 0^- to denote that the corresponding limit tends to 0 from the left. Finally, in the extreme right side the final result was $-\infty$ because $\cos\left(\frac{(\pi-\beta)\pi}{\pi+\alpha}\right) > 0$ since $\frac{(\pi-\beta)\pi}{\pi+\alpha} \in \left(0, \frac{\pi}{2}\right)$.
Working in the same way we can find that

$$A'_{\left(\frac{\alpha}{\pi+\alpha}\right)^-}C = \lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-} A'_yC = a\cdot \frac{\sin\left(\frac{(\pi-\beta)\pi}{\pi+\alpha}\right)\cos\left(\frac{(\pi-\gamma)\pi}{\pi+\alpha}\right)}{\sin(\pi^+)} = +\infty, \tag{A.2}$$

because this time $\sin(\pi^-) = 0^+$ and $\cos\left(\frac{(\pi-\gamma)\pi}{\pi+\alpha}\right) < 0$ because $\frac{(\pi-\gamma)\pi}{\pi+\alpha} \in \left(\frac{\pi}{2}, \pi\right)$.
When the extended BC, from B to C, is taken as the positive semi-axis, the two results found for $BA'_{\left(\frac{\alpha}{\pi+\alpha}\right)^-} = -\infty$ and $A'_{\left(\frac{\alpha}{\pi+\alpha}\right)^-}C = +\infty$ show that the point A'_y goes to $-\infty$, as $y \to \frac{\alpha}{\pi+\alpha}$ from the left. Below we show that despite the infinites involved the sum $BA'_{\left(\frac{\alpha}{\pi+\alpha}\right)^-} + A'_{\left(\frac{\alpha}{\pi+\alpha}\right)^-}C$ remains constant and equal to $BC = a$ as is expected.
From (A.1) and (A.2) we can successively obtain

$$\begin{aligned}
BA'_{\left(\frac{\alpha}{\pi+\alpha}\right)^-} + A'_{\left(\frac{\alpha}{\pi+\alpha}\right)^-}C &= a\cdot\left(\lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-} BA'_y + \lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-} A'_yC\right) \\
&= \lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-}\left(BA'_y + A'_yC\right) \\
&= a\cdot \lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-}\left(\frac{\sin((\pi-\gamma)(1-y))\cos((\pi-\beta)(1-y))}{\sin((\pi+\alpha)(1-y))}\right. \\
&+ \left.\frac{\sin((\pi-\beta)(1-y))\cos((\pi-\gamma)(1-y))}{\sin((\pi+\alpha)(1-y))}\right) \\
&= a\cdot \lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-}\frac{\sin((\pi+\alpha)(1-y))}{\sin((\pi+\alpha)(1-y))} \\
&= a\cdot \lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^-}(1) = a\cdot 1 = a > 0.
\end{aligned} \tag{A.3}$$

Case (b): We work in exactly the same way as in Case (a) except that now we have that $y \to \left(\frac{\alpha}{\pi+\alpha}\right)^+$. So (A.1), (A.2), and (A.3) will eventually give the following results corresponding one by one to those we have found.

$$\begin{array}{rcccl} BA'_{\left(\frac{\alpha}{\pi+\alpha}\right)^+} & = & a\cdot\dfrac{\sin\left(\frac{(\pi-\gamma)\pi}{\pi+\alpha}\right)\cos\left(\frac{(\pi-\beta)\pi}{\pi+\alpha}\right)}{\sin(\pi^-)} & = & +\infty, \\ A'_{\left(\frac{\alpha}{\pi+\alpha}\right)^+}C & = & a\cdot\dfrac{\sin\left(\frac{(\pi-\beta)\pi}{\pi+\alpha}\right)\cos\left(\frac{(\pi-\gamma)\pi}{\pi+\alpha}\right)}{\sin(\pi^-)} & = & -\infty, \\ BA'_{\left(\frac{\alpha}{\pi+\alpha}\right)^+}+A'_{\left(\frac{\alpha}{\pi+\alpha}\right)^+}C & = & a\cdot\lim_{y\to\left(\frac{\alpha}{\pi+\alpha}\right)^+}(1) & = & a>0. \end{array} \tag{A.4}$$

So, as y approaches $\frac{\alpha}{\pi+\alpha}$ from the right, the point A'_y goes to $+\infty$.

As a conclusion we may say that when y passes through $\frac{\alpha}{\pi+\alpha}$ from smaller values to larger ones the vertex A'_y of the "*exterior twin Hofstadter triangle* $A'_yB'_yC'_y$" makes an infinite leap and goes from $-\infty$ to $+\infty$ on the axis whose origin is B and positive direction that from B to C. We may also say that the cartesian coordinates of A'_y just before the infinite leap takes place are $(-\infty, 0)$ and just after the leap $(+\infty, 0)$. On the other hand, the corresponding vertex A_y of the "*exterior Hofstadter triangle* $A_yB_yC_y$", whose projection onto BC is A'_y, lies in the extended straight semi-line $A_{y=\frac{\pi}{\pi+\alpha},ex}A$, with origin $A_{y=\frac{\pi}{\pi+\alpha},ex}$, and has limiting coordinates $(-\infty, +\infty)$ as $y \to \left(\frac{\pi}{\pi+\alpha}\right)^-$ along the above straight semi-line. Then, as $y = \frac{\pi}{\pi+\alpha}$ goes on increasing, $y \to \left(\frac{\pi}{\pi+\alpha}\right)^+$, A_y makes an infinite leap and reappears on the extended straight semi-line $A_{y=\frac{\pi}{\pi+\alpha},ex}A$ with origin $A_{y=\frac{\pi}{\pi+\alpha},ex}$ and has limiting coordinates $(+\infty, -\infty)$.

From the discussion so far it has become clear that the extended straight line $A_{\left(\frac{\pi}{\pi+\alpha}\right),ex}A$ constitutes an asymptote of the two branches of the arcs that are drawn from the point A_y when $y \in \left(0, \left(\frac{\pi}{\pi+\alpha}\right)^-\right)$ and $y \in \left(\left(\frac{\pi}{\pi+\alpha}\right)^+, 1\right)$, respectively. Obviously, the former branch lies in the same half-plane with the vertex A *wrt* to the extended straight line BC and begins at the point $2R\sin\beta\sin\gamma : 0 : 0$ while the latter one lies in the other half-plane *wrt* BC and ends at the point $0 : \frac{\pi-\beta}{\sin\beta} : \frac{\pi-\gamma}{\sin\gamma}$.

As in the case of A'_y and A_y, something similar can be said for the corresponding vertices C'_y and C_y, B'_y and B_y of the "*exterior twin Hofstadter triangle*" and of the "*exterior Hofstadter triangle*". The only difference will be that the particular points $y = \frac{\gamma}{\pi+\gamma}$ and $y = \frac{\beta}{\pi+\beta}$, as well as all the entities associated with them, will replace the ones associated with the particular point $y = \frac{\alpha}{\pi+\alpha}$ whose study of the behavior and its associated vertices A'_y of the "*exterior twin Hofstadter triangle*" and A_y of the "*exterior Hofstadter triangle*" has been studied exhaustively in the present section.

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