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TWO TRIPLES OF LINES RELATED TO STEINER'S ELLIPSE

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ABSTRACT. In this article we explore the role of two triples of lines naturally related to Steiner's point, the in- and circum- Steiner ellipses, the Brocard points, the Brocard triangles, and the Euler line of the triangle.

1. INTRODUCTION

The aim of this article is to explore the role of two triples of lines I encountered in a recent study ([1]) involving *"Stother's quintic"*, a curve of the fifth degree ([2], [3, p. 219]), (See Figure 1). The first triple of lines appears in a natural procedure of the determination of the *"nodal tangents"* at the centroid G of the triangle of reference ABC, which is a singular point of the quintic. In barycentric coordinates or *"barycentrics"* $\{u, v, w\}$ relative

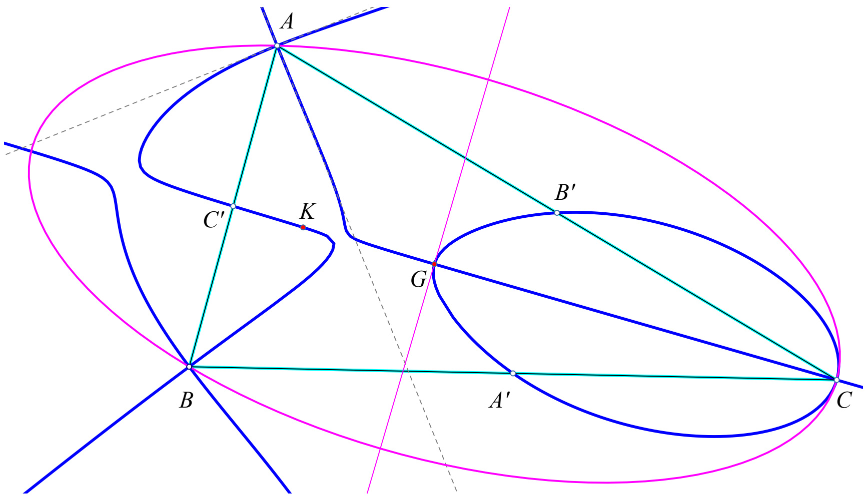

Figure 1. Stother's quintic

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to $\triangle ABC$ ([4, p.25], [5]), which I use throughout this article, the quintic is defined by the equation

$$a^2vw(v-w)(vw-u^2)+b^2wu(w-u)(wu-v^2)+c^2uv(u-v)(uv-w^2)=0, \qquad (1.1)$$

$\{a=|BC|, b=|CA|, c=|AB|\}$ denoting the side-lengths of the triangle. Besides the centroid G, at which the nodal tangents coincide with the axes of the *"Steiner ellipse"* ([6, p.383]), the ellipse circumscribing $\triangle ABC$ with center G, the quintic has singular points also at the vertices of the triangle, the nodal tangents there coinciding with the bisectors of the triangle's angles. The first partial derivatives of the function $f(u,v,w)$ on the left hand side of equation (1.1) vanish at the singular points $\{A,B,C,G\}$ and the *"Hessian matrix"* of the second derivatives at $G(1,1,1)$ is

$$H_f(G) = \begin{pmatrix} \frac{\partial^2 f}{\partial u^2} & \frac{\partial^2 f}{\partial v \partial u} & \frac{\partial^2 f}{\partial w \partial u} \\ \frac{\partial^2 f}{\partial u \partial v} & \frac{\partial^2 f}{\partial v^2} & \frac{\partial^2 f}{\partial w \partial v} \\ \frac{\partial^2 f}{\partial u \partial w} & \frac{\partial^2 f}{\partial v \partial w} & \frac{\partial^2 f}{\partial w^2} \end{pmatrix}(G) = 2\begin{pmatrix} b^2-c^2 & a^2-b^2 & c^2-a^2 \\ a^2-b^2 & c^2-a^2 & b^2-c^2 \\ c^2-a^2 & b^2-c^2 & a^2-b^2 \end{pmatrix}. \qquad (1.2)$$

The first triple $\{\nu_1,\nu_2,\nu_3\}$ of our study are lines with coefficients corresponding to the three rows of this symmetric and singular matrix. They are permutations of the coefficients of the line

$$\xi_1 \;:\; (b^2-c^2)u \;+\; (c^2-a^2)v \;+\; (a^2-b^2)w \;=\; 0,$$

which is the *"trilinear polar"* or *"tripolar"* of the *"Steiner point"* S of $\triangle ABC$. Latter is the fourth intersection point of Steiner's ellipse and the circumcircle, with barycentrics:

$$S \;\cong\; \left(\frac{1}{b^2-c^2} \;:\; \frac{1}{c^2-a^2} \;:\; \frac{1}{a^2-b^2}\right). \qquad (1.3)$$

The second triple $\{\xi_1,\xi_2,\xi_3\}$ of our study are lines resulting from ξ_1 by cyclically permuting its coefficients. Figure 2 shows the configuration of the two triples of lines, the

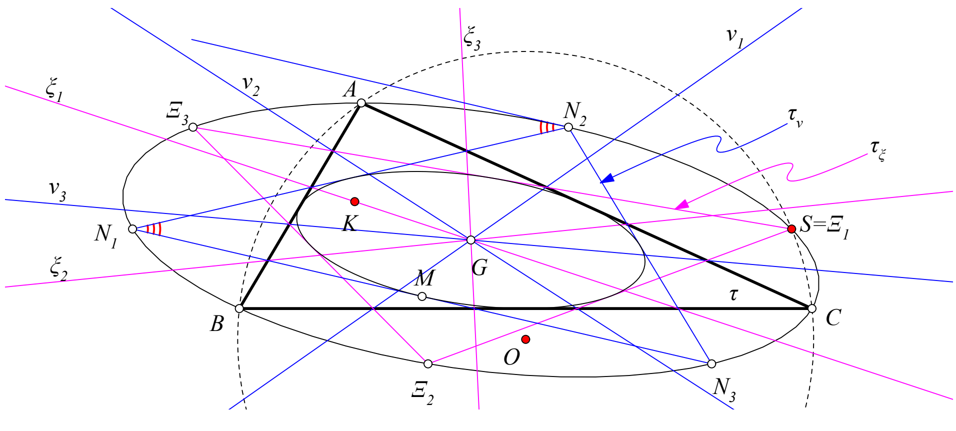

Figure 2. The Steiner ellipses and the two triples $\{\nu_1,\nu_2,\nu_3\}$ and $\{\xi_1,\xi_2,\xi_3\}$

Steiner in- and circum- ellipse, suggesting also some properties to be discussed below. The vertices of the triangles $\{\tau_\nu = N_1N_2N_3, \tau_\xi = \Xi_1\Xi_2\Xi_3\}$ are respectively the *"trilinear poles"* or *"tripoles"* of the corresponding lines with the same labels.

These triangles, as well as the triangle of reference $\tau = ABC$, are simultaneously inscribed in the Steiner ellipse and circumscribed to the *"Steiner in-ellipse"*, which is homothetic to the preceding w.r.t. their common center G by the factor $1/2$. These triangles touch the Steiner in-ellipse at the middles M of their sides, which are parallel to the tangents of the Steiner ellipse at the opposite vertices. The tripoles of all the lines which pass through the centroid lie on the Steiner ellipse. The Steiner in-ellipse is tangent to the tripolars of all *"points at infinity"*, i.e. points satisfying $u + v + w = 0$.
Regarding the background of the discussed material, we could say that it touches several ideas pertaining to *triangle geometry*. Definitions and fundamental properties of the shapes involved in this area can be found in the books by Gallatly [7], Court [8], Casey [9], Johnson [10] and Yiu [4]. For general methods and formulas in barycentrics the books by Loney [11, vol.II], Montesdeoca [12] and Kimberling's list [13] of *"triangle centers"* are also useful. In some places appear well known *triangle centers* in Kimberling's notation $X(n)$ ([13]).
Regarding the organization, in section 2 we discuss the connection of the lines $\{\nu_1, \nu_2, \nu_3\}$ to the axes of the Steiner ellipse. In section 3 we study general properties of triples of lines passing through the centroid and resulting from each other by cyclic permutations of their coefficients, our triples being special cases of such a structure. In section 4 we study a structure related to the Steiner point and a projectivity mapping the Steiner ellipse onto the circumcircle. In section 5 we study the relation of triangles inscribed in Steiner's ellipse and circumscribed to Steiner's in-ellipse and their images via the projectivity f of the preceding section, detecting also a quadrangle circumscribing both the Steiner in-ellipse and the Brocard ellipse of $\triangle ABC$. In section 6 we apply the results of the preceding sections to the study of two pairs of triangles naturally associated to the two triples of lines $\{\nu_1, \nu_2, \nu_3\}$ and $\{\xi_1, \xi_2, \xi_3\}$. In section 7 we study the relations of the pairs of triangles of the preceding section to the two Brocard triangles of $\triangle ABC$. In section 8 we study some relations of the tripolar ξ_1 of the Steiner point to the Steiner ellipse and the relations of the triple $\{\xi_1, \xi_2, \xi_3\}$ to the Brocard points of the triangle. Finally, in section 9 we study the relation of the triple $\{\xi_1, \xi_2, \xi_3\}$ of lines to the third Brocard point and the Euler line of $\triangle ABC$.

2. Three lines through the centroid

We continue here with the notation introduced in the previous section, denoting the medians of the triangle from corresponding vertices $\{A, B, C\}$ by $\{\mu_1, \mu_2, \mu_3\}$. Obviously $G(1,1,1)$ lies on each ν_i, hence these lines pass through the centroid of the triangle. Next lemma clears their connection with the axes of the Steiner ellipse (See Figure 3).

Lemma 2.1. *Each pair $\{(\mu_i, \nu_i),\ i = 1,2,3\}$ consists of two lines symmetric w.r.t. the axes of the Steiner ellipse.*

Proof. This results by the explicit determination of the nodal tangents of the quintic at its node G, which coincide with the axes of the Steiner ellipse. The product of these two lines is the degenerate conic with matrix H_f, and there is a standard method of detecting them, described in analytic geometry books ([11, I, p.95]). By this method, setting

$$A = b^2 - c^2, \quad B = c^2 - a^2, \quad C = a^2 - b^2 \quad \Rightarrow$$

$$h(u,v,w) = (u,v,w)H_f(u,v,w)^t = Au^2 + Bv^2 + Cw^2 + 2Avw + 2Bwu + 2Cuv = 0.$$

Completing the square we have

$$Ah = A^2u^2 + 2ACuv + 2ABuw + ABv^2 + 2A^2vw + ACw^2$$
$$= (Au + Cv + Bw)^2 + (AB - C^2)v^2 + 2(A^2 - BC)vw + (AC - B^2)w^2.$$

Then, since $A + B + C = 0$, we have $A^2 - BC = B^2 - AC = C^2 - AB = k^2$, say, noticing that $A^2 - BC = (B + C)^2 - BC = B^2 + BC + C^2 > 0$. Thus,

$$Ah = (Au + Cv + Bw)^2 - k^2(v - w)^2 = \nu_1^2 - k^2\mu_1^2 . \qquad (2.1)$$

This proves the theorem for the pair (μ_1, ν_1). In fact, $v - w = 0$ represents the medial

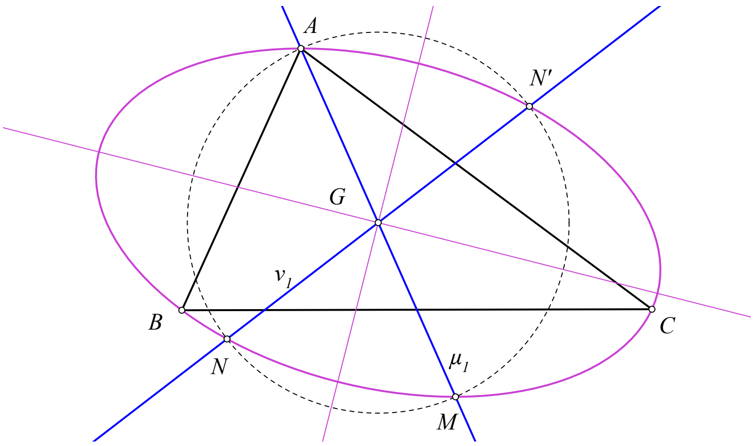

Figure 3. The median μ_1 and the line ν_1

line μ_1 of the triangle. $h(u, v, w) = 0$ represents the nodal tangents of the quintic at G, known to coincide with the axes of the Steiner ellipse. Since latter are orthogonal lines and $\{\nu_1, \mu_1\}$ are harmonic cojugate w.r.t. the lines $\{\nu_1 + k\mu_1, \nu_1 - k\mu_1\}$, latter bisect the angle of the former. Analogously is proved the claim for the other pairs of lines. □

Corollary 2.1. *The circle with center G and radius AG intersects the Steiner ellipse in the diametral point M of A and in two other diametral points $\{N, N'\}$ lying on line ν_1. Analogous statements hold also for the lines $\{\nu_2, \nu_3\}$ (See Figure 3).*

The constant k in (2.1) is easily seen to satisfy

$$k^2 = a^4 + b^4 + c^4 - (b^2c^2 + c^2a^2 + a^2b^2) = S_\omega^2 - 3S^2 ,$$

where S is twice the area of $\triangle ABC$ and $S_\omega = S\cot(\omega)$, where ω is the *"Brocard angle"* of the triangle satisfying $\cot(\omega) = \cot(\widehat{A}) + \cot(\widehat{B}) + \cot(\widehat{C})$ and $2S_\omega = a^2 + b^2 + c^2$. This proves next corollary.

Corollary 2.2. *With the notation and conventions adopted so far, the axes of the Steiner ellipse can be expressed in the form $\{\nu_i \pm k\mu_i,$ for $i = 1, 2, 3\}$ with $k^2 = S_\omega^2 - 3S^2$.*

Remark 2.1. *Throughout this discussion we assume that the triangle of reference ABC is non-equilateral, the Steiner ellipse for equilaterals coinciding with the circumcircle of the triangle. In the equilateral case the quintic is reducible to the product of the three inner bisectors and the circumcircle of the triangle:*

$$f(u, v, w) = (u - v)(v - w)(w - u)(uv + vw + wu) = 0.$$

For non-equilateral triangles the known fact that the angle ω *is less than* $60°$ *(*[14, p.102]*) is equivalent with*

$$\cot^2(\omega) > 3 \quad \Leftrightarrow \quad k^2 = S_\omega^2 - 3S^2 = S^2(\cot^2(\omega) - 3) > 0.$$

3. Cyclic sets of lines through the centroid

The centroid G of the triangle of reference ABC has a remarkable property regarding the lines through it. Their coefficients in barycentrics (u, v, w) satisfy

$$pu + qv + rw = 0 \quad \text{passes through} \quad G \quad \Leftrightarrow \quad p + q + r = 0.$$

This implies, that for any such line, permuting the coefficients $\{p, q, r\}$ produces another line also passing through G. In particular, doing cyclic permutations of such a triple of numbers $\{p, q, r\}$ produces a total of three lines through G which I call a *"cyclic set"*. Cyclic sets possess properties, some of which are relevant for our study, therefore we discuss them in this section.
First, considering the tripoles $(1/p, 1/q, 1/r)$ of lines through G, we see that they are points of the Steiner ellipse,

$$v \cdot w \; + \; w \cdot u \; + \; u \cdot v \; = \; 0,$$

which is the affine image $\sigma = h(\kappa)$ of the circumcircle κ of an equilateral. The affinity h

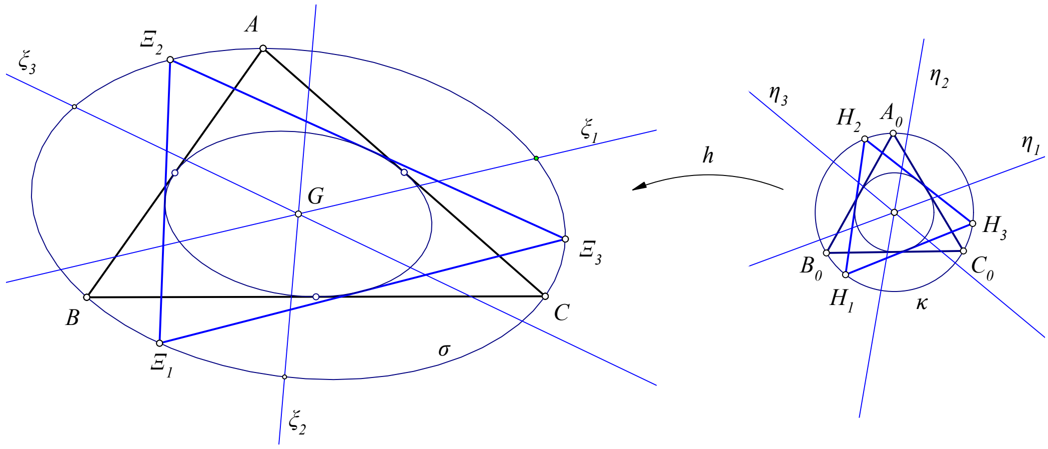

Figure 4. The cyclic set of lines $\{\xi_1, \xi_2, \xi_3\}$

involved being defined by its property to map the vertices of the equilateral $\triangle\, A_0B_0C_0$ to corresponding vertices of $\triangle\, ABC$, and, as a consequence, mapping also the center of the equilateral to the centroid G of $\triangle\, ABC$. Figure 4 shows such a cyclic set $\xi = (\xi_1, \xi_2, \xi_3)$ of lines together with the triangle $\triangle \Xi_1\Xi_2\Xi_3$, whose vertices are the tripoles of the corresponding lines. Next propositions use the notation and conventions adopted so far.

Lemma 3.1. *The lines* $\{\xi_i\}$ *of a cyclic set and their corresponding tripoles* $\{\Xi_i\}$ *are the images* $\{\xi_i = h(\eta_i)\,,\; \Xi_i = h(H_i)\}$ *of a cyclic set* $\eta = (\eta_1, \eta_2, \eta_3)$ *and its tripoles w.r.t. the equilateral* $\triangle\, A_0B_0C_0$*, and* $\triangle\, H_1H_2H_3$ *is also equilateral.*

Proof. Obviously a permutation applied to the tripolar ξ_i and its tripole Ξ_i maps to a tripolar ξ_j and its tripole Ξ_j. Since affinities, preserve the relation *tripole-tripolar*, a cyclic permutation in $\{\Xi_1, \Xi_2, \Xi_3\}$ induces a corresponding cyclic permutation on $\{H_1, H_2, H_3\}$, which, by the symmetry of the equilateral, is a rotation by an angle of measure $2\pi/3$, showing that $\triangle\ H_1H_2H_3$ is an equilateral. □

Corollary 3.1. *The lines* $\{\eta_i\}$ *of an equilateral's* $A_0B_0C_0$ *cyclic set intersect at* $60°$.

Next propositions result from lemma 3.1 and the affine properties of h : (i) to map the circumcircle and incircle of $\triangle\ A_0B_0C_0$ respectively to the Steiner elliplse and in-ellipse of $\triangle\ ABC$, (ii) to preserve parallelity and (iii) to preserve ratios defined by three collinear points.

Corollary 3.2. *All* $\triangle\ \Xi_1\Xi_2\Xi_3$ *have the same centroid G, are inscribed in the Steiner ellipse and circumscribed to the Steiner in-ellipse of* $\triangle\ ABC$. *Also the tangent to the Steiner ellipse at a vertex* Ξ_i *is parallel to the opposite side* $\Xi_j\Xi_k$, *which is tangent to the Steiner in-ellipse at its middle.*

Corollary 3.3. *All triangles* $\{\Xi_1\Xi_2\Xi_3\}$, *which are simultaneously inscribed to the Steiner ellipse and circumscribed to the Steiner in-ellipse are characterized by the property of the barycentrics* (u_i, v_i, w_i) *of their vertices, to result from those of one vertex by cyclically permutating its components.*

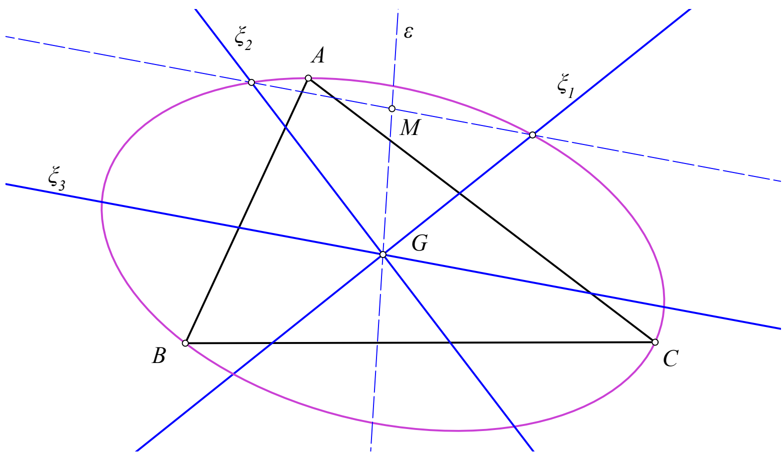

Figure 5. Conjugate lines of a cyclic system

Lemma 3.2. *For a cyclic set* $\{\xi_1, \xi_2, \xi_3\}$ *of* $\triangle\ ABC$ *the concepts of conjugation of each* ξ_i *w.r.t. the other two and the conjugation of diameters w.r.t. the Steiner ellipse coincide.*

Proof. In the case of ξ_3 say, this means that the conjugate line ε w.r.t. the pair (ξ_1, ξ_2) coincides with the conjugate diameter w.r.t. the Steiner ellipse (See Figure 5). This follows from the corresponding property of cyclic sets of the equilateral, which by corollary 3.1, consists of lines intersecting at $60°$. Since a cyclic set of $\triangle\ ABC$ is an $h-$image of a cyclic set $\{\eta_1, \eta_2, \eta_3\}$ of $\triangle\ A_0B_0C_0$, the property follows from the corresponding property of cyclic sets of the equilateral. In fact, for such a set the conjugate of η_3 w.r.t. both concepts of conjugacy coincides with the line ε_0 through the center of the circumcircle of $\triangle A_0B_0C_0$, which is orthogonal to η_3. The result follows by the preservation of both concepts of conjugacy by affinities applied to $\varepsilon = h(\varepsilon_0)$. □

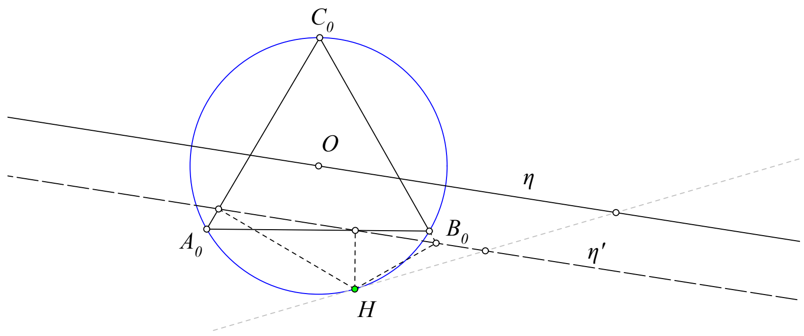

Figure 6. Steiner lines η as tripolars of equilaterals

Remark 3.1. *Notice that in the case of equilaterals the tripolar of a point H on their circumcircle coincides with the Steiner line η of H (See Figure 6), which, by definition, is the parallel through the orthocenter to the Simson line η' of H. This gives an alternative proof of the angles of η_i being equal to 60°, since, as is well known, the angles of Simson lines corresponding to two points $\{H, H'\}$ equals half the measure of the central angle $\widehat{HOH'}$. Notice also that the Simson lines, as H moves on the circle κ, rotate in the opposite direction. This implies, that $\triangle H_1H_2H_3$ is inversely oriented to $\triangle A_0B_0C_0$ and, consequently, also $\triangle \Xi_1\Xi_2\Xi_3$ is inversely oriented to $\triangle ABC$.*

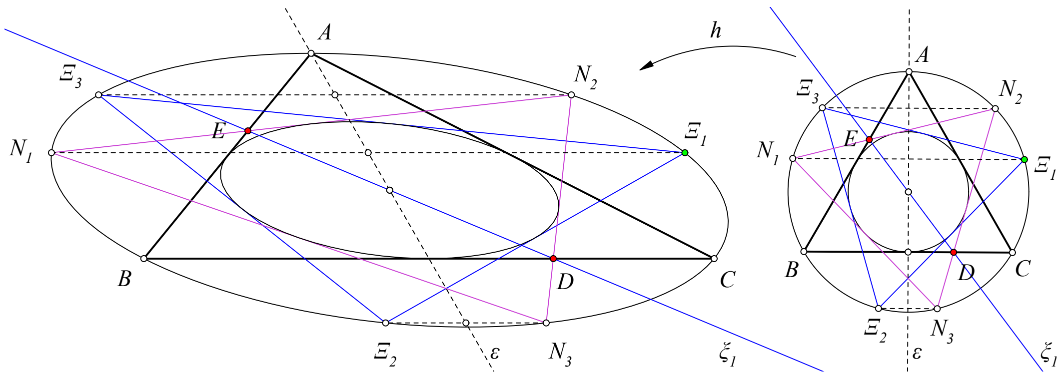

Figure 7. Relation of the triangles of tripolars

Figure 7 illustrates a relation of the triangles with vertices the tripoles of two particular cyclic systems $\{\nu_i\}$ and $\{\xi_i\}$. The figure on the left is produced as the affine image via h of the figure on the right. Corresponding under h elements have the same letter labels. The system $\{\xi_i\}$ is arbitrary while, by definition, the tripoles of the system $\{\nu_i\}$ result by considering the barycentrics (p, q, r) of the tripole of ξ_1 and first interchanging the two last components $(p, q, r) \rightarrow (p, r, q)$ then permuting cyclically the components of the resulting triple. On the right part of the figure, this is applied to the equilateral $\triangle \Xi_1\Xi_2\Xi_3$ and produces the equilateral $\triangle N_1N_2N_3$, which is symmetric to the former w.r.t. the medial line ε of BC. By the preceding remark, the tripolar ξ_1 of Ξ_1 is the Steiner line of Ξ_1 w.r.t. the equilateral $\triangle ABC$ and it is easily verified that it passes through the intersection points $\{D = BC \cap N_2N_3,\ E = AB \cap N_1N_2\}$ of the sides of the two triangles. The affinity h, preserving all the relevant properties, implies next property of the left part of figure 7.

Lemma 3.3. *The side BC of $\triangle ABC$ is parallel to lines $\{\Xi_1N_1\ ,\ \Xi_2N_3\ ,\ \Xi_3N_2\}$. Analogous properties hold also for the other sides of $\triangle ABC$. In addition, the tripolars $\{\xi_i\}$ pass through intersection points of two pairs of sides of the triangles $\{\triangle ABC\ ,\ \triangle N_1N_2N_3\}$. Analogously the tripolars $\{\nu_i\}$ pass through intersection points of two pairs of sides of the triangles $\{\triangle ABC\ ,\ \triangle \Xi_1\Xi_2\Xi_3\}$.*

Remark 3.2. *The mapping defined by $\{\Xi_1 \mapsto N_1, \Xi_2 \mapsto, \Xi_3 \mapsto N_2\}$ on the right of figure 7 is a "reflection" interchanging the two triangles and is conjugate under h to the "Affine reflection"* ([15, p. 203]) *w.r.t. the median line ε of ABC from A, interchanging the equally labeled triangles on the left part of the figure.*

4. Some properties of the Steiner point

One of the general properties of projective transformations ([16, I, p.213]) guarantees the existence of a unique *projectivity* f mapping the Steiner ellipse to the circumcircle and fixing the vertices of the triangle of reference ABC. This transformation is represented in barycentrics by non-zero multiples of a matrix M_f, and the identity assumptions for the vertices $f(A) = A,\ f(B) = B, f(C) = C$ imply that the matrix is of the diagonal form

$$M_f = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}.$$

The condition on f to map the Steiner ellipse onto the circumcircle implies that a point

$$\begin{aligned} U &= (u, v, w) && \text{satisfying} \quad vw + wu + uv = 0 \quad \text{maps to} \\ U' &= (k_1u, k_2v, k_3w) && \text{satisfying} \quad a^2k_2k_3vw + b^2k_3k_1wu + c^2k_1k_2uv = 0. \end{aligned}$$

From these follows that $\{k_1 = \lambda a^2, k_2 = \lambda b^2, k_3 = \lambda c^2\}$ for a constant $\lambda \neq 0$ and we may assume that

$$M_f = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}.$$

Lemma 4.1. *The projectivity f maps the centroid G to the symmedian point of ABC and for each point U on the Steiner ellipse and its image $U' = f(U)$ on the circumcircle the line UU' passes through the Steiner point S.*

Proof. The proof of the first claim is trivial, since M_f maps $G(1, 1, 1)$ to $K(a^2, b^2, c^2)$, which represents the *symmedian point* in barycentrics. The second claim follows by computing the determinant of the matrix

$$\begin{vmatrix} u & v & w \\ a^2u & b^2v & c^2w \\ (b^2-c^2)^{-1} & (c^2-a^2)^{-1} & (a^2-b^2)^{-1} \end{vmatrix} = -(vw + wu + uv).$$

The second and the third row of the matrix represent respectively the points $U' = f(U)$ and the Steiner point. The last expression vanishes for points $U(u, v, w)$ on the Steiner ellipse. □

Theorem 4.1. *Let $\kappa(O)$ be the circumcircle of $\triangle ABC$, λ be its Steiner ellipse and K be its Symmedian center. The following are valid properties (See Figure 8):*

(1) The projectivity f maps the Steiner point S to the other intersection point $V = \kappa \cap t_S$ of the circumcircle with the tangent t_S of λ at S. Consequently the tangent t_S to λ maps to the tangent t_V to κ.
(2) Point V coincides with the focus $X(110)$ of Kiepert's parabola of ABC whose directrix is the Euler line of $\triangle ABC$.
(3) The lines $\{SG, VK\}$ intersect on κ in the "Parry" point $P = X(111)$ of ABC.
(4) The lines $\{SO, VG\}$ intersect in the "Tarry" point $W = X(98)$, the diametral of the Steiner point S.
(5) Points $\{O, G, K, P\}$ are concyclic and the angle $\widehat{GOK}$ is equal to the angle of lines $\{t_S, t_V\}$.

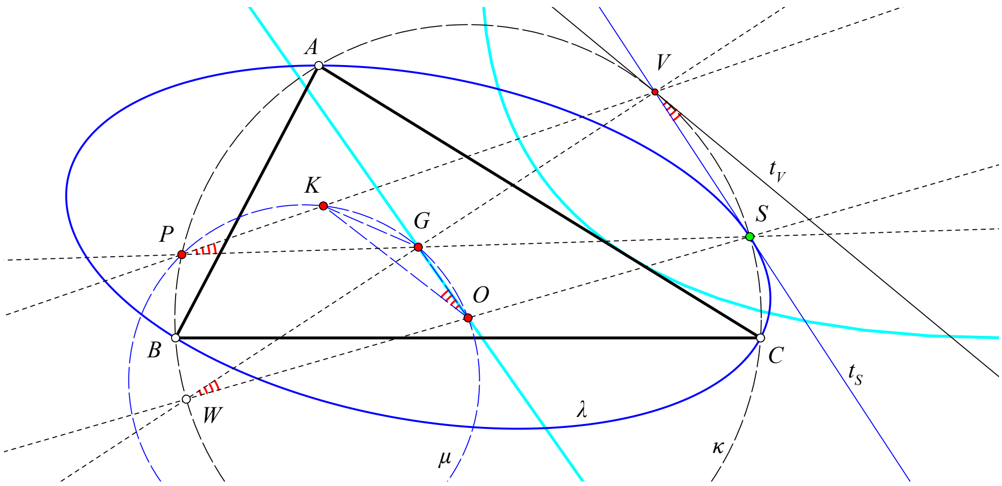

Figure 8. Properties of the projectivity f mapping Steiner's ellipse to the circumcircle

Proof. *Nr-1* is a direct corollary of lemma 4.1.
Nr-2 follows from the representation of S in barycentrics given by equation (1.3). This implies for $V = f(S)$ the corresponding representation

$$V = \left(\frac{a^2}{b^2 - c^2} : \frac{b^2}{c^2 - a^2} : \frac{c^2}{a^2 - b^2} \right). \tag{4.1}$$

In Kimberling's triangle-centers notation ([13]) this is point $X(110)$ and is known to coincide with the focus of *Kiepert's parabola* ([17]) of ABC. The *Simson line* of V is parallel to the *Euler line* and bisects the distance VG.
Nr-3 follows also from known collinearities of triangle centers ([13]). It can be also seen directly using the barycentrics of $P(p_1 : p_2 : p_3) = X(111)$:

$$(p_1 : p_2 : p_3) = \left(\frac{a^2}{a^2 - 2S_A} : \frac{b^2}{b^2 - 2S_B} : \frac{c^2}{c^2 - 2S_C} \right)$$

and verifying the collinearity conditions expressed through determinants with columns the barycentrics of the points: $|P,G,S| = |P,K,V| = 0$, where the *Conway triangle symbols* ([18]) are defined by

$$S_A = (b^2+c^2-a^2)/2, \qquad S_B = (c^2+a^2-b^2)/2, \qquad S_C = (a^2+b^2-c^2)/2\,. \tag{4.2}$$

Nr-4 results also using barycentrics for

$$W = X(98) \;=\; (\, b^4+c^4-a^2(b^2+c^2) \;:\; c^4+a^4-b^2(c^2+a^2) \;:\; a^4+b^4-c^2(a^2+b^2)\,)$$

and the determinants $|W,G,V| = |W,O,S| = 0$.
Nr-5, the second part is a consequence of the first one, since the angle of lines $\{t_S, t_V\}$ is equal to $\widehat{KPG}$. The first part can be proved by applying the criterion for four concyclic points. This results by a somewhat tedious calculation using the barycentrics formula ([5]) for the circle through three points. The formula, considered for the first three points and applied to the fourth, reduces to the vanishing of a determinant in the form:

$$\begin{vmatrix} o_1 & o_2 & o_3 & s(O) \\ g_1 & g_2 & g_3 & s(G) \\ k_1 & k_2 & k_3 & s(K) \\ p_1 & p_2 & p_3 & s(P) \end{vmatrix} = 0\,,$$

where in each row appear the barycentrics of the corresponding point and the expression

$$s(X) \;=\; \frac{a^2x_2x_3 + b^2x_3x_1 + c^2x_1x_2}{x_1+x_2+x_3}\,.$$

The barycentrics of the points $\{G, K, P\}$ were considered above and those of the circumcenter O are given by $O(\, a^2S_A \;:\; b^2S_B \;:\; c^2S_C\,)$. □

5. Circumquadrangle of Brocard- and Steiner in-ellipse

We could give an alternative definition of f, using the fundamental property of projectivities ([16, I, p.96]), by which such a map is completely determined by prescribing its images at four points in general position. In fact, from lemma 4.1 follows, that f is uniquely determined as the projectivity fixing $\{A, B, C\}$ and mapping G to K. Figure 9 shows two perspective triangles $\{\, \tau = T_1T_2T_3\,,\ \beta = V_1V_2V_3\,\}$. The first is of the kind discussed in the previous section, i.e. simultaneously inscribed/circumscribed to the Steiner-/Steiner-in-ellipse. The second is its image via f. By lemma 4.1 the two triangles are perspective from the Steiner point S. The line σ, seen in the figure, is the perspective axis of the two triangles. Although τ and consequently its image $\beta = f(\tau)$ may vary continuously the perspective axis of the triangles $\{\tau, \beta\}$ remains the same line.

Theorem 5.1. *The perspective axis σ of the triangles $\{\, T_1T_2T_3\,,\ V_1V_2V_3\,\}$ is a tangent to the Steiner in-ellipse coinciding with the trilinear polar of the point at infinity of the Lemoine axis of $\triangle\, ABC$.*

Proof. By definition, the *"Lemoine axis"* is the trilinear polar of the symmedian point K. Thus, it has the coefficients $\{a^{-2}, b^{-2}, c^{-2}\}$ and its point at infinity is

$$(\, b^{-2}-c^{-2} \;:\; c^{-2}-a^{-2} \;:\; a^{-2}-b^{-2}\,),$$

producing the trilinear polar with equation

$$\sigma \ : \ \frac{b^2c^2}{b^2-c^2}u \ + \ \frac{c^2a^2}{c^2-a^2}v \ + \ \frac{a^2b^2}{a^2-b^2}w \ = \ 0. \tag{5.1}$$

On the other side, for a point $T_1(x,y,z)$ on the Steiner ellipse, the vertex T_2 of the

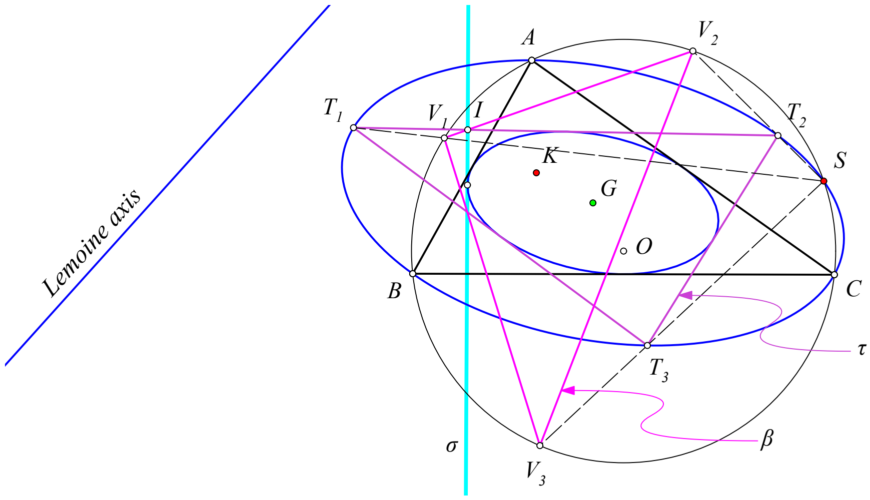

Figure 9. Two triangles perspective from the Steiner point

triangle $T_1T_2T_3$, according to corollary 3.3, can be considered to be $T_2(y,z,x)$. The corresponding points on the circle are then

$$V_1 \ = \ f(T_1) \ = \ (a^2x : b^2y : c^2z) \ , \quad V_2 \ = \ f(T_2) \ = \ (a^2y : b^2z : c^2x).$$

The lines $\{T_1T_2, V_1V_2\}$ are then respectively equal to

$$(yx-z^2)u \ + \ (zy-x^2)v \ + \ (xz-y^2)w \ = \ 0, \tag{5.2}$$

$$b^2c^2(yx-z^2)u \ + \ c^2a^2(zy-x^2)v \ + \ a^2b^2(xz-y^2)w \ = \ 0. \tag{5.3}$$

The determinant of the coefficients of the three lines is then equal to

$$\begin{vmatrix} b^2c^2/(b^2-c^2) & c^2a^2/(c^2-a^2) & a^2b^2/(a^2-b^2) \\ yx-z^2 & zy-x^2 & xz-y^2 \\ b^2c^2(yx-z^2) & c^2a^2(zy-x^2) & a^2b^2(xz-y^2) \end{vmatrix}$$
$$= \ a^2b^2c^2(yz+xz+xy)(yz+xz+xy-x^2-y^2-z^2).$$

Since the last expression vanishes if $T_1(x,y,z)$ is on the Steiner ellipse, this proves that the three lines are concurrent, hence the second part of the theorem. The first claim follows from the well known general fact that the tripolars of points at infinity are all tangent to the inner Steiner ellipse. □

Remark 5.2. *In* [19] *I studied configurations like those of the triangle* $T_1T_2T_3$ *considering it as an orbit* $\{T_1, T_2 = p(T_1), T_3 = p(T_2)\}$ *of a projectivity* p *of period 3, preserving the conic, in which the triangle is inscribed. The prototype for this is the rotation* r *of the equilateral by* $2\pi/3$ *about its center which permutes cyclically its vertices. Using the map* h *of the equilateral to* ABC*, of section 2, we obtain* p *by conjugation:* $p = h \circ r \circ h^{-1}$*. Using further the map*

f, mapping the Steiner ellipse to the circumcircle, we obtain again by conjugation a projectivity $q = f \circ p \circ f^{-1}$ of period 3, preserving the circle and permutting cyclically the vertices of the triangle $V_1V_2V_3$. The sides of the triangles $V_1V_2V_3$ are seen to be tangent to the "Brocard ellipse" (See Figure 10), identified with the image of the Steiner in-ellipse under f and whose focals are

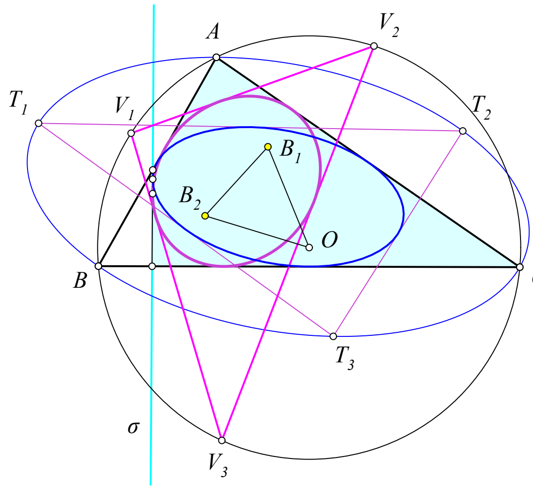

Figure 10. The Brocard ellipse

the "Brocard points" $\{B_1, B_2\}$ of the triangle ABC. It is there also proved that the triangle OB_1B_2 is an orbit of the periodic projectivity q. By the discussion so far it follows that the via f corresponding sides $\{T_iT_j, V_iV_j\}$ intersect on line σ and from this, the more general formulated property, that the tangents t and $f(t)$, respectively of the Steiner-in- and the Brocard ellipse, intersect on line σ. This implies easily that σ is simultaneously tangent to these two ellipses. Thus, line σ complements the three sides of ABC, defining the fourth side of a "quadrangle which simultaneously circumscribes the Steiner-in- and the Brocard ellipse" of $\triangle ABC$.

6. Reflected and similar pairs of triangles

In this section we apply the general results for *cyclic sets*, established in the preceding two sections, to the triples of lines $\{\nu_1, \nu_2, \nu_3\}$ and $\{\xi_1, \xi_2, \xi_3\}$ introduced in section 1. Points $\{N_1, N_2, N_3\}$ are the respective tripoles of $\{\nu_1, \nu_2, \nu_3\}$ and $\{V_1, V_2, V_3\}$ are their corresponding images via the projectivity f studied in section 4 (See Figure 11). By lemma 4.1 the triples $\{(N_i, V_i, S),\ i = 1,2,3\}$ consist of collinear points, S being the Steiner point of the triangle of reference.

Theorem 6.1. *With the notation and conventions adopted so far, the following are valid properties.*

(1) Lines $\{SN_i,\ i = 1,2,3\}$ are parallel to corresponding sides $\triangle ABC$.
(2) The triangle $V_1V_2V_3$ is equal to $\triangle ABC$.
(3) Triangles $\{ABC, V_1V_2V_3\}$ are, each, the reflection of the other w.r.t. the Brocard axis OK.

Proof. *Nr-1*, for lines $\{SN_1, BC\}$, follows from the vanishing of the determinant

$$\begin{vmatrix} (b^2-c^2)^{-1} & (c^2-a^2)^{-1} & (a^2-b^2)^{-1} \\ (b^2-c^2)^{-1} & (a^2-b^2)^{-1} & (c^2-a^2)^{-1} \\ 0 & -1 & 1 \end{vmatrix} = 0,$$

in which the rows of the matrix are respectively the barycentrics of $\{S, N_1\}$ and the point at infinity of line BC. Analogously is proved the parallelity of the other pairs of lines.

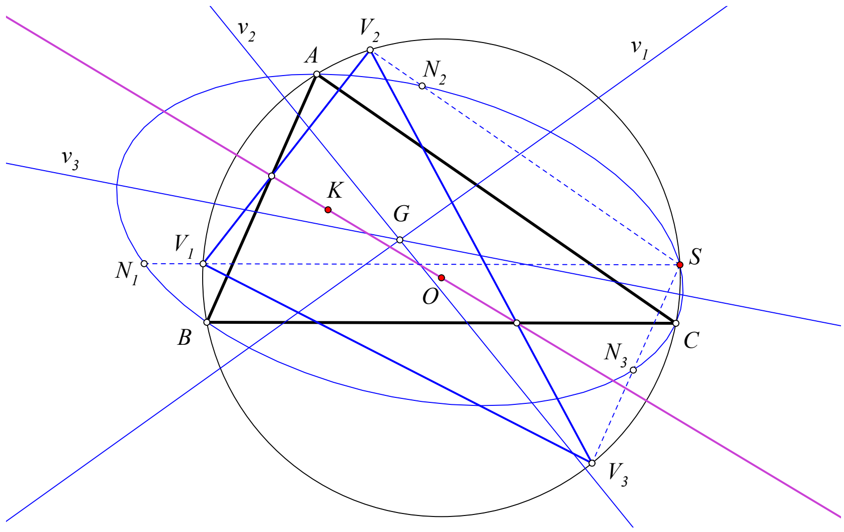

Figure 11. The triangle $V_1V_2V_3$ corresponding to $\{v_1, v_2, v_3\}$

Nr-2 is a trivial consequence of *nr-1* and is valid more generally for the configuration resulting from three parallels to the sides of ABC from any point S of its circumcircle. This is discussed in the next section.
Nr-3 is proved by showing that lines $\{AV_1, BV_2, CV_3\}$ are orthogonal to the Brocard axis OK. This is equivalent to showing that these lines are parallel to the, orthogonal to Brocard, *Lemoine axis*, which is the trilinear polar of the symmedian point K. The method, for the line AV_1, again as in *nr-1*, is to show that $\{A, V_1\}$ and the point at infinity of the Lemoine axis define through their barycentrics a vanishing determinant:

$$\begin{vmatrix} 1 & 0 & 0 \\ a^2(b^2-c^2)^{-1} & b^2(a^2-b^2)^{-1} & c^2(c^2-a^2)^{-1} \\ b^{-2}-c^{-2} & c^{-2}-a^{-2} & a^{-2}-b^{-2} \end{vmatrix} = 0.$$

Analogously is proved the orthogonality to OK of the other lines $\{BV_2, CV_3\}$. □

Remark 6.2. *The theorem provides a method to find the Steiner point S without to consider the Steiner ellipse. For this it suffices to draw (i) line KO (ii) reflect ABC on KO to $V_1V_2V_3$ (iii) draw the parallel V_1S to BC intersecting the circumcircle at S.*

Theorem 6.3. *The triangle $\Xi_1\Xi_2\Xi_3$ with vertices correspondingly the tripoles of the three lines $\{\xi_1, \xi_2, \xi_3\}$ and its image $V_1V_2V_3$ via the projectivity f are similar.*

Proof. By lemma 4.1 the points $\{\Xi_i, V_i, S\}$ are collinear (See Figure 12), Ξ_1V_1 being tangent to the ellipse κ at the Steiner point $\Xi_1 = S$ according to theorem 4.1. This implies that the angles of the trianlges $\widehat{\Xi_1} = \widehat{V_1}$, since both points view the same arc V_2V_3 of the circumcircle from its two points $\{\Xi_1, V_1\}$. For the same reason we have also $\widehat{V_2} = \widehat{\Xi_3 S V_1}$. By the tangency to the ellipse κ of SV_1 at S and a general property of the in-ellipse κ',

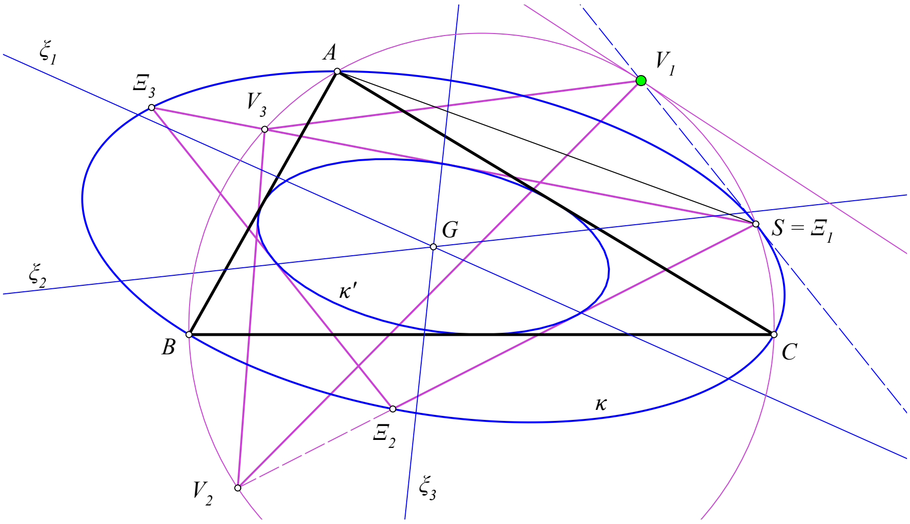

Figure 12. The similar triangles $\{\Xi_1\Xi_2\Xi_3\, , V_1V_2V_3\}$

the opposite to Ξ_1 side $\Xi_2\Xi_3$ of the circumscribed to κ' triangle $\Xi_1\Xi_2\Xi_3$ is parallel to SV_1 (corollary 3.2). Thus, $\widehat{\Xi_3} = \widehat{\Xi_3 S V_1} = \widehat{V_2}$, proving the similarity of the two triangles. □

7. Relations to the two Brocard triangles

The parallels to the sides of ABC, alluded to in *nr-1* of theorem 6.1, can be drawn through any point S of the circumcircle of ABC and define a configuration related to *"Brocard geometry"* ([8, p. 274]). Next theorem formulates some properties of this configuration. Definitions and related fundamental properties of *"Brocard angle"*, *"Brocard points"*, *"Brocard circle"* and 1st and 2nd *"Brocard triangles"* can be found in the aforementioned reference.

Theorem 7.1. *From an arbitrary point S on the circumcircle $\kappa(O)$ of triangle ABC we draw parallels to its sides, intersecting κ a second time at the vertices of the triangle $A'B'C'$ (See Figure 13). The following are valid properties of the resulting configuration.*

(1) *The triangles $\{ABC, A'B'C'\}$ are congruent and each is the reflection of the other on a line passing through O and the middle M of AA'.*
(2) *The intersections of AA' with lines $\{BC', CB'\}$ define triangle $A''B''C''$, which is isosceles with base angles of measure $\widehat{A}$, hence of similarity type independent of the position of point S on the circumcircle.*
(3) *The circles $\{\alpha_1 = (ACB''),\ \alpha_2 = (ABC''),\ \alpha_3 = (BCA'')\}$ pass through the same point Q. The first two are tangent at A respectively to the sides $\{AB, AC\}$. The circle α_3 passes through the circumcenter O.*
(4) *The circles $\{\alpha_1, \alpha_2\}$ pass respectively through the Brocard points $\{B_1, B_2\}$.*

(5) *Point Q is on the Brocard circle (OB_1B_2) and also on the symmedian line AK of ABC coinciding with a vertex of the second Brocard triangle of ABC.*

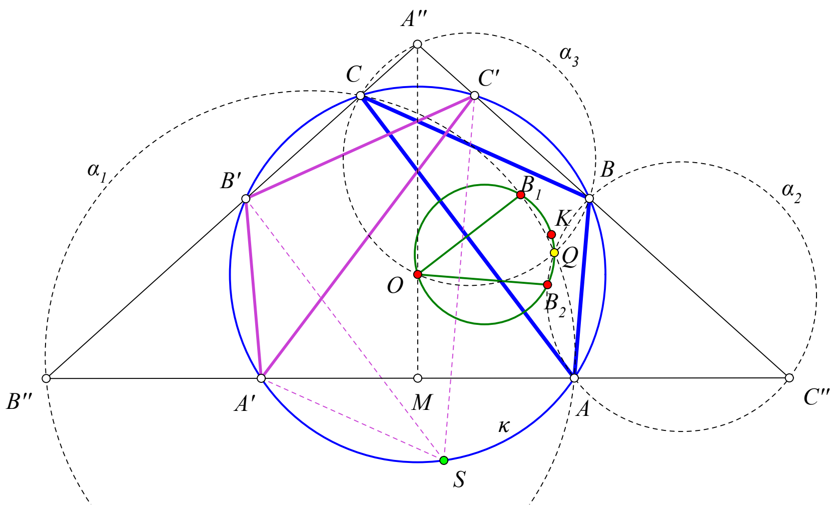

Figure 13. Configuration related to *Brocard geometry* of ABC

Proof. *Nr-1* results by the easily deducible equality of angles and the inverse orientation to ABC.
Nr-2 follows from *nr-1*, since the angle $\widehat{A''C''B''} = \widehat{A''BB'} = \widehat{C'A'B'} = \widehat{A}$.
Nrs 3-4: AB is tangent to α_1 since $\widehat{A} = \widehat{B''}$. By the definition of Brocard points, this implies also that α_1 passes through B_1 and the angle $\widehat{BAB_1} = \omega$, latter being the *Brocard angle*. Analogously is seen the tangecy of α_2 to AC and the equality of measures $\widehat{CAB_2} = \omega$. That $OBCA''$ is cyclic follows from its opposite angles $\widehat{A''} + \widehat{OBC} = \pi$. This implies also that the three circles pass through the same point Q.
Nr-5: A simple angle chasing argument shows that triangles $\{QB''C'', QAB, QCA\}$ are similar. This implies that the ratio of distances of Q from the sides $\{AB, AC\}$ is equal to the length-ratio of these sides, which is something characterizing the points of the symmedian line from A. Thus, Q is on the line AK. The other claim of this *nr* is related to the known fact, that OK is a diameter of the Brocard circle and $\{B_1, B_2\}$ are symmetric w.r.t. it, hence OK bisects the angle $\widehat{B_1OB_2}$, which is 2ω in measure. Considering the inscribed in α_1 quadrangle CB_1QA we see that the angle $\widehat{B_1QK} = \widehat{B_1CA} = \omega$. This impies that Q is on the Brocard circle and finishes the proof, since, by its definition, a vertex of the 2nd Brocard triangle of ABC is characterized as the other than K intersection point of a symmedian of ABC with the Brocard circle. □

Remark 7.2. *In the previous theorem we singled out the line AA' and proceeded to the construction of a configuration which can be repeated by singling out one of the other lines $\{BB', CC'\}$ and repeating the same procedure. This leads to similar statements leading also to relations involving the other vertices of the 2nd Brocard triangle. Next theorem gives another relation of this triangle to the lines $\{v_i\}$ and the first Brocard triangle.*

Theorem 7.3. *The lines $\{v_1, v_2, v_3\}$ pass respectively through the vertices $\{A_1, B_1, C_1\}$ of the 1st Brocard triangle and the centroid of ABC is also the centroid of the 1st Brocard triangle and perspectivity center of the two Brocard triangles $\{A_1B_1C_1, A_2B_2C_2\}$.*

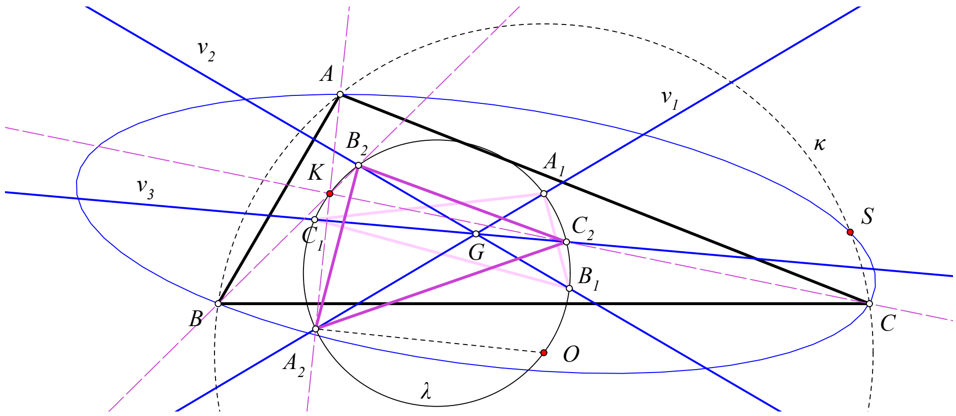

Figure 14. Lines $\{v_1, v_2, v_3\}$ through vertices of the Brocard triangles

Proof. The barycentrics of $\{A_1, B_1, C_1\}$ are respectively ([7, p.100]):

$$(a^2\ ,\ c^2\ ,\ b^2)\ ,\ (c^2\ ,\ b^2\ ,\ a^2)\ ,\ (b^2\ ,\ a^2\ ,\ c^2)$$

and seen readily to be on lines $\{v_1, v_2, v_3\}$, whose coefficients in barycentrics are given by the rows of the matrix in equation (1.2). The statement on the coincidence of the centroids of $\{ABC, A_1B_1C_1\}$ is a well known fact ([8, p.281]). The fact that the two brocard triangles are perspective w.r.t the centroid of $\triangle ABC$ is also well known ([10, p.280]). For the sake of completeness of exposition I include a proof of this using barycentrics and showing that these lines pass through the vertices of the second Brocard triangle. □

Theorem 7.4. *The lines $\{v_1, v_2, v_3\}$ pass respectively through the vertices $\{A_2, B_2, C_2\}$ of the 2nd Brocard triangle.*

Proof. The proof for v_1 results by showing the vanishing of the determinant with rows consisting respectively of the barycentric coefficients of the lines $\{v_1, OA_2, AK\}$. The coefficients of v_1 are given by the first row of the matrix in equation (1.2). Those of AK are easily seen to be $(0, -c^2, b^2)$. To compute the coefficients of line OA_2 we use the fact that it is orthogonal to KA, whose point at infinity, defining its *direction*, is seen to be $(-c^2 - b^2, b^2, c^2)$. The orthogonal to it direction, or point at infinity, is the seen to be

$$A^* \ = \ (\ S_A(c^2 - b^2)\ ,\ b^2(c^2 + S_A)\ ,\ -c^2(b^2 + S_A)\ ).$$

Line OA_2 coinciding with OA^* is calculated by the vector product of the corresponding barycentrics of A^* and O :

$$OA^* \ = \ O \times A^* \ = \ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \ = \ \begin{pmatrix} -b^2c^2(S_B(b^2 + S_A) + S_C(c^2 + S_A)) \\ c^2(c^2 - b^2)S_AS_C + a^2c^2S_A(b^2 + S_A) \\ a^2b^2S_A(c^2 + S_A) - b^2(c^2 - b^2)S_AS_B \end{pmatrix}.$$

Thus, the proof for ν_1 amounts to showing that the determinant

$$\begin{vmatrix} b^2-c^2 & a^2-b^2 & c^2-a^2 \\ a_1 & a_2 & a_3 \\ 0 & -c^2 & b^2 \end{vmatrix} = 0,$$

which is indeed verified, showing that line ν_1 passes through A_2. Analogously are proved the statements about $\{\nu_2, \nu_3\}$. □

8. THE TRIPOLAR OF THE STEINER POINT

The tripolar $\xi_1 = GK$ of the Steiner point S of $\triangle\, ABC$ generates through the cyclic permutations of its coefficients$\{b^2-c^2, c^2-a^2, a^2-b^2\}$ the other triple of lines $\{\xi_1, \xi_2, \xi_3\}$. Line ξ_1 is though a distinguished member of this triple and has some properties related to the three lines $\{\nu_1, \nu_2, \nu_3\}$.

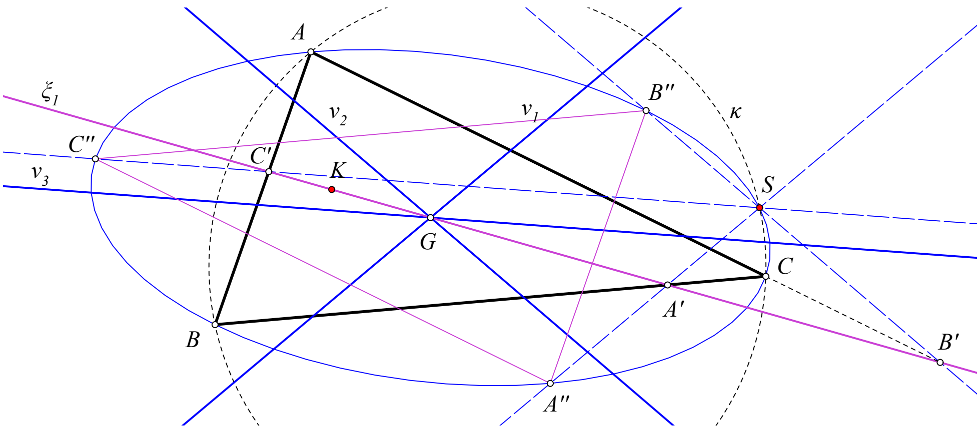

Figure 15. Relation of line $\xi_1 = GK$ to $\{\nu_1, \nu_2, \nu_3\}$

Theorem 8.1. *The parallels from the Steiner point S of $\triangle\, ABC$ to lines $\{\nu_1, \nu_2, \nu_3\}$ intersect respectively the sides $\{BC, CA, AB\}$ at three points $\{A', B', C'\}$ lying on the trilpolar $\xi_1 = GK$ of S (See Figure 15).*

Proof. As usual, the proof reduces to the vanishing of a determinant. In the case of A' this is the determinant with rows the coefficients in barycentrics of the lines BC, GK and $S\nu_1^*$, where ν_1^* is the point at infinity of ν_1 :

$$\nu_1^* = (\, 2a^2-b^2-c^2 \,,\, 2c^2-a^2-b^2 \,,\, 2b^2-a^2-c^2 \,) \quad \Rightarrow$$
$$S\nu_1^* \;:\; d_1 u + d_2 v + d_3 w = 0, \quad \text{where}$$
$$d_1 = \frac{2d}{(b^2-a^2)(c^2-a^2)}, \quad d_2 = \frac{d}{(b^2-a^2)(c^2-b^2)}, \quad d_3 = -\frac{d}{(c^2-a^2)(c^2-b^2)},$$
$$\text{with} \quad d = a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2, \tag{8.1}$$

equal to the constant k^2 of equation (2.1). The proof follows from the vanishing of

$$\begin{vmatrix} 1 & 0 & 0 \\ b^2-c^2 & c^2-a^2 & a^2-b^2 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0,$$

which shows that A' is on GK. Analogous is the proof for $\{B', C'\}$. □

Theorem 8.2. *The second intersections of the Steiner ellipse with the parallels from the Steiner point S of $\triangle ABC$ to lines $\{\nu_1, \nu_2, \nu_3\}$ define a triangle $A''B''C''$ symmetric to $\triangle ABC$ w.r.t. the center G of the ellipse (See Figure 15).*

Proof. To prove that A'' is symmetric to A w.r.t. G it suffices to show that the middle $M(m_1, m_2, m_3)$ of the segment SA is on line ν_1.

$$(b^2-c^2)m_1 + (a^2-b^2)m_2 + (c^2-a^2)m_3 = 0. \tag{8.2}$$

The coordinates of M, are

$$M = d(1,0,0) + (\ (c^2-a^2)(a^2-b^2)\ ,\ (b^2-c^2)(a^2-b^2)\ ,\ (b^2-c^2)(c^2-a^2)\ ),$$

with d given by equation (8.1). It is readily seen that equation (8.2) is indeed satisfied, thereby proving that A'' is the symmetric of A w.r.t. G. Analogous is the proof of the corresponding property for the vertices $\{B'', C''\}$. □

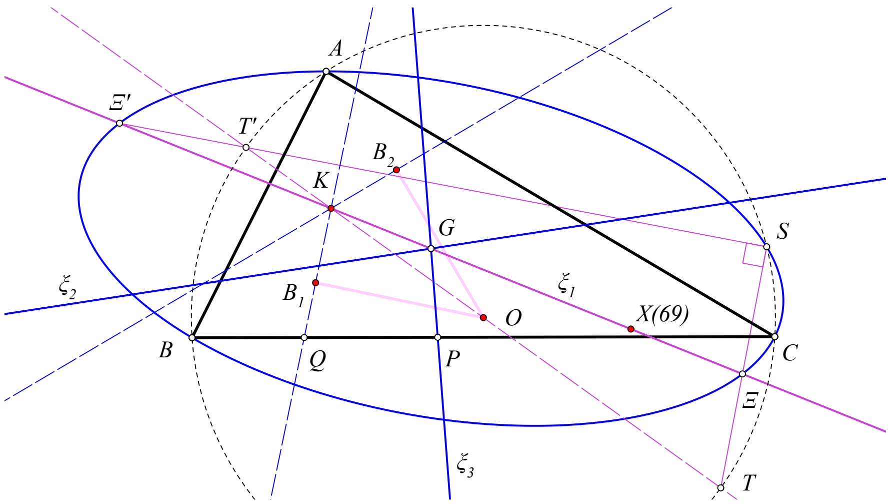

Figure 16. The images of lines $\{f(\xi_1), f(\xi_2), f(\xi_3)\}$

Theorem 8.3. *The projectivity f maps line ξ_1 to the Brocard axis KO and lines $\{\xi_2, \xi_3\}$ to $\{KB_2, KB_1\}$, where $\{B_1, B_2\}$ are the Brocard points of the triangle ABC (See Figure 16).*

Proof. Since, by its definition in section 4, f maps G to K, the proof for $\xi_1 = GK$ results by observing that the point X of this line, whose distances $|XK|/|XG| = 3/2$ and whose barycentrics are

$$X(69) = (a^2+b^2+c^2)G - 2K = (a^2+b^2+c^2)(1,1,1) - 2(a^2,b^2,c^2),$$

maps via f to O.

Since f leaves invariant the side-lines of ABC, in order to show that ξ_3 maps to KB_1 it suffices to show that the intersection $P = BC \cap \xi_3$ maps via f to the intersection $Q = KB_1 \cap BC$. The barycentrics of $\{K, B_1, P, Q\}$ and the determinant with rows that of $\{K, B_1, Q\}$ are seen to be correspondingly:

$$K = (a^2 : b^2 : c^2), \quad B_1 = (b^{-2} : c^{-2} : a^{-2}), \quad P = (0 : c^2 - a^2 : c^2 - b^2),$$

$$Q = (0 : b^2(c^2 - a^2) : c^2(c^2 - b^2)), \quad \begin{vmatrix} a^2 & b^2 & c^2 \\ b^{-2} & c^{-2} & a^{-2} \\ 0 & b^2(c^2 - a^2) & c^2(c^2 - b^2) \end{vmatrix} = 0,$$

thereby proving that ξ_3 maps to KB_1. Analogously is seen that ξ_2 maps to KB_2. □

Corollary 8.1. *Line ξ_1 intersects the Steiner ellipse at points $\{\Xi, \Xi'\}$, such that lines $\{S\Xi, S\Xi'\}$ are parallel to the axes of the ellipse.*

Proof. By theorem 8.3 the line ξ_1 maps via f to the Brocard axis (See Figure 16) and by theorem 4.1 its intersection points $\{\Xi, \Xi'\}$ map to diametral points $\{T, T'\}$ of the circumcircle of ABC lying also on the respective lines $\{S\Xi, S\Xi'\}$. Thus, point S is viewing the diameter $\Xi\Xi'$ of the ellipse under a right angle. □

9. Euler line and third Brocard point

The *"third Brocard point"* $B_3(a^{-2}, b^{-2}, c^{-2})$ of the triangle ABC, in Kimberling's notation $X(76)$, is the *"isotomic conjugate"* of the symmedian point K. It results in a similar way the lines $\{\xi_i\}$ are produced from ξ_1 through cyclic permutations and completes the *Brocard points* to a *cycle* $\{\ B_1(b^{-2}, c^{-2}, a^{-2})\ ,\ B_2(c^{-2}, a^{-2}, b^{-2})\ ,\ B_3(a^{-2}, b^{-2}, c^{-2})\ \}$. Next theorem reveals an intimate relation between the point B_3 and the lines $\{\xi_1, \xi_2, \xi_3\}$ (See Figure 17).

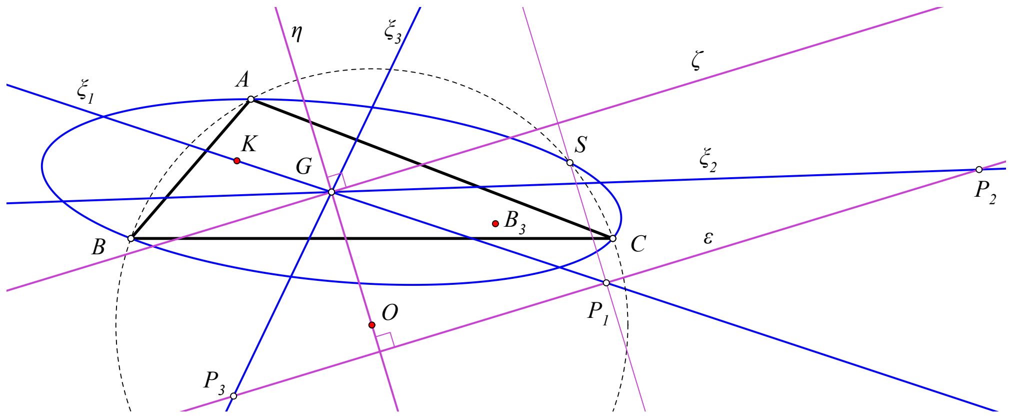

Figure 17. Relation of the Euler line to $\{\xi_1, \xi_2, \xi_3\}$

Theorem 9.1. *With the preceding notation and conventions, the following are valid properties:*

(1) The trilinear polar ε of B_3 maps via f to the line at infinity.
(2) The harmonic conjugate line ζ of ξ_1 w.r.t. $\{\xi_2, \xi_3\}$ is parallel to ε.
(3) The lines $\{\xi_1, \xi_2, \xi_3\}$ intersect ε at points $\{P_1, P_2, P_3\}$ so that P_1 is the middle of P_2P_3.

(4) *The Euler line* η *of* ABC *is orthogonal to* ε.
(5) *The line* SP_1 *joining the Steiner point* S *with* P_1 *is orthogonal to* ε *hence parallel to the Euler line.*

Proof. *Nr-1* is obvious, since the trilinear polar of B_3 given in Barycentrics by equation

$$\varepsilon \;:\; a^2u \;+\; b^2v \;+\; c^2w \;=\; 0\,,$$

implies that point (u, v, w) on this line maps via f to $(u', v', w') = (a^2u, b^2v, c^2w)$ on the line at infinity, satisfying $u' + v' + w' = 0$.
Nr-2: Obviously the lines satisfy the relation:

$$\xi_1 \;=\; \xi_2 \;+\; \xi_3,$$

hence the harmonic conjugate of ξ_1 w.r.t. the other two is

$$\zeta \;=\; \xi_2 - \xi_3 \quad \Leftrightarrow$$
$$\zeta \;:\; (2a^2 - b^2 - c^2)u \;+\; (2b^2 - c^2 - a^2)v \;+\; (2c^2 - a^2 - b^2)w \;=\; 0.$$

Its point at infinity $(b^2 - c^2 : c^2 - b^2 : a^2 - b^2)$ coincides with that of ε, which shows that the two lines $\{\varepsilon, \zeta\}$ are parallel. Alternatively, one could start with lemma 3.2 showing that ζ is the conjugate diameter of ξ_1 w.r.t. the ellipse.
Nr-3 is an immediate consequence of *nr-2*.
Nr-4: The *Euler line* is represented in barycentrics by equation

$$\eta \;:\; (b^2S_B - c^2S_C)u \;+\; (c^2S_C - a^2S_A)v \;+\; (a^2S_A - b^2S_B)w \;=\; 0\,.$$

Its *point at infinity* $X_{30} = (\eta_1 : \eta_2 : \eta_3)$ is

$$(\eta_1 : \eta_2 : \eta_3) \;=\; (\; 2a^2S_A - b^2S_B - c^2S_C \;:\; 2b^2S_B - c^2S_C - a^2S_A \;:\; 2c^2S_C - a^2S_A - b^2S_B\;)\,.$$

On the other side, the point at infinity of ε is given by

$$(\varepsilon_1 : \varepsilon_2 : \varepsilon_3) \;=\; (b^2 - c^2 \;:\; c^2 - a^2 \;:\; a^2 - b^2\;)\,.$$

Hence the orthogonality condition amounts to showing

$$S_A\eta_1\varepsilon_1 \;+\; S_B\eta_2\varepsilon_2 \;+\; S_C\eta_3\varepsilon_3 \;=\; 0\,,$$

which is indeed seen to be valid.
Nr-5 results by showing that $\{P_1, S\}$ and the point at infinity of the Euler line are collinear. Point P_1 is represented in barycentrics by the vector product of the corresponding line coefficients of ξ_1 and ε:

$$P_1 \;=\; \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \;=\; \begin{pmatrix} a^2b^2 + a^2c^2 - b^4 - c^4 \\ b^2c^2 + b^2a^2 - c^4 - a^4 \\ c^2a^2 + b^2c^2 - a^4 - b^4 \end{pmatrix}\,.$$

The aforementioned collinearity is expressible by the vanishing of the determinant:

$$\begin{vmatrix} (b^2 - c^2)^{-1} & (c^2 - a^2)^{-1} & (a^2 - b^2)^{-1} \\ p_1 & p_2 & p_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix} \;=\; 0\,,$$

which is indeed seen to be true. □

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