



DARBOUX CURVES IN THREE DIMENSIONAL WALKER MANIFOLD

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ABSTRACT. In this paper, we investigate the characterization for darboux curves in a strict Walker 3-manifold and a classification for ruled surface, which is generated by darboux curve in Walker 3-manifold is obtained. An example is given at the end.

1. INTRODUCTION AND MOTIVATIONS

The study of submanifolds of a given ambient space is a naturel interesting problem which enriches our knowledge and understanding of the geometry of the space itself. Here the ambient space we will consider is a Lorentzian three-manifold admitting a parallel null vector field called strict Walker manifold. It is known that Walker metrics have served as a powerful tool of constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics. For details see [1, 2].

Darboux curves in the Euclidean space were studied by Saban [9] and were generalised by Ergin [4]. For a curve α on a surface in the Euclidean 3-space the function $\mathbb{D} = \langle \alpha''', U \rangle = \kappa'_n - \kappa_g \tau_g$ is called Darboux function of α where U is normal vector field of surface, κ_n , κ_g and τ_g are normal curvature, geodesic curvature and geodesic torsion. Darboux function is equal to zero for Darboux curves. For more informations about Darboux curves see [6, 8].

In Minkowski 3-space timelike Darboux curves on a timelike surface were studied by Ergin [5]. In [7], Surođlu et al. study and classify modified translation surfaces in $Heis3$ and investigate conditions of being minimal surface. Also, they obtain the characterizations of points on this surface.

The study of ruled surfaces of a given ambient space is a naturel and interesting problem. A surface S in a manifold M is said to be ruled if every point of S is on (a open geodesic segment) in M that lies in S (see [3]). Locally a ruled surface is made by a one parameter family of geodesic segments.

In this paper we study the darboux curves in a three dimensional Walker manifold. We give the characterization for darboux curves in a strict Walker 3-manifold and a classification for ruled surface, which is generated by darboux curve in Walker 3-manifold is obtained.

The paper is organised as follow: in section 2 we give some preliminaries results about

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Walker 3-manifold and basic notions about curves lying on a surfaces. In section 3 we give the main results of this paper with an example for illustrate the obtained results.

2. PRELIMINARIES

A Walker n -manifold is a pseudo-Riemannian manifold, which admits a field of null parallel r -planes, with $r \leq \frac{n}{2}$. The canonical forms of the metrics were investigated by A. G. Walker ([1]). Walker has derived adapted coordinates to a parallel plan field. Hence, the metric of a three-dimensional Walker manifold (M, g_f^ϵ) with coordinates (x, y, z) is expressed as

$$g_f^\epsilon = dx \circ dz + \epsilon dy^2 + f(x, y, z) dz^2 \quad (2.1)$$

and its matrix form as

$$g_f^\epsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & f \end{pmatrix} \quad \text{with inverse} \quad (g_f^\epsilon)^{-1} = \begin{pmatrix} -f & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

for some function $f(x, y, z)$, where $\epsilon = \pm 1$ and thus $D = \text{Span} \partial_x$ as the parallel degenerate line field. Notice that when $\epsilon = 1$ and $\epsilon = -1$ the Walker manifold has signature $(2, 1)$ and $(1, 2)$ respectively, and therefore is Lorentzian in both cases.

It follows after a straightforward calculation that the Levi-Civita connection of any metric (2.1) is given by:

$$\begin{aligned} \nabla_{\partial_x} \partial_z &= \frac{1}{2} f_x \partial_x, & \nabla_{\partial_y} \partial_z &= \frac{1}{2} f_y \partial_x, \\ \nabla_{\partial_z} \partial_z &= \frac{1}{2} (f f_x + f_z) \partial_x + \frac{1}{2} f_y \partial_y - \frac{1}{2} f_x \partial_z \end{aligned} \quad (2.2)$$

where ∂_x , ∂_y and ∂_z are the coordinate vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$, respectively. Hence, if (M, g_f^ϵ) is a strict Walker manifolds i.e., $f(x, y, z) = f(y, z)$, then the associated Levi-Civita connection satisfies

$$\nabla_{\partial_y} \partial_z = \frac{1}{2} f_y \partial_x, \quad \nabla_{\partial_z} \partial_z = \frac{1}{2} f_z \partial_x - \frac{\epsilon}{2} f_y \partial_y. \quad (2.3)$$

Note that the existence of a null parallel vector field (i.e $f = f(y, z)$) simplifies the non-zero components of the Christoffel symbols and the curvature tensor of the metric g_f^ϵ as follows:

$$\Gamma_{23}^1 = \Gamma_{32}^1 = \frac{1}{2} f_y, \quad \Gamma_{33}^1 = \frac{1}{2} f_z, \quad \Gamma_{33}^2 = -\frac{\epsilon}{2} f_y \quad (2.4)$$

Starting from local coordinates (x, y, z) for which (2.1) holds, it is easy to check that

$$e_1 = \partial_y, \quad e_2 = \frac{2-f}{2\sqrt{2}} \partial_x + \frac{1}{\sqrt{2}} \partial_z, \quad e_3 = \frac{2+f}{2\sqrt{2}} \partial_x - \frac{1}{\sqrt{2}} \partial_z \quad (2.5)$$

are local pseudo-orthonormal frame fields on (M, g_f^ϵ) , with $g_f^\epsilon(e_1, e_1) = \epsilon$, $g_f^\epsilon(e_2, e_2) = 1$ and $g_f^\epsilon(e_3, e_3) = -1$. Thus the signature of the metric g_f^ϵ is $(\epsilon, 1, -1)$.

Let now u and v be two vectors in M . Denoted by $(\vec{i}, \vec{j}, \vec{k})$ the canonical frame in \mathbb{R}^3 .

The vector product of u and v in (M, g_f^ϵ) with respect to the metric g_f^ϵ is the vector denoted by $u \times v$ in M defined by

$$g_f^\epsilon(u \times v, w) = \det(u, v, w) \quad (2.6)$$

for all vector w in M , where $\det(u, v, w)$ is the determinant function associated to the canonical basis of \mathbb{R}^3 . If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ then by using (2.6), we have:

$$u \times v = \left(\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} - f \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \right) \vec{i} - \epsilon \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \vec{k} \quad (2.7)$$

Lemma 2.1. *The Walker cross product in M has the following properties:*

- (1) *The Walker cross product is bilinear and anti-symmetric.*
- (2) *$X \times_f Y$ is perpendicular both of X and Y .*
- (3) *The frame defined in (2.5) verify the following: $e_1 \times_f e_2 = -e_3$, $e_2 \times_f e_3 = -e_1$ and $e_3 \times_f e_1 = e_2$.*

Let $\alpha : I \subset \mathbb{R} \rightarrow (M, g_f^\epsilon)$ be a curve parametrized by its arc-length s .

The Frenet frame of α is the vectors T , N and B along α where T is the tangent, N the principal normal and B the binormal vector. They satisfied the Frenet formulas

$$\begin{cases} \nabla_T T &= \epsilon_2 \kappa N \\ \nabla_T N &= -\epsilon_1 \kappa T - \epsilon_3 \tau B \\ \nabla_T B &= \epsilon_2 \tau N \end{cases} \quad (2.8)$$

where κ and τ are respectively the curvature and the torsion of the curve α , with $\epsilon_1 = g_f(T; T)$; $\epsilon_2 = g_f(N; N)$ and $\epsilon_3 = g_f(B, B)$.

Now let $\alpha : I \subset \mathbb{R} \rightarrow (M, g_f^\epsilon)$ be a timelike curve lying in a surface S timelike in M . Let U be the unit normal of S , the Darboux frame is given by $\{T, Y, U\}$, where T is the tangent vector of the curve $\alpha(s)$ and $Y = U \times T$.

The usual transformations between the Walker Frenet frame and the Darboux takes the form

$$Y = \cos \theta N - \sin \theta B \quad (2.9)$$

$$U = \sin \theta N + \cos \theta B, \quad (2.10)$$

where θ is an angle between the surface normal vector N and the binormal vector B of α . Derivating Y along the curve alpha we get

$$\nabla_T Y = \cos \theta \nabla_T N - \theta' \sin \theta N - \sin \theta \nabla_T B - \theta' \cos \theta B.$$

Using the Frenet equation in (2.7) we have

$$\nabla_T Y = \cos \theta (-\epsilon_1 \kappa T - \epsilon_3 \tau B) - \theta' \sin \theta N - \sin \theta (\epsilon_2 \tau N) - \theta' \cos \theta B.$$

Now we suppose that the principal normal and the binormal have the same sign. then we get

$$\nabla_T Y = -\epsilon_1 \kappa \cos \theta T - (\theta + \epsilon_2 \tau) U \quad (2.11)$$

The same calculus gives

$$\nabla_T U = -\epsilon_1 \kappa \sin \theta T + (\theta + \epsilon_2 \tau) Y. \quad (2.12)$$

Then the Walker Darboux equation is expressed as

$$\begin{cases} \nabla_T T = -\epsilon_2 \kappa_g Y - \epsilon_2 \kappa_n U \\ \nabla_T Y = \kappa_g T - \tau_g U \\ \nabla_T U = \kappa_n T + \tau_g Y, \end{cases} \quad (2.13)$$

where κ_g , κ_n and τ_g are the geodesic curvature, normal curvature and geodesic torsion of $\alpha(s)$ on S , respectively. Also, (2.13) implies

$$\begin{aligned} \kappa_g &= \kappa \cos \theta, & \kappa_n &= \kappa \sin \theta \\ \kappa^2 &= \kappa_g^2 + \kappa_n^2, & \tau &= \tau_g - \frac{\kappa_g \kappa_n' - \kappa_n \kappa_g'}{\kappa^2}. \end{aligned} \quad (2.14)$$

3. MAIN RESULTS

Let S be a spacelike regular surface and $\alpha : I \subset \mathbb{R} \rightarrow M_f$ be a unit speed curve lying on the surface S .

Theorem 3.1. *Let $\alpha : I \subset \mathbb{R} \rightarrow M_f$ be a unit speed curve in a Walker 3-manifold. If α is a Darboux curve on the surface S , then*

$$\epsilon_2 \kappa^2 \kappa_n + 2\kappa_g' \tau_g + \tau_g^2 \kappa_n + \kappa_g \tau_g' - \kappa_n'' = 0. \quad (3.1)$$

Proof. The function \mathbb{D} of a Darboux curve on the surface S is given by

$$\mathbb{D} = g^\epsilon(\nabla_T^3 T, U), \quad (3.2)$$

where g^ϵ is the Walker metric of M_f .

Now let us compute $\nabla_T^3 T$.

Using the equation in (2.13), we get

$$\nabla_T(\nabla_T T) = -\epsilon_2 \kappa^2 T - \epsilon_2(\kappa_g' + \tau_g \kappa_n)Y + \epsilon_2(\tau_g \kappa_g - \kappa_n')U. \quad (3.3)$$

Differentiating (3.3) with respect to s and using (2.13), we get

$$\begin{aligned} \nabla_T^3 T &= -\epsilon_2(\kappa \kappa' + \kappa_g' \kappa_g + \kappa_n' \kappa_n)T \\ &\quad + (\kappa^2 \kappa_g - \epsilon_2 \kappa_g'' - \epsilon_2 \tau_g' \kappa_n - 2\tau_g \kappa_n' + \epsilon_2 \kappa_g \tau_g^2)Y \\ &\quad + (\kappa^2 \kappa_n + 2\epsilon_2 \kappa_g' \tau_g + \epsilon_2 \tau_g^2 \kappa_n + \epsilon_2 \kappa_g \tau_g' - \epsilon_2 \kappa_n'')U. \end{aligned} \quad (3.4)$$

Finally combine the equations (3.2) and (3.4) and we get

$$\mathbb{D} = \kappa^2 \kappa_n + 2\epsilon_2 \kappa_g' \tau_g + \epsilon_2 \tau_g^2 \kappa_n + \epsilon_2 \kappa_g \tau_g' - \epsilon_2 \kappa_n''$$

and that end the proof. \square

Proposition 3.1. *Let α be a unit speed curve and S be a ruled surface in (M_f, g_f^ϵ) which is parametrized as*

$$S(s, u) = \alpha(s) + uT(s). \quad (3.5)$$

If α is a darboux curve, then

$$-\epsilon_2 \kappa^2 \kappa_g + \kappa_g'' + \tau_g' \kappa_n + 2\epsilon_2 \tau_g \kappa_n' - \kappa_g \tau_g^2 = 0 \quad (3.6)$$

Proof. We recall that the normal unit Y of the surface S is given by the formula

$$U = \frac{S_s \times S_u}{\|S_s \times S_u\|}, \quad (3.7)$$

where \times is the vector product defined in (2.7).

From equation (3.5) we have

$$\begin{aligned} S_s(s, u) &= T(s) + u\nabla_T T \\ &= (1 - \epsilon_2 \kappa_g u)T - \epsilon_2 \kappa_n U \end{aligned}$$

and

$$S_u(s, u) = T(s).$$

By the vector product defined in (2.7), we have

$$S_s \times S_u = \epsilon_2 \kappa_n Y \quad (3.8)$$

and

$$\|S_s \times S_u\| = \epsilon \kappa_n. \quad (3.9)$$

From equations (3.8)-(3.9) and the formula (3.7), the unit normal vector field of the surface S is

$$U = -\epsilon_2 Y. \quad (3.10)$$

If α is a darboux curve, from (3.2)-(3.3) and (3.10), we have (3.6). \square

Example 3.1. We will consider that the function $f = f(s, u)$ wich defines the geometry of the strict Walker manifold is given by

$$f(s) = -2ae^{-2s}, \quad a \in \mathbb{R}, \quad 0 < a < 1. \quad (3.11)$$

We consider the curve α given by

$$\alpha(s) = (-ae^{-s}, s, e^s), \quad s \in \mathbb{R}. \quad (3.12)$$

So we have

$$T(s) = (ae^{-s}, 1, e^s), \quad s \in \mathbb{R}. \quad (3.13)$$

An easy computation show that $g_f^\epsilon(\alpha'(s), \alpha'(s)) = \epsilon = \epsilon_1$.

From (3.12) and (3.13), the ruled surface which is parametrized in (3.5) is

$$S(s, u) = (ae^{-s}(u-1), s+u, e^s(u+1)). \quad (3.14)$$

The unit normal vector field of the surface S is

$$U = \epsilon \epsilon_u \left(-\frac{3}{2}e^{-s}, \epsilon, -\frac{1}{2a}e^s \right). \quad (3.15)$$

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