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A STUDY OF A SINGULAR CASE BY $1/0 = 0$

HIROSHI OKUMURA

Abstract. We consider a configuration of four circles in the plane in the case where one of the circles degenerates to a point or a line using recently made the definition of division by zero $1/0 = 0$. The result shows that the new concept is very powerful for the study of such a case and also shows that it would open an entirely new world of mathematics.

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1. Introduction

It seems to be a custom that we do not consider singular cases in our mathematics, or to be more accurate, we have not been able to consider such cases for a long time. In this article we show that we can consider such cases by introducing recently made the definition of division by zero $1/0 = 0$ [2], [24]. The result will also show that the new concept is very powerful for the study of such cases and also shows that it would open an entirely new world of mathematics.

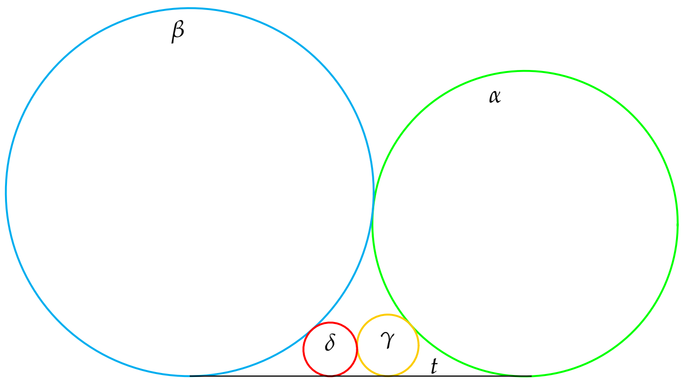

Figure 1: The configuration $\mathcal{S}$.

Let α and β be external touching circles in the plane with an external common tangent t. If γ and δ are external touching circles lying inside of the curvilinear triangle made by α, β and t such that γ and δ touch α and β, respectively, and both touch t. We denote the configuration consisting of the four circles and t by $\mathcal{S}$ (see Figure 1). The configuration was considered in Wasan geometry, where Wasan means Japanese mathematics developed in Edo era. We consider the limiting cases of this configuration, in which each of the circles α and γ degenerates to a point or a line by introducing $z/0 = z \cdot 0 = 0$ for any real number z.

2. Problem

Let a, b, c and d be the radii of α, β, γ and δ, respectively. Firstly we consider the next problem in [25].

Problem 1. *Show that the following relation holds for $\mathcal{S}$.*

$$d = a\left(\frac{\sqrt{b}-\sqrt{c}}{\sqrt{b}+\sqrt{c}}\right)^2. \tag{1}$$

The same configuration can also be found in [26]. We give a simple solution using the next fact:

Proposition 2.1. *If two externally touching circles of radii p and q touch one of their external common tangents at points P and Q, then $|PQ| = 2\sqrt{pq}$ holds.*

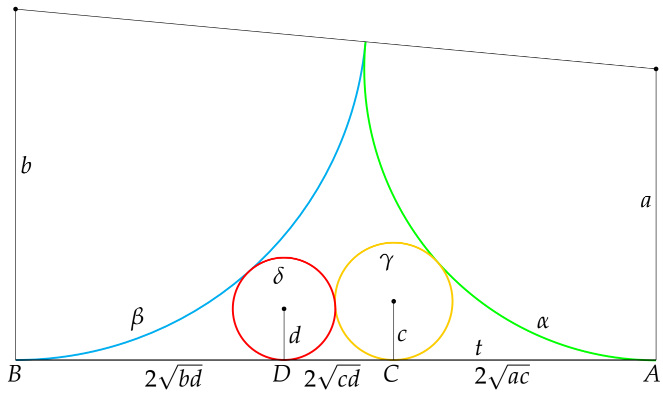

Figure 2.

Solution. From $|AC| = 2\sqrt{ac}$ and so on, we have $2\sqrt{ab} = 2\sqrt{ac} + 2\sqrt{bd} + 2\sqrt{cd}$. This implies

$$1 = \sqrt{\frac{c}{b}} + \sqrt{\frac{d}{a}} + \sqrt{\frac{c}{b}\frac{d}{a}}.$$

Therefore we have the following equation.

$$\left(\sqrt{\frac{d}{a}}+1\right)\left(\sqrt{\frac{c}{b}}+1\right) = 2. \tag{2}$$

Solving the equation for d, we have (1). □

3. Division by zero $1/0 = 0$

To consider the case $a = 0$ or $c = 0$, we need recently made the definition of division by zero $1/0 = 0$ [2], [24]. Therefore we briefly introduce this at first. For a field F, we consider the following canonical bijection $\psi : F \to F$:

$$\psi(a) = \begin{cases} a^{-1} & \text{if } a \neq 0 \\ a & \text{if } a = 0. \end{cases}$$

It is a custom to denote $z\psi(a)$ by z/a if $a \neq 0$, i.e., $z\psi(a) = a/z$ for $a \neq 0$. Following to this, we also write

$$z\psi(0) = \frac{z}{0} \;\; for \;\forall z \in F.$$

Then we have

$$\frac{z}{a} = z \cdot \psi(a) \;\; for \;\forall a, \forall z \in F.$$

Especially we have

$$\frac{z}{0} = z \cdot 0 = 0 \ \ for \ \forall z \in F. \tag{3}$$

Therefore *to divide by zero is to multiply by zero* [15]. Notice that the reduction to common denominator can not be made for $z/0$, i.e., we have the following relation in general in the case $b = 0$ or $d = 0$:

$$\frac{a}{b} + \frac{c}{d} \neq \frac{ad + bc}{bd}.$$

From now on we assume (3).
To consider the value of a non-simple function when division by zero occurs in it, we need a generalization of (3). For a meromorphic function $f(z)$, we consider the Laurent expansion of f around $z = a$:

$$f(z) = \sum_{n=-\infty}^{n=-1} C_n (z-a)^n + C_0 + \sum_{n=1}^{n=\infty} C_n (z-a)^n.$$

Then we *define* $f(a) = C_0$. The definition is called *division by zero calculus* [24]. The following example may be of help to see the need of this definition. For a function $f(x) = (e^x - 1)/x$, the reader may consider $f(0) = 0$ by (3). However from the Laurent expansion of f around 0:

$$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots,$$

we get $f(0) = 1$, which is natural in mathematical sense. By the Laurent expansion of $\tan x$ around $\pi/2$

$$\tan x = \frac{-1}{x - \frac{\pi}{2}} + \frac{x - \frac{\pi}{2}}{3} + \frac{(x - \frac{\pi}{2})^3}{45} + \cdots,$$

we have

$$\tan \frac{\pi}{2} = 0. \tag{4}$$

We get the next theorem [24].

Theorem 3.1. *The following statements are true.*
(i) *A line can be considered to be a circle of radius* 0*.*
(ii) *Two orthogonal figures can be considered to touch each other.*

Proof. Since most mathematicians are unfamiliar with applying (3) at the time of writing, we give a proof of the theorem here. A line or a circle is represented by the equation

$$e(x^2 + y^2) + 2fx + 2gy + h = 0.$$

If this represents a circle, it radius equals

$$\sqrt{\frac{f^2 + g^2 - eh}{e^2}}.$$

Hence (i) follows from (3), for a line is represented if $e = 0$. This proves (i).
Assume two figures have a point P in common and the angle between the tangent lines at P equals θ. Then the two figures are said to touch at P if and only if $\tan\theta = 0$. While the angle between the tangent lines at the point of intersection equals $\pi/2$ for two orthogonal figures. This proves (ii) by (4). □

From now on, we consider that the following two figures touch: (t1) overlapping two points or two lines or two circles, (t2) a point and a line or a circle passing through the point, (t3) two parallel lines, (t4) orthogonal two lines or a line and a circle or two circles.

For two touching proper circles, we distinguish internal touching and external touching. However for the mentioned above figures, the words "internal" and "external" have no sense. Therefore when we consider propositions using the words "internal touching" or "external touching" for the above figures, we construe that it means merely touching.

4. THE CASE $a = 0$ WITH FIXED CIRCLES γ AND δ

In this section we fix the circles γ and δ and consider the case in which the circle α has radius 0. Notice that the shape of a circle is uniquely determined by its radius if it is proper. However there are two figures for circles of radius 0, a point circle and a line by Theorem 3.1(i). Therefore we would obtain several configurations $\mathcal{S}$ in the case $a = 0$. We use a rectangular coordinate system with origin at the midpoint of CD so that the center of γ has coordinates $(\sqrt{cd}, c)$. The point coincides with the point of intersection of t and the internal common tangent of γ and δ.

4.1. **The circle α in the case $a = 0$.** The circle α has an equation $(x - \sqrt{cd} - 2\sqrt{ac})^2 + (y - a)^2 = a^2$. This can be arranged in three ways as follows:

$$\begin{aligned}(x-\sqrt{cd})^2 + y^2 - 4\sqrt{c}(x-\sqrt{cd})\sqrt{a} - 2(y-2c)a &= 0,\\ \frac{(x-\sqrt{cd})^2 + y^2}{\sqrt{a}} - 4\sqrt{c}(x-\sqrt{cd}) - 2(y-2c)\sqrt{a} &= 0,\\ \frac{(x-\sqrt{cd})^2 + y^2}{a} - \frac{4\sqrt{c}(x-\sqrt{cd})}{\sqrt{a}} - 2(y-2c) &= 0.\end{aligned}$$

Therefore we have $(x - \sqrt{cd})^2 + y^2 = 0$, $x = \sqrt{cd}$, $y = 2c$ if $a = 0$ by (3). The three equations represent the point C, the perpendicular to t at C, and the tangent of γ parallel to t, which are denoted by α_0, α_1 and α_2, respectively (see the green figures in Figure 4).

4.2. **The circle β in the case $a = 0$.** The circle β has an equation $(x + \sqrt{cd} + 2\sqrt{bd})^2 + (y - b)^2 = b^2$. While solving (2) for b, we have

$$b = c\left(\frac{\sqrt{a}+\sqrt{d}}{\sqrt{a}-\sqrt{d}}\right)^2.$$

Eliminating b from the two equations and rearranging we have

$$\frac{b_1(x,y) + b_2(x,y)\sqrt{a} + b_3(x,y)a}{(\sqrt{a}-\sqrt{d})^2} = 0, \tag{5}$$

where

$$\begin{aligned}b_1(x,y) &= d((x-\sqrt{cd})^2 + (y-c)^2 - c^2),\\ b_2(x,y) &= -2\sqrt{d}((x+\sqrt{cd})^2 + (y+c)^2 - c(c+4d)),\\ b_3(x,y) &= (x+3\sqrt{cd})^2 + (y-c)^2 - c^2.\end{aligned}$$

From (5) we have $b_1 + b_2\sqrt{a} + b_3 a = 0$, $b_1/\sqrt{a} + b_2 + b_3\sqrt{a} = 0$, and $b_1/a + b_2/\sqrt{a} + b_3 = 0$. Therefore $a = 0$ implies $b_1 = b_2 = b_3 = 0$ by (3). Hence if $a = 0$, we have

$$\begin{aligned}(x-\sqrt{cd})^2 + (y-c)^2 &= c^2,\\ (x+\sqrt{cd})^2 + (y+c)^2 &= c(c+4d),\\ (x+3\sqrt{cd})^2 + (y-c)^2 &= c^2.\end{aligned}$$

The three equations represent the circle γ, the circle of radius $\sqrt{c(c+4d)}$ and the center of coordinates $(-\sqrt{cd}, -c)$, and the circle of radius c and the center of coordinates $(-3\sqrt{cd}, c)$, which are denoted by β_1, β_2, and β_3, respectively (see the light blue circle(s) in Figure 3 for β_2 and Figure 4 for β_1 and β_3). Notice that β_1 and β_3 are symmetric about the perpendicular to t at the point D.

4.3. **The circle β_2.** The circle β touches δ and t, however the circle β_2 does not touch them. Therefore β_2 is not eligible to be the circle β in the case $a = 0$. However it is remarkable that β_2 still intersects δ and t with the same angle θ_b, where $\cos\theta_b = \sqrt{c/(c+4d)}$, and intersects γ with the angle $\pi - \theta_b$ (see Figure 3).

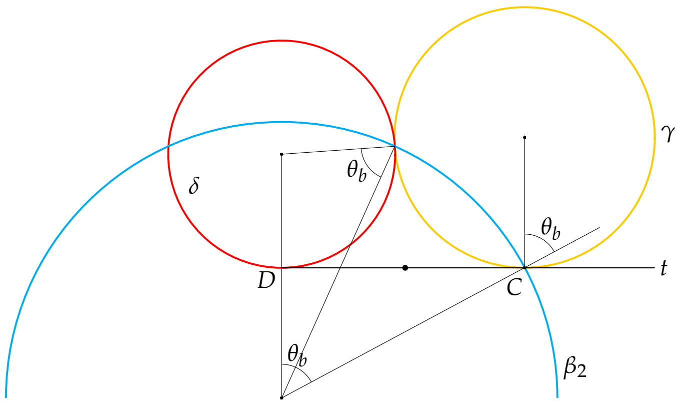

Figure 3: The circle β_2.

The point of tangency of γ and δ is the internal center of similitude of the two circles and has coordinates $((d-c)\sqrt{cd}/(c+d), 2cd/(c+d))$. With this fact we can see that this point lie on β_2. The point C also lies on β_2.

4.4. **The configuration $\mathcal{S}$ in the case $a = 0$.** We now explicitly denote the configuration $\mathcal{S}$ by $\mathcal{S}_{\alpha_i\beta_j}$ if $\alpha = \alpha_i$ and $\beta = \beta_j$. By the remarks stated in the two paragraphs at the end of section 3, we get four configurations $\mathcal{S}_{\alpha_0\beta_1}$, $\mathcal{S}_{\alpha_1\beta_1}$, $\mathcal{S}_{\alpha_2\beta_1}$, $\mathcal{S}_{\alpha_2\beta_3}$. Since $a = 0$ and $b = c$, (2) still holds for those configurations.

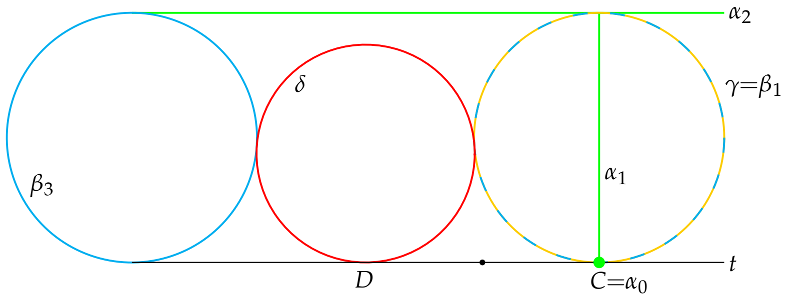

Figure 4: $\mathcal{S}_{\alpha_0\beta_1}$, $\mathcal{S}_{\alpha_1\beta_1}$, $\mathcal{S}_{\alpha_2\beta_1}$, $\mathcal{S}_{\alpha_2\beta_3}$.

5. The case $c = 0$ with fixed circles α and β

In this section we fix the circles α and β and consider the case $c = 0$ with a rectangular coordinate system with origin at the midpoint of AB so that the center of α has coordinates $(\sqrt{ab}, a)$.

5.1. **The circle γ in the case $c = 0$.** The circle γ has an equation $(x - \sqrt{ab} + 2\sqrt{ac})^2 + (y - c)^2 = c^2$. This can be arranged in three ways as follows:

$$\begin{aligned}
(x - \sqrt{ab})^2 + y^2 + 4\sqrt{a}(x - \sqrt{ab})\sqrt{c} - 2(y - 2a)c &= 0, \\
\frac{(x - \sqrt{ab})^2 + y^2}{\sqrt{c}} + 4\sqrt{a}(x - \sqrt{ab}) - 2(y - 2a)\sqrt{c} &= 0, \\
\frac{(x - \sqrt{ab})^2 + y^2}{c} + \frac{4\sqrt{a}(x - \sqrt{ab})}{\sqrt{c}} - 2(y - 2a) &= 0.
\end{aligned}$$

Therefore $c = 0$ implies $(x - \sqrt{ab})^2 + y^2 = 0$, $x = \sqrt{ab}$ and $y = 2a$. The three equations represent the point A, the perpendicular to t at A and the tangent of α parallel to t, which are denoted by γ_0, γ_1 and γ_2, respectively (see the yellow figures in Figure 6).

5.2. **The circle δ in the case $c = 0$.** The circle δ has an equation $(x + \sqrt{ab} - 2\sqrt{bd})^2 + (y - d)^2 = d^2$. Substituting (1) in this equation and rearranging we have

$$\frac{d_1(x, y) + d_2(x, y)\sqrt{c} + d_3(x, y)c}{(\sqrt{b} + \sqrt{c})^2} = 0, \tag{6}$$

where

$$\begin{aligned}
d_1(x, y) &= b((x - \sqrt{ab})^2 + (y - a)^2 - a^2), \\
d_2(x, y) &= 2\sqrt{b}((x + \sqrt{ab})^2 + (y + a)^2 - a(a + 4b)), \\
d_3(x, y) &= (x + 3\sqrt{ab})^2 + (y - a)^2 - a^2.
\end{aligned}$$

From (6) we have $d_1 + d_2\sqrt{c} + d_3 c = 0$, $d_1/\sqrt{c} + d_2 + d_3\sqrt{c} = 0$, $d_1/c + d_2/\sqrt{c} + d_3 = 0$. Therefore if $c = 0$ we get $d_1 = d_2 = d_3 = 0$. Hence $c = 0$ implies

$$\begin{aligned}
(x - \sqrt{ab})^2 + (y - a)^2 &= a^2, \\
(x + \sqrt{ab})^2 + (y + a)^2 &= a(a + 4b), \\
(x + 3\sqrt{ab})^2 + (y - a)^2 &= a^2.
\end{aligned}$$

The three equations represent the circle α, the circle of radius $\sqrt{a(a + 4b)}$ and the center of coordinates $(-\sqrt{ab}, -a)$, and the circle of radius a and the center of coordinates $(-3\sqrt{ab}, a)$, which are denoted by δ_1, δ_2, and δ_3, respectively (see the red circle(s) in Figure 5 for δ_2 and Figure 6 for δ_1 and δ_3). Notice that δ_1 and δ_3 are symmetric about the perpendicular to t at the point D.

5.3. **The circle δ_2.** The circle δ touches β and t, however the circle δ_2 does not touch them. Hence it is not eligible to be the circle δ in the case $c = 0$. However it is remarkable that δ_2 still intersects β and t with the same angle θ_d, where $\cos\theta_d = \sqrt{a/(a + 4b)}$, and intersects α with the angle $\pi - \theta_d$ (see Figure 5). The point of tangency of α and β has coordinates $((b - a)\sqrt{ab}/(a + b), 2ab/(a + b))$, and lies on δ_2. The point A also lies on δ_2.

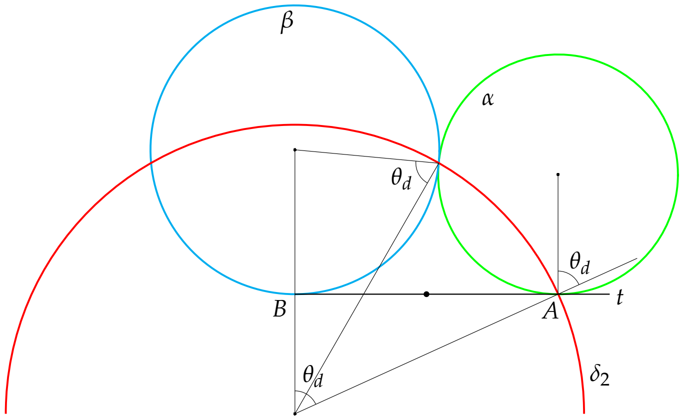

Figure 5: The circle δ_2.

5.4. **The configuration $\mathcal{S}$ in the case $c = 0$.** We
We now explicitly denote the configuration $\mathcal{S}$ by $\mathcal{S}_{\gamma_i\delta_j}$ if $\gamma = \gamma_i$ and $\delta = \delta_j$. Then we get four configurations $\mathcal{S}_{\gamma_0\delta_1}$, $\mathcal{S}_{\gamma_1\delta_1}$, $\mathcal{S}_{\gamma_2\delta_1}$, $\mathcal{S}_{\gamma_2\delta_3}$ (see Figure 6). Since $c = 0$ and $a = d$, (2) still holds for those configurations. We see that the result in this section is very similar to that of the previous section.

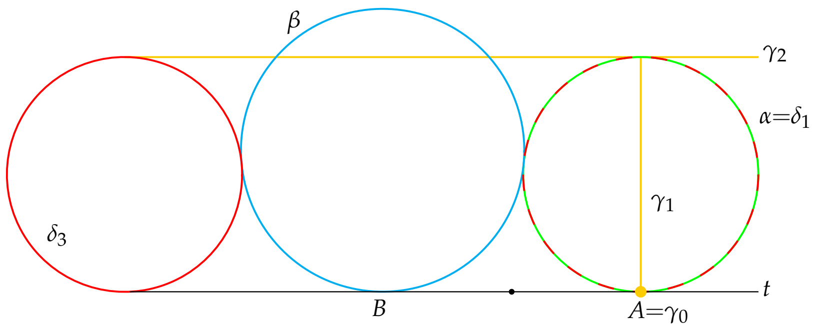

Figure 6: $\mathcal{S}_{\gamma_0\delta_1}$, $\mathcal{S}_{\gamma_1\delta_1}$, $\mathcal{S}_{\gamma_2\delta_1}$, $\mathcal{S}_{\gamma_2\delta_3}$.

6. Conclusion

The singular case has long been ignored in our mathematics, since we have no idea to handle. However the result in this paper shows that we can consider the case by division by zero and division by zero calculus moreover the case is worthy of being considered because of its variety. The definition of division by zero $1/0 = 0$ was founded by Saburou Saitoh in 2014. He has been making a list of successful example applying division by zero and division by zero calculus. There are more than 1200 examples in his list, which are evidences showing that an entirely new world of mathematics would be opened by introducing them. For an extensive reference of division by zero and division by zero calculus including those evidences, see [24]. For applications to Euclidean spaces see [4]. For applications of division by zero calculus to plane geometry see [7]. Especially applications to Wasan geometry and circle geometry of division by zero and division by zero calculus, see [1], [3], [5, 6, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19], [20, 21, 22, 23].

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MAEBASHI GUNMA 371-0123, JAPAN
Email address: `hokmr@yandex.com`