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BULGING TRIANGLES II: ON THEIR CIRCUMSCRIBED CIRCLES

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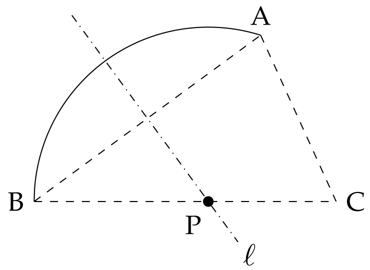
ABSTRACT. In [4], bulging triangles were introduced as one generalization of the Reuleaux triangle. We prove that any bulging triangle is inscribed in the circumscribed circle of the original triangle. Furthermore, we study geometrical properties that hold between two arcs as a supplement.

1. INTRODUCTION

Generalized Reuleaux triangles called **bulging triangles** were introduced in [4] and some geometrical properties of them were also given there. (See, e.g., [1, 2, 3, 4] for more information on Reuleaux triangles and their applications.)

"Bulging triangles $\widehat{\triangle}\mathrm{ABC}$ (stemmed) from $\triangle\mathrm{ABC}$" are defined only for acute-angled or right-angled triangles as follows:



Let $\triangle\mathrm{ABC}$ be an acute-angled or right-angled triangle. Consider the perpendicular bisector ℓ of side $\overline{\mathrm{AB}}$, and find out the intersection point P of another side and ℓ. Then, remark that P lies on the longer ($\overline{\mathrm{BC}}$ in the case of the above figure) of the remaining two sides $\overline{\mathrm{BC}}$ and $\overline{\mathrm{CA}}$. Determine intersection points Q and R in the same way for sides $\overline{\mathrm{BC}}$ and $\overline{\mathrm{CA}}$. Note that

$$|\overline{\mathrm{AP}}| = |\overline{\mathrm{BP}}|,$$
$$|\overline{\mathrm{BQ}}| = |\overline{\mathrm{CQ}}|,$$
$$|\overline{\mathrm{CR}}| = |\overline{\mathrm{AR}}|$$

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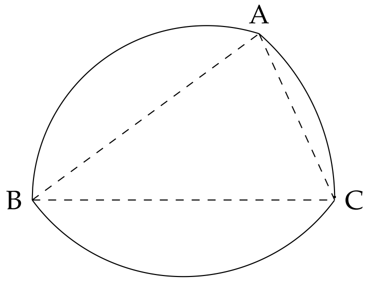
where $|\cdot|$ denotes the absolute value, i.e., the length of a segment. Here each P, Q, R is uniquely determined. P, Q and R are called the AB-**center**, BC-**center** and CA-**center** of $\hat{\triangle}$ABC, respectively. We obtain the "round side" called **edge** of $\hat{\triangle}$ABC by drawing the arc whose center and (the length of) radius are P and $|\overline{AP}|$ ($= |\overline{BP}|$), respectively. The other edges are similarly obtained. This is how bulging triangles can be defined. (See the figure below for the shape.) Vertices of a bulging triangle can be regarded as them of the original triangle. In other words, A, B and C are vertices of $\hat{\triangle}$ABC. The edges of $\hat{\triangle}$ABC are denoted by $\widetilde{AB}$, $\widetilde{BC}$ and $\widetilde{CA}$ and their lengths are written by $|\widetilde{AB}|$, $|\widetilde{BC}|$ and $|\widetilde{CA}|$, respectively. $\overline{AP}$ (or equivalently, $\overline{BP}$) is called the AB-**radius** of $\hat{\triangle}$ABC, and the others are similar.

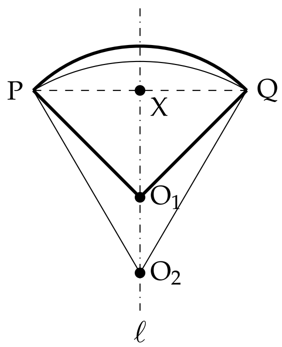


If $\triangle$ABC is an equilateral triangle, the AB-center is C, the BC-center is A and the CA-center is B. Hence, the bulging triangle defined above is certainly an extension of the Reuleaux triangle.
We saw, in [4], that bulging triangles satisfy some properties for usual triangles. Any usual triangle is inscribed in the circumscribed circle, but we are wondering if this property is also true for any bulging triangle. It particularly proved, in Theorem 2.3 of [4], that any right-angled bulging triangle is inscribed in the circumscribed circle of the original right-angled triangle, but we did not mention the general bulging triangle. In this short paper, we prove that any bulging triangle is inscribed in the circumscribed circle of the original triangle in a simpler way.

2. Theorem and the Proof

Let P and Q be any points in $\mathbb{R}^2$. Consider two arcs $\widehat{A_1}$ and $\widehat{A_2}$, which both pass through P and Q.
In general, an arc uniquely determines a circle as the whole of itself. We can thus obtain circles C_1 and C_2 determined from $\widehat{A_1}$ and $\widehat{A_2}$, respectively. We say, in this paper, that C_j **is reproduced from** $\widehat{A_j}$ for $j = 1, 2$.
Denote by O_j the center of C_j for $j = 1, 2$. We often change the name of O_j to the **center of** $\widehat{A_j}$, for the reasons above. Let ℓ be the perpendicular bisector of P and Q. Then, O_1 and O_2 obviously lie on ℓ. We put the following notation for convenience.

Definition 2.1. *Let* X *be a intersection point of ℓ and segment* $\overline{PQ}$. *We write* $\mathrm{O}_1 \preccurlyeq \mathrm{O}_2$ *if* $|\overline{\mathrm{XO}_1}| \leq |\overline{\mathrm{XO}_2}|$. *The negation of this is denoted by* $\mathrm{O}_1 \succ \mathrm{O}_2$.



The following lemma which seems trivial at first glance but should be proved is the core of our theorem.

Lemma 2.1. *Let* P *and* Q *be any points in* $\mathbb{R}^2$*. Consider two arcs* $\widehat{A_1}$ *and* $\widehat{A_2}$*, which both pass through* P *and* Q*, and let* O_j *be the center of* $\widehat{A_j}$ *for* $j = 1, 2$*. Write* D *for the area (i.e., open set) consisting of* $\widehat{A_1}$ *and segment* $\overline{PQ}$*. If* $O_1 \preccurlyeq O_2$*, then* $\widehat{A_2}$ *lies on* $D \cup \widehat{A_1}$*.*

Proof. ℓ represents the perpendicular bisector of $\overline{PQ}$. Without loss of generality, we can set $\ell = y$-axis, $P = (-p, q)$, $Q = (p, q)$, $O_1 = (0, 0)$ and $O_2 = (0, -c)$ because of the assumption "$O_1 \preccurlyeq O_2$", where $c, p, q > 0$. (See the figure above.) Moreover, we can represent, by Cartesian coordinates,

$$\begin{aligned} \widehat{A_1} &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = p^2 + q^2,\ y \geq q\}, \\ \widehat{A_2} &= \{(x, y) \in \mathbb{R}^2 : x^2 + (y + c)^2 = p^2 + (q + c)^2,\ y \geq q\}, \\ D &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < p^2 + q^2,\ y > q\}. \end{aligned}$$

Note that D is an open set. It suffices to prove that $\widehat{A_2} \subset D \cup \widehat{A_1}$. Let $(u, v) \in \widehat{A_2}$ and $v \neq q$. We then have

$$u^2 + (v + c)^2 = p^2 + (q + c)^2 \quad \text{and} \quad -2cv < -2cq$$

since $v > q$. Therefore,

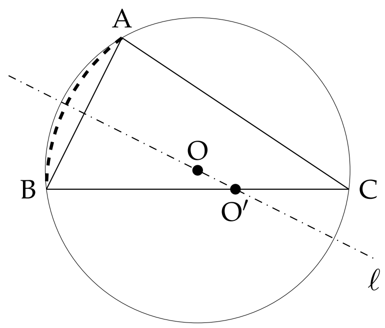
$$\begin{aligned} u^2 + v^2 &= u^2 + (v + c)^2 - 2cv - c^2 \\ &= p^2 + (q + c)^2 - 2cv - c^2 \\ &< p^2 + (q^2 + 2cq + c^2) - 2cq - c^2 \\ &= p^2 + q^2. \end{aligned}$$

So, we have $(u, v) \in D$. On the other hand, if $(u, v) \in \widehat{A_2}$ and $v = q$, then it obviously holds that $(u, v) \in \widehat{A_1}$ since $u = \pm p$. These imply that $(u, v) \in D \cup \widehat{A_1}$. Hence the proof has been finished. □

Theorem 2.1. *Any bulging triangle* $\widehat{\triangle}$ABC *is inscribed in the circumscribed circle of* $\triangle$ABC*.*

Proof. Without loss of generality, we can suppose that $|\overline{AB}| < |\overline{CA}| < |\overline{BC}|$ for $\triangle$ABC. Let O be a center of the circumscribed circle C of $\triangle$ABC and r the length of its radius. Consider the perpendicular bisector ℓ of segment $\overline{AB}$ and the intersection point O′ of ℓ

and $\overline{\mathrm{BC}}$. Denote by C' the circle with center O$'$ and passing through A and B. Moreover, write $\widehat{\mathrm{AB}}$ for the arc of C. Edge $\widetilde{\mathrm{AB}}$ of $\widehat{\triangle}$ABC can be considered as the arc of C'.



It suffices to prove that $\widetilde{\mathrm{AB}}$ lies on $D := \Omega \cup \widehat{\mathrm{AB}}$, where Ω is the area consisting of $\widehat{\mathrm{AB}}$ and $\overline{\mathrm{AB}}$. We then have $\mathrm{O} \preccurlyeq \mathrm{O}'$, so Lemma 2.1 implies that $\widetilde{\mathrm{AB}}$ lies on D. This completes the proof. □

3. Supplements

We discuss in more depth the two arcs of different lengths passing through two common points and their positions.

Lemma 3.1. *The function $f : (0, \pi) \to \mathbb{R}$ defined as $f(x) = \sin x / x$ is monotone non-increasing.*

Proof. The derivative of f is represented as

$$\frac{df}{dx}(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x (x - \tan x)}{x^2} \leq 0,$$

since

- $\cos x > 0$ and $x - \tan x < 0$ if $x \in (0, \pi/2)$,
- $x \cos x \leq 0$ and $\sin x > 0$ if $x \in [\pi/2, \pi)$.

Hence the desired result is obtained. □

Proposition 3.1. *Let* P *and* Q *be any points in $\mathbb{R}^2$. Consider two arcs $\widehat{A_1}$ and $\widehat{A_2}$, which both pass through* P *and* Q. *Denote by O_j the center of the circle reproduced from $\widehat{A_j}$ for $j = 1, 2$. Then, $|\widehat{A_1}| \geq |\widehat{A_2}|$ if and only if $\mathrm{O}_1 \preccurlyeq \mathrm{O}_2$. Here $|\widehat{A_j}|$ denotes the length of $\widehat{A_j}$.*

Proof. Assume that $|\widehat{A_1}| \geq |\widehat{A_2}|$ and $\mathrm{O}_1 \succ \mathrm{O}_2$. Let $r := |\overline{\mathrm{PO}_1}| = |\overline{\mathrm{QO}_1}|$, $\rho := |\overline{\mathrm{PO}_2}| = |\overline{\mathrm{QO}_2}|$, $\theta := \angle \mathrm{PO}_1\mathrm{Q}$ and $\phi := \angle \mathrm{PO}_2\mathrm{Q}$. Then, it holds that

$$0 < \theta < \phi < \pi \tag{3.1}$$

by the assumption. Note that $\angle \mathrm{PO}_1\mathrm{O}_2 = \angle \mathrm{QO}_1\mathrm{O}_2 = \theta/2$ and $\angle \mathrm{PO}_2\mathrm{X} = \angle \mathrm{QO}_2\mathrm{X} = \phi/2$, where X is the intersection point of the prolongation of $\overline{\mathrm{O}_1\mathrm{O}_2}$ and $\overline{\mathrm{PQ}}$. We first have

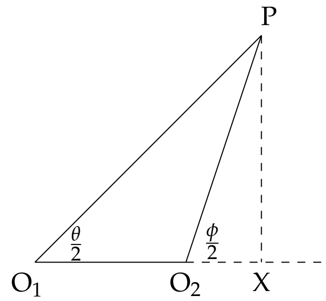
$$|\widehat{A_1}| = r\theta. \tag{3.2}$$

By the sine theorem for $\triangle \mathrm{PO_1O_2}$, we next have

$$\frac{r}{\sin(\pi - \phi/2)} = \frac{\rho}{\sin(\theta/2)}$$

and so

$$|\ \widehat{A_2}\ | = \rho\phi = \frac{\sin(\theta/2)}{\sin(\phi/2)} r\phi. \tag{3.3}$$



Since $|\ \widehat{A_1}\ | \geq |\ \widehat{A_2}\ |$ by the assumption, (3.2) and (3.3) imply that

$$\frac{\sin(\theta/2)}{\theta} \leq \frac{\sin(\phi/2)}{\phi}, \quad \text{i.e.,} \quad \frac{\sin(\theta/2)}{\theta/2} \leq \frac{\sin(\phi/2)}{\phi/2}.$$

However, by (3.1) and Lemma 3.1, this is a contradiction. Hence it have to be hold that $\mathrm{O_1} \preccurlyeq \mathrm{O_2}$, and the necessity has been proved.
The sufficiency is also easily proved by the same method. □

By virtue of this proposition, Lemma 2.1 can be rewritten as a statement with a clear meaning as follows.

Corollary 3.1. *Let* P *and* Q *be any points in* $\mathbb{R}^2$*. Consider two arcs* $\widehat{A_1}$ *and* $\widehat{A_2}$*, which both pass through* P *and* Q *and satisfy* $|\ \widehat{A_1}\ | \geq |\ \widehat{A_2}\ |$*. Write* D *for the area consisting of* $\widehat{A_1}$ *and segment* $\overline{\mathrm{PQ}}$*. Then,* $\widehat{A_2}$ *lies on* $D \cup \widehat{A_1}$*.*

4. Conclusions

This paper showed that any bulging triangle has the circumscribed circle as in the case of usual triangles (see Section 2). This study was the "homework" assigned to us in [4]. In addition, the results of a study on the positional relationship of arcs of different lengths passing through two common fixed points were given (see Section 3).
The mathematical propositions (in particular Lemma 2.1, Proposition 3.1 and Corollary 3.1) that was given in this paper appear to be fundamental. They should not however be accepted as sensibly obvious, but should be rigorously proved. This paper used analytic geometry, set and topological theory, and calculus to do so.
In the future, we would like to try to study not only geometric properties of bulging triangles but also their applications.

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