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SOME METRIC PROPERTIES OF GENERAL SEMI-REGULAR POLYGONS

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ABSTRACT. If we construct isosceles polygon P_k^a with $(k-1)$ sides of the same length over each side of regular polygon P_n^b, with n sides, we get a new equilateral polygon $P_N^{a,\delta}$ with $N=(k-1)n$ sides of the same length and with different inner angles. Such polygon is called equilateral semi-regular polygon. This paper deals with metric properties of general semi-regular polygons P_N^a, with given side a and angle δ, defined by equation $\delta = \angle(a, d_1) = \angle(d_i, d_{i+1})$, $i = 1, 2, \ldots, k-3$, as an angle between diagonals d_i drawn from the apex A_j, $j = 1, 2, \ldots, n$ of the regular polygon with sides $d_{k-2} = A_jA_{j+1} = b$ as well as the application of the obtained results.

1. INTRODUCTION

Given the set of points $A_j \in E^2$,$j = 1, 2, \ldots, n$ in Euclidian plane E^2, such that any three successive points do not lie on a line p and for which we have a rule: if $A_j \in p$ and $A_{j+1} \in p$ for each j point A_{j+2} does not belong to the line p.

1. Polygon P_n or closed polygonal line is the union along $A_1A_2, A_2A_3, \ldots, A_nA_{n+1}$, and write short

$$P_n = \bigcup_{j+1}^{n} A_jA_{j+1}, (n+1 \equiv 1 \mod n) \tag{1.1}$$

Points A_j are vertices, and lines A_jA_{j+1} are sides of polygon P_n.

2. The angles on the inside of a polygon formed by each pair of adjacent sides are angles of the polygon

3. If no pair of polygon's sides, apart from the vertex, has no common points, that is , if $A_jA_{j+1} \cap A_{j+l}A_{j+l+1} = \emptyset, l \neq 1$
polygon is simple, otherwise it is complex. This paper deals with simple polygons only.

4. Simple polygons can be convex and non-convex. Polygon is convex if it all lies on the same side of any of the lines A_jA_{j+1}, otherwise it is non-convex. Polygon P_n divides plane E^2 into two disjoint subsets,U and V. Subset U is called interior, and subset V is exterior area of the polygon. Union of polygon P_n and its interior area U_n makes *polygonal area* S_n, which is:

$$S_n = P_n \cup U_n \tag{1.2}$$

Key words and phrases. semi regular polygons, regular polygons,polygons .
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5. Given polygon P_n with vertices $A_j, j = 1, 2, \ldots, n, (n + 1 \equiv 1 \mod n)$ lines of which A_jA_i are called polygonal diagonals if indices are not consecutive natural numbers, that is, $j \neq i$. We can draw $n - 3$ diagonals from each vertex of the polygon with n number of vertices.
6. Exterior angle of the polygon P_n with vertex A_j is the angle $\angle A_{v,j}$ with one side $A_{j+1}A_j$, and vertex A_j, and the other one is extension of the side A_jA_{j-1} through vertex A_j
7. Sum of all exterior angles of the given polygon P_n is equal to multiplied number or product of tracing around the polygons in a certain direction and 2π, that is, the rule is

$$\sum_{j=1}^{n}(\angle A_{v,j}) = 2k\pi, k \in \mathrm{Z} \tag{1.3}$$

In which k is number of turning around the polygon in certain direction.
8. The interior angle of the polygon with vertex A_j is the angle $\angle A_{u,j}, j = 1, 2, \ldots, n$ for which $\angle A_{u,j} + \angle A_{v,j} = \pi$. That is the angle with one side $A_{j-1}A_j$, and the other side A_jA_{j+1}. Sum of all interior angles of the polygon is defined by equation

$$\sum_{j=1}^{n}\angle A_{u,j} = (n - 2k)\pi, n \in \mathrm{N}, k \in \mathrm{Z}. \tag{1.4}$$

In which k is number of turning around the polygon in certain direction.
9. A regular polygon is a polygon that is equiangular (all angles are equal in measure) and equilateral (all sides have the same length). Regular polygon with n sides of b length is marked as P_n^b. The formula for interior angles γ of the regular polygon P_n^b with n sides is $\gamma = \frac{(n-2)\pi}{n}$. A non-convex regular polygon is a regular star polygon.For more about polygons in [4,5,6].
10. Polygon that is either equiangular or equilateral is called *semi-regular polygon*. Equilateral polygon with different angles within those sides are called *equilateral semi-regular* polygons, whereas polygons that are *equiangular* and with sides different in length are called *equiangular* semi regular polygons. For more about in [1,2,3].

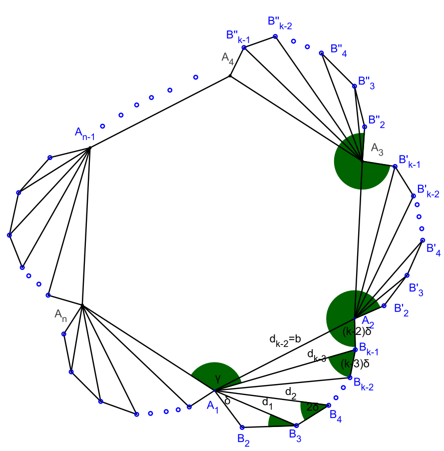

FIGURE 1. Convex semi-regular polygon P_N with $N = (k-1)n$ sides constructed above the regular polygon P_n^b

2. BASIC TERMS, SIGNS AND DEFINITIONS

1. If we construct a polygon P_k with $(k-1)$ sides, $k \geq 3, k \in \mathrm{N}$ with vertices $B_i, i = 1,2,\ldots,k$ over each side of the convex polygon $P_n, n \geq 3, n \in \mathrm{N}$ with vertices $A_j, j = 1,2,\ldots,n$, $(n+1) \equiv 1 \mod n$, that is $A_j = B_1, A_{j+1} = B_k$, we get new polygon with $N = (k-1)n$ sides, (figure 1) marked as P_N.
Here are the most important elements and terms related to constructed polygons:

(1) Polygon P_k with vertices $B_1B_2\ldots B_{k-1}B_k$, $B_1 = A_j, B_k = A_{j+1}$ constructed over each side $A_jA_{j+1}, j = 1,2,\ldots,n$ of polygon P_n with which it has one side in common is called edge polygon for polygon P_n.
(2) $A_jB_2, B_2B_3, \ldots, B_{k-1}A_{j+1}, j = 1,2,\ldots,n$ the sides polygon P_k.
(3) $A_jB_2A_jB_3, \ldots, A_jB_{k-1}$ are diagonals $d_i, i = 1,2,\ldots,k-2$,of the polygon P_k^a drawn from the top A_j and that implies $d_{k-2} = A_jA_{j+1} = b$.
(4) Angles $\angle B_{u,i}$ are interior angles of vertices $B_{u,i}$ of the polygon P_N and are denote as β_i. Interior angle $\angle A_{u,j}$ of the polygon of the vertices A_j are denoted as α_j.
(5) Polygon P_k of the side a constructed over the side b of the polygon P_n is isosceles, with $(k-1)$ equal sides, is denoted as P_k^a.
(6) With $\delta = \angle(d_i, d_{i+1}), i = 1,2,\ldots,k-2$ denotes the angle between its two consecutive diagonals drawn from the vertices $A_j, j = 1,2,\ldots,n$ for which it is true

$$\delta = \angle(a, d_1) = \angle(d_id_{i+1}), i = 1,2,\ldots,k-3, d_{k-2} = b \tag{2.1}$$

(7) If the isosceles polygon P_k^a is constructed over each side of the b regular polygon P_n^b with n sides, then the constructed polygon with $N = (k-1)n$ of equal sides is called equilateral *semi-regular polygon* which is denoted as P_N^a.

2. We analyzed here some metric characteristics of the general equilateral semi-regular polygons, if side a is given, and angle is $\delta = \angle(d_i, d_{i+1}), i = 1,2,\ldots,(k-2)$, in between the consecutive diagonals of the polygon P_k^a drawn from the vertex P_k^a of the regular polygon P_n^b. Such semi-regular polygon with $N = (k-1)n$ sides of a length and angle δ defined in (2.1) we denote as $P_N^{a,\delta}$.
3. Regular polygon P_n^b polygon is called corresponding regular polygon of the semi-regular polygon $P_N^{a,\delta}$.
4. Interior angles of the semi-regular equilateral polygon is divided into two groups
- angles at vertices $B_i, i = 2,\ldots,k-1$ we denote as β,
- angles at vertices $A_j, j = 1,2,\ldots,n$ we denote as α.

5. K_N stands for the sum of the interior angles of the semi-regular polygon $P_N^{a,\delta}$.
6. $S_{A_j}^{\gamma}$ stands for the sum of diagonals comprised by angle γ and drawn from the vertex A_j, and with $\varepsilon_{A_j}^{\gamma}$ we denote the angle between the diagonals drawn from vertex A_j comprised by angle γ.
7. We denote the radius with r_N of the inscribed circle of the semi-regular polygon $P_N^{a,\delta}$.

3. SOME METRIC PROPERTIES OF SEMI-REGULAR POLYGONS $P_N^{a,\delta}$

3.1. **Interior angles of the semi-regular polygon.** Let on each side of the regular polygon P_n^b, be constructed polygon P_k^a, with $(k-1)$ equal sides, and let $d_l = A_jB_i, l = 1,2,\ldots,k-2, d_{k-2} = A_jA_{j+1} = b, j = 1,2,\ldots,n; i = 3,4,\ldots,k; B_k = A_{j+1}$ diagonals drawn from the vertex $A_j, A_jA_{j+1} = b, j = 1,2,\ldots,n$ to the vertices B_i of the polygon P_k^a.

The following lemma is valid for interior angles at vertices $B_i, i = 3, 4, \ldots, k, B_k = A_{j+1}$ of triangle $\triangle A_j B_{i-1} B_i$ determined by diagonals d_i.

Lemma 3.1. *Ratio of values of interior angles $\triangle A_j B_{i-1} B_i; i = 3, 4, \ldots, k$ at vertex B_i of the base $A_j B_i = d_{i-2}$ from the given angle $A_j B_i = d_{i-2}$ δ is defined by relation $\angle B_i = (i-2)\delta$.*

Proof. The proof is done by induction on i, $(i \geq 3), i \in \mathrm{N}$. Let us check this assertion for $i = 4$ because for $i = 3$ the claim is obvious because the triangle erected on the sides of the regular polygon is isosceles and angles at the base b are equal as angle δ.
If $i = 4$ and isosceles rectangle is constructed on side b of the regular polygon P_n^b (figure 2a) with vertices $A_1 B_2 B_3 B_4$, i $B_4 = A_2$ where $A_1 A_2 = b$ side of the regular polygon.
Diagonals constructed from the vertex A_1 divide polygon $A_1 B_2 B_3 B_4$, into triangles $\triangle A_1 B_2 B_3$ and $\triangle A_1 B_3 B_4$. According to the definition of the angle δ we have:

$$\angle B_2 A_1 B_3 = \angle B_2 B_3 A_1 = \angle B_3 A_1 B_4 = \delta$$

Intersection of the centerline of the triangle's base $\triangle A_1 B_2 B_3$ i $A_1 B_4 = b$ is point S_1.Since $A_1 B_2 = B_2 B_3 = a$, a and construction of the point S_1 leads to conclusion that $\square A_1 S_1 B_2 B_3$ is a rhombus with side a. Since $B_3 S_1 = B_3 B_4 = a$ a triangle $\triangle B_3 S_1 B_4$ is isosceles, and its interior angle at vertex S_1 is exterior angle of the triangle $\triangle A_1 B_3 S_1$, thus $\angle S_1 = 2\delta$, as well as $\angle B_4 = 2\delta$.

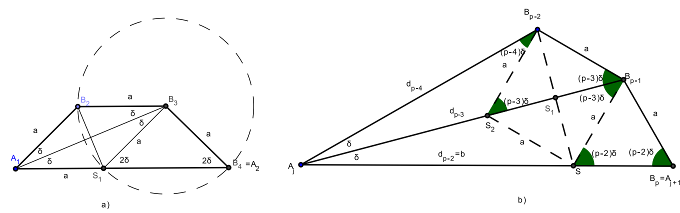

FIGURE 2. a. Rectangle $A_1 B_2 B_3 B_4$ b.Rectangle $A_j B_{p-2} B_{p-1} B_p$

Let us presume that the claim is valid for an arbitrary integer $(p-1), (p \geq 4), p \in \mathrm{N}$, that is $i = (p-1)$ interior angle of the triangle $\triangle A_j B_{p-2} B_{p-1}$ at the vertex B_{p-1} has value $\angle B_{p-1} = (p-3)\delta$.
Let us show now that this ascertain is true for integer p, that is for $i = p$. Also, interior angle of the triangle $\triangle A_j B_{p-1} B_p$ at vertex B_p has value $\angle B_p = (p-2)\delta$.
Let us note $\square A_j B_{p-2} B_{p-1} B_p$ which is split into triangles $\triangle A_j B_{p-2} B_{p-1}$ and $\triangle A_j B_{p-1} B_p$ by diagonal d_{p-3}, and that $\angle B_{u,p-1} = (p-3)\delta$ according to presumption.
Since interior angles of triangles are congruent at vertex A_j, by definition of angle δ,and $B_{p-2} B_{p-1} = B_{p-1} B_p = a$, it is easily proven that there is point S such that triangle $\triangle S B_{p-1} B_p$ is isosceles triangle(figure 2), and rectangle $\square A_j B_{p-2} B_{p-1} S$ is rectangle with perpendicular diagonals. Congruence of triangles $\triangle A_j B_{p-2} B_{p-1} \simeq \triangle A_j B_{p-1} S$ leads us to conclusion that $\angle A_j B_{p-1} S = (p-3)\delta$. Angle at vertex S is the exterior angle of triangle $\triangle A_j B_{p-2} B_{p-1}$. And thus we have $\angle S = \delta + (p-3)\delta = (p-2)\delta$.
Since triangle $S B_{p-1} B_p$ isosceles, $\angle B_p = (p-2)\delta$ which we were supposed to prove. So, for each $i \in \mathrm{N}, i \geq 3$ interior angle of triangle $\triangle A_j B_{i-1} B_i$ at vertex B_i is $\angle B_i = (i-2)\delta$.

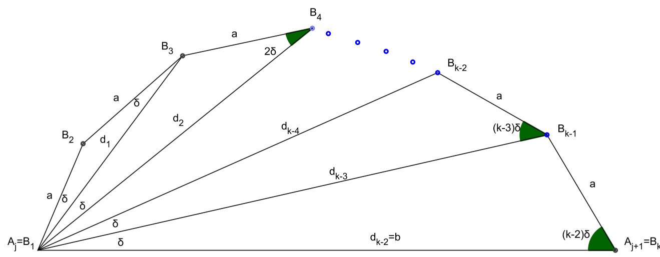

FIGURE 3. Isosceles polygon P_k^a constructed on side b of the regular polygon P_n^b

Lemma 3.2. *Semi regular equilateral polygon $P_{(k-1)n}^{a,\delta}$ with given side a and angle δ defined with* (2.1)*,has n interior angles equal to an that angle*

$$\alpha = \frac{(n-2)\pi}{n} + 2(k-2)\delta \tag{3.1}$$

and $(k-2)n$ interior angles equal to an that angle

$$\beta = \pi - 2\delta, \delta > 0, k \geq 3, n \geq 3, k, n \in \mathrm{N} \tag{3.2}$$

Proof. Using figure 3 and results of lemma 3.1, it is easily proven that polygon P_k^a constructed on side b of the regular polygon P_n^b has $(k-2)$ interior angles with value P_n^b $\pi - 2\delta$, and which are at the same time interior angles of the semi-regular polygon $P_N^a, N = (k-1)$.
So indeed, for $k = 3$ the constructed polygon P_k is isosceles triangle with interior angle at vertex $B_2 = \pi - 2\delta$, and for $k = 4$ constructed polygon is isosceles rectangle (figure 2). That rectangle is drawn by diagonal d_1 from vertex A_j,$j = 1, 2, \ldots, n$, $B_1 \equiv A_1$,$B_4 \equiv A_{j+1}$ and $A_jA_{j+1} = b$ split into triangles $A_jB_2B_3$ and $A_jB_3A_{j+1}$ with interior angles at vertices $\angle B_2 = \angle B_3 = \pi - 2\delta$.
Similarly is proven that for every rectangle $A_jB_{i-2}B_{i-1}B_i, i = 4, 5, \ldots, k$; $(B_1 = A_j, B_k = A_{j+1}$,$A_jA_{j+1} = b)$ and the value of its vertex B_{i-1},

$$\angle B_{i-1} = (i-3)\delta + \pi - [(i-2)\delta + \delta] = \pi - 2\delta.$$

So, in every isosceles polygon P_k^a exist $k-2$ interior angles with measure $\pi - 2\delta$. Since isosceles polygon P_k^a, is constructed on each side of regular polygon P_n^b, it follows that equilateral semi-regular polygon P_N^a has total of $(k-2)n$ angles, which we were supposed to prove.
When interior angle of the semi-regular equilateral polygon at vertex A_j, $j = 1, 2, \ldots, n$ is equal to sum of interior angle of the regular polygon P_n^b and double value of the interior angle of the polygon P_k^a at vertex B_k, (Lema 3.1)is valid $\angle A_{u,j} = \frac{(n-2)\pi}{n} + 2(k-2)\delta$ which we were supposed to prove.

Corollary 3.3. *Sum of interior angles of the equilateral semi-regular polygon $P_N^{a,\delta}$ is given in*

$$K_N = [N-2]\pi \tag{3.3}$$

Proof. On the basis of Lema 3.2. and relation (3.1) and (3.2) it is true that
$K_N = n\alpha + \beta(k-2)n = n[\frac{(n-2)\pi}{n} + 2\delta(k-2)] + n(k-2)(\pi - 2\delta) = (n-2)\pi + n(k-2)\pi = (nk-n-2)\pi = [n(k-1)-2]\pi = [N-2]\pi.$

3.2. **Convexity semi regular equilateral polygons $P_N^{\alpha,\delta}$.** Condition of convexity of the semi-regular equilateral polygon $P_N^{\alpha,\delta}$ and the values of its angle δ is given in the theorem.

Theorem 3.4. *Equilateral semi-regular polygon $P_N^{\alpha,\delta}$, $N = (k-1)n$ is convex if the following is true for the angle δ*

$$\delta \in \left(0; \frac{\pi}{(k-2)n}\right) \quad k, n \in \mathrm{N}, n, k \geq 3. \tag{3.4}$$

Proof. Let us write values of the interior angles of the semi-regular polygon $P_N^{\alpha,\delta}$ defined by relations (3.1),(3.2) in the form of linear functions

$$f(\delta) = \frac{(n-2)\pi}{n} + 2(k-2)\delta, g(\delta) = \pi - 2\delta, \dot{k}, n \in \mathrm{N}, \dot{k}, n \geq 3. \tag{3.5}$$

As the polygon is convex if all its interior angles are smaller than π, to prove the theorem it is enough to show that for $\forall\delta \in \left(0; \frac{\pi}{(k-2)n}\right)$ all interior angles of the semi-regular polygon $P_N^{\alpha,\delta}$ are smaller than π.
Indeed, from this relation $\beta = g(\delta) = \pi - 2\delta$ follows that $\beta = 0$ for $\delta = \frac{\pi}{2}$, (figure 4). On the basis of this and demands $\beta > 0$ and $\delta > 0$, we find that $\beta \in (0, \pi)$ and $0 < \delta < \frac{\pi}{2}$, and thus we have $\delta \in \left(0; \frac{\pi}{(k-2)n}\right), k \geq 3$.

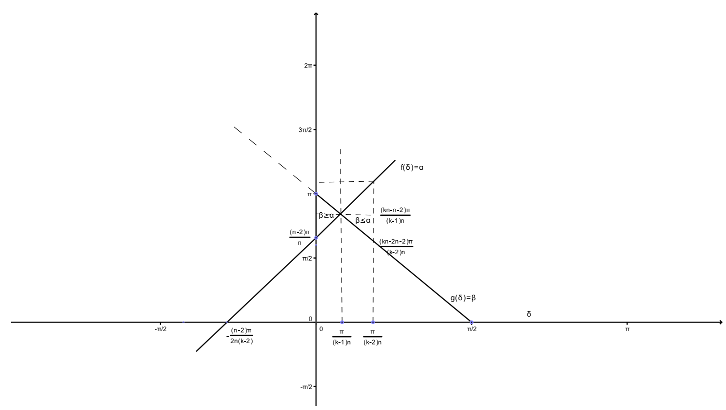

FIGURE 4. Semi-regular polygon and convexity

It is similar for interior angles equal to angle α, (figure 4). If we multiply the inequality $0 < \delta < \frac{\pi}{(k-2)n}$ with $2(k-2)$, and then add to the left and right side $\frac{(n-2)\pi}{n}$, we get the inequality

$$\frac{(n-2)\pi}{n} < \frac{(n-2)\pi}{n} + 2(k-2)\delta < \frac{2\pi}{n} + \frac{(n-2)\pi}{n} \Leftrightarrow$$
$$\frac{(n-2)\pi}{n} < \alpha < \pi, \Rightarrow \alpha \in \left(\frac{(n-2)\pi}{n}, \pi\right)$$

for $\delta \in \left(0; \frac{\pi}{(k-2)n}\right), k \geq 3$.
So, for every $\delta \in \left(0, \frac{\pi}{(k-2)n}\right)$ interior angles of the semi-regular polygon $P_N^{a,\delta}$ are smaller than π. That is, semi-regular equilateral polygon $P_N^{a,\delta}$ is convex for $\delta \in \left(0; \frac{\pi}{(k-2)n}\right)$.
Values of the interior angles of the convex semi-regular equilateral polygon $P_N^{a,\delta}$ depend on the interior angle of the corresponding regular polygon $\gamma = \frac{(n-2)\pi}{n}$ as well as the angle δ. Which means that the following theorem is true:

Theorem 3.5. *Values of the interior angles of the equilateral semi-regular polygon $P_{(k-1)n}^{a,\delta}$ are within the interval $\left(\frac{(n-2)\pi}{n}; \pi\right)$ if $\beta \leq \alpha$ or within the interval $\left(\frac{(kn-2n-2)\pi}{(k-2)n}; \pi\right)$ if $\beta \leq \alpha$ for $k, n \geq 3, k, n \in \mathrm{N}$.*

Proof. Let $\beta \geq \alpha$. We find from the inequality $\pi - 2\delta \geq \frac{(n-2)\pi}{n} + 2(k-2)\delta$ that $\delta \leq \frac{\pi}{(k-1)n}$. Let us determine values of the interior angles, equal to angles α, β defined in relations (3.1) and (3.2) if $\delta \in \left(0, \frac{\pi}{(k-1)n}\right]$.
Let us multiply the inequality $0 < \delta \leq \frac{\pi}{(k-1)n}$ with -2, and then add π to the left and right side, we get inequalities from which we determine value interval of the angles equal to angle β.

$$\pi > \pi - 2\delta \geq \pi - \frac{2\pi}{(k-1)n} \Leftrightarrow \pi > \beta \geq \frac{(kn-n-2)\pi}{(k-1)n} \Rightarrow \beta \in \left[\frac{(kn-n-2)\pi}{(k-1)n}, \pi\right).$$

We can similarly determine values of the interior angles equal to angle α. If we multiply the initial inequality $0 < \delta \leq \frac{\pi}{(k-1)n}$ is first by $2(k-2), k \geq 3$ and then we add $\frac{(n-2)\pi}{n}, n \geq 3$ to the left and right side.
We get a series of inequalities from which we can determine value interval (figure 4)

$$\frac{(n-2)\pi}{n} < \frac{(n-2)\pi}{n} + 2(k-2)\delta \leq \frac{(n-2)\pi}{n} + \frac{2(k-2)\pi}{(k-1)n}$$
$$\Leftrightarrow \frac{(n-2)\pi}{n} < \alpha \leq \frac{(kn-n-2)\pi}{(k-1)n}$$
$$\Rightarrow \alpha \in \left(\frac{(n-2)\pi}{n}, \frac{(kn-n-2)\pi}{(k-1)n}\right].$$

So,

$$\alpha, \beta \in \left(\frac{(n-2)\pi}{n}, \pi\right). \tag{3.6}$$

Now, let us determine values of the interior angles if $\beta \leq \alpha$.We have the inequality $\pi - 2\delta \leq \frac{(n-2)\pi}{n} + 2(k-2)$ from the condition $\delta \geq \frac{\pi}{n(k-1)}$.

Since the semi-regular polygon is convex we have $\delta \in \left[\frac{\pi}{n(k-1)}, \frac{(n-2)}{\pi}\right)$. If we multiply the inequality $\frac{\pi}{n(k-1)} \leq \delta < \frac{\pi}{(k-2)n}$ with -2, and then add π to the left and right side , we get

$$\pi - \frac{2\pi}{n(k-1)} \geq \pi - 2\delta > \pi - \frac{2\pi}{n(k-2)} \Leftrightarrow \frac{(nk-2n-2)\pi}{n(k-2)} < \beta \leq \frac{(nk-n-2)\pi}{n(k-1)}$$
$$\Rightarrow \beta \in \left(\frac{(nk-2n-2)\pi}{n(k-2)}, \frac{(nk-n-2)\pi}{n(k-1)}\right].$$

Similarly, if we multiply the inequality $\frac{\pi}{(k-1)n} \leq \delta < \frac{\pi}{(k-2)n}$ with $2(k-2)$, and then we add $\frac{(n-2)\pi}{n}, n \geq 3$ we get

$$\frac{(n-2)\pi}{n} + \frac{2(k-2)\pi}{(k-1)n} \leq \frac{(n-2)\pi}{n} + 2(k-2)\delta < \frac{2\pi}{n} + \frac{(n-2)\pi}{n} \Leftrightarrow$$
$$\frac{(nk-n-2)\pi}{n(k-1)} \leq \alpha < \pi \Rightarrow \alpha \in \left[\frac{(nk-n-2)\pi}{n(k-1)}, \pi\right)$$

So, if $\beta \leq \alpha$ then

$$\alpha.\beta \in \left(\frac{(nk-2n-2)\pi}{n(k-2)}, \pi\right) \tag{3.7}$$

which was supposed to be proven.

Corollary 3.6. *Convex semi-regular equilateral polygon* $P_N^{a,\delta}$ *is regular for* $\delta = \frac{\pi}{(k-1)n}; k, n \in$ N$, n, k \geq 3, \delta > 0$ *and the values of its interior angles are given in the relation*

$$\alpha = \beta = \frac{(nk-n-2)\pi}{n(k-1)}. \tag{3.8}$$

Proof. According to the definition of the regular polygon, its every angle has to be equal, thus from $\alpha = \beta$ and the relation (3.1) ,(3.2) we have the equation

$$\frac{(n-2)\pi}{n} + 2(k-2)\delta = \pi - 2\delta$$

out of which we find out that the sought value of the angle is $\delta = \frac{\pi}{(k-1)n}$ for which the semi-regular equilateral polygon $P_N^{a,\delta}$ is regular. On this basis we find that the value of the interior angles is

$$\alpha = \beta = \frac{(nk-n-2)\pi}{n(k-1)}.$$

Example 1 Values of the angle δ and of the interior angles for different values k and n for which the semi-regular equilateral polygon is regular (Table 1).

Table 1.

k	n	$\delta = \frac{\pi}{(k-1)n}$	$\frac{(nk-n-2)\pi}{n(k-1)}$
3	3	$\delta = \frac{\pi}{6}$	$\frac{2\pi}{3}$
3	4	$\delta = \frac{\pi}{8}$	$\frac{3\pi}{4}$
4	3	$\delta = \frac{\pi}{9}$	$\frac{7\pi}{9}$
5	3	$\delta = \frac{\pi}{12}$	$\frac{5\pi}{6}$
4	4	$\delta = \frac{\pi}{12}$	$\frac{5\pi}{6}$

Theorem 3.7. *Semi-regular equilateral polygon, $P_N^{a,\delta}$ with positively oriented angles is non-convex for $k,n \in \mathrm{N}, n,k \geq 3$, $\delta \in \left(\frac{\pi}{(k-2)n}; \frac{\pi}{2}\right)$, and for $\delta \geq \frac{\pi}{2}$ is not defined.*

Proof. It is sufficient to prove that there are interior angles of the semi-regular polygon, $P_N^{a,\delta}$,for those values of angle δ, which are bigger than π. Starting from the inequality $\frac{\pi}{(k-2)n} < \delta < \frac{\pi}{2}$, we find that the value interval of those angles, for the interior angles equal to angle β (Lema 3.2), is

$$0 < \pi - 2\delta < \pi - \frac{2\pi}{(k-2)n} \Leftrightarrow 0 < \beta < \frac{(kn-n-2)\pi}{(k-2)n} \Rightarrow \beta \in \left(0; \frac{(kn-n-2)\pi}{(k-2)n}\right)$$

We can apply the similar procedure here. If we multiply the initial inequality with $2(k-2)$, and then we add $\frac{(n-2)\pi}{n}$ to the left and right side , after solving the equation we find that the values of the interior angles equal to angle α are within the interval $\left(\pi; \frac{(kn-n-2)\pi}{n}\right)$.

Since $\alpha > \pi$ for all values $\delta \in \left(\frac{\pi}{(k-2)n}; \frac{\pi}{2}\right)$ semi-regular equilateral polygon $P_N^{a,\delta}$ is non-convex. If

$$\delta \geq \frac{\pi}{2} \Rightarrow 2\delta \geq \pi \Rightarrow \pi - 2\delta \leq 0 \Rightarrow \beta \leq 0$$

Which is opposite to the presumption that interior angles are positively oriented, thus there is no convex semi-regular polygon for $\delta \geq \frac{\pi}{2}$.

3.3. **Diagonals of the semi-regular equilateral polygons $P_N^{a,\delta}$.**

Theorem 3.8. *Length of the diagonal $d_i, i = 1,2,\ldots,k-2$,and $d_{k-2} = b$, drawn from the vertex A_j, $j = 1,2,\ldots,n, A_jA_{j+1} = b$ of the edging polygon $P_k^{a,\delta}$, constructed on side b of the regular polygon P_n^b, defined by the recurrent equations.*

$$d_1 = 2a\cos\delta, d_2 = a(1+2\cos 2\delta), d_i = d_{i-2} + 2acos(i\delta), i \geq 3. \tag{3.9}$$

Proof. This will be proven by mathematical induction with i. Let us check the recurrent formula for $i = 3$. If the formula is valid then

$$d_3 = d_1 + 2a\cos 3\delta = 2a\cos\delta + 2acos3\delta = 2a(cos\delta + cos3\delta) = 4acos2\delta cos\delta. \tag{3.10}$$

In order to prove the equality (3.10) let us presume that isosceles polygon $A_jB_2B_3B_4A_{j+1}$ is constructed on side $b = A_jA_{j+1}$ of the regular polygon P_n^b.(Figure 5a) Diagonals d_1, d_2, d_3 are drawn from the vertex A_j for the rectangle $\square A_jB_3B_4A_{j+1}$ and on the basis of the definition of the angle δ it is valid that $\delta = \angle(a, d_1) = \angle(d_1d_2) = \angle(d_2d_3)$, $d_3 = b$.

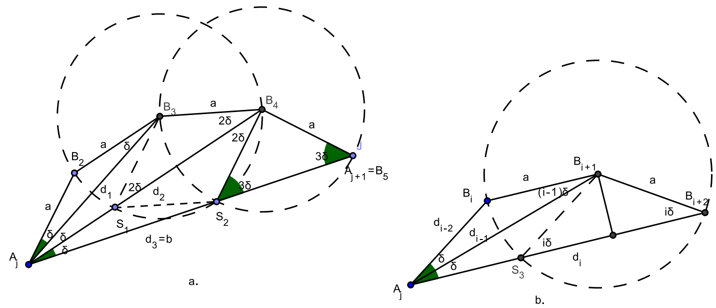

FIGURE 5. Isosceles polygon P_k^a constructed on the side b of the regular polygon P_n^b

Let us note that triangle $\triangle A_jB_2B_3$ is isosceles triangle for which $\angle A_j = \angle B_3 = \delta$, $A_jB_2B_2B_3 = a$ and $A_jB_3 = d_1$. On the basis of Lemma 3.1. we have $\angle B_3 = \delta$, $\angle B_4 = 2\delta$, $\angle A_{j+1} = 3\delta$.
Let us construct point S_1 on the base of the triangle $\triangle A_jB_3B_4$ so that the triangles $\triangle A_jB_3S_1$ and $\triangle S_1B_3B_4$ are isosceles. It is valid that $\angle S_1 = 2\delta$ for it is exterior angle of the triangle $\triangle A_jB_3S_1$.
Similarly, we will construct point S_2 on the base $A_jA_{j+1} = b$ of the triangle $\triangle A_jB_4A_{j+1}$ so that triangle $\triangle S_2B_4A_{j+1}$ is isosceles with interior angles at the base $\angle S_2 = \angle A_{j+1} = 3\delta$.
Further on, $\angle B_4S_2A_{j+1} = 3\delta$ is the exterior angle of the triangle $\triangle A_jB_4S_2$, for which $\angle A_j = \delta$ and $\overline{S_2B_4} = \overline{B_3B_4} = a$ so it is valid that $\angle A_jB_4S_2 = 2\delta$.
Since $\overline{B_3B_4} = \overline{B_4B_2} = \overline{B_4A_{j+1}} = a$ and $\angle A_jB_4B_3 = 2\delta$ triangles $\triangle A_jB_3B_4$ and $\triangle A_jS_2B_4$ are congruent, thus we have $\overline{A_jB_3} = \overline{A_jS_2} = d_1 = 2a\cos\delta$.
Triangle $\triangle S_2B_4A_{j+1}$ is isosceles thus we have $\overline{S_2A_{j+1}} = 2a\cos 3\delta$. So,

$$\begin{aligned} d_3 = A_jA_{j+1} = A_jS_2 + S_2A_{j+1} &= 2a\cos\delta + 2a\cos 3\delta \\ &= 2a(\cos\delta + \cos 3\delta) = 4a\cos\delta\cos 3\delta. \end{aligned}$$

This proves that formulas (3.10) is valid.We can similarly check if this is valid for $i = 4$. That is, it is true for diagonal d_4 drawn from the vertex A_j

$$\begin{gathered} d_4 = d_2 + 2a\cos 4\delta = a + 2a\cos 2\delta + 2a\cos 4\delta = \\ a + 2a(\cos 2\delta + \cos 4\delta) = a(1 + 4\cos\delta\cos 3\delta). \end{gathered}$$

Let us presume that the recurrent formula (3.9) is valid for natural number $p-1$, $p \geq 4$, $p \in \mathrm{N}$ that is for $i = p-1$ and that

$$d_{p-1} = d_{p-3} + 2a\cos(p-1)\delta. \tag{3.11}$$

Let us show that the formula is valid for natural number p, that is for $i = p$. We can take $A_jB_pB_{p+1}B_{p+2}$ from the isosceles polygon P_k^a (figure 5b.) constructed on the side b of the regular polygon P_n^b. This is valid for triangle $\triangle A_jB_pB_{p+1}$ according to definition (2.1), and according to Lemma 3.1, $\angle B_{p+1} = (p-1)\delta$ and $B_pB_{p+1} = a$ is the side of the regular polygon $P_N^{a,\delta}$. Let us construct isosceles triangle $\triangle B_{p+1}S_3B_{p+2}$, so that $\overline{S_3B_{p+1}} = \overline{B_{p+1}B_{p+2}} = a$. Then, according to Lemme 3.1 $\angle B_{p+1}S_3B_{p+2} = p\delta$.

Since $\angle B_{p+1}S_3B_{p+2} = 3\delta$ is the exterior angle of the triangle $\triangle A_jB_{p+1}S_3$ and $\angle A_j = \delta$ it follows that $\angle A_jB_{i+1}S_3 = (p-1)\delta$. So, triangles $\triangle A_jB_iB_{i+1}$ and $\triangle A_jB_{p+1}S_3$ are congruent, thus we have $\overline{A_jB_p} = \overline{A_jS_3} = d_{p-2}$.
We find that $\overline{S_3B_{p+2}} = 2a\cos(i\delta)$ from the isosceles triangle $\triangle B_{p+1}S_3B_{p+2}$. Hence

$$d_p = A_jB_{p+2} = A_jS_3 + S_3B_{p+2} = d_{p-2} + 2a\cos(p\delta).$$

Thus,the recurrent formula is valid for every $i = 3, 4, \ldots, k-2$.

Theorem 3.9. *Length of the diagonal $d_i, i = 1, 2, \ldots, k-2$ of the edging polygons $P_k^{a,\delta}$ drawn from the vertex $A_j, j = 1, 2, \ldots, n$, in which $A_jA_{j+1} = b$ is the side of the regular polygon P_n^b for $i = 2p$, $p \in \mathrm{N}$, is given in the formula*

$$d_{2p} = a + 2a\frac{\cos(p+1)\delta \sin p\delta}{\sin\delta} \tag{3.12}$$

and for $i = 2p-1, p \in \mathrm{N}$ with formula

$$d_{2p-1} = a\frac{\sin(2p\delta)}{\sin\delta} \tag{3.13}$$

Proof. Using Theorem (3.8) for $i = 2p$ we have

$$\begin{aligned} d_{2p} = d_{2p-2} + 2a\cos 2p\delta &= d_{2p-4} + 2a\cos(2p-2)\delta + 2a\cos 2p\delta \\ &= d_{2p-6} + 2a\cos(2p-4)\delta + 2a\cos(2p-2)\delta + 2a\cos 2p\delta \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &= d_2 + 2a\cos 4\delta + 2a\cos 6\delta + \cdots + 2a\cos 2p\delta \\ = a + 2a(\cos 2\delta + \cos 4\delta + \cdots + \cos 2p\delta) &= a + 2a\frac{\cos(p+1)\delta \ldots \sin p\delta}{sin\delta}. \end{aligned}$$

We find similar thing for $i = 2p-1$,

$$\begin{aligned} d_{2p-1} = d_{2p-3} + 2a\cos(2p-1)\delta &= d_{2p-5} + 2a\cos(2p-3)\delta + 2a\cos(2p-1)\delta \\ &= d_{2p-7} + 2a\cos(2p-5)\delta + 2a\cos(2p-3)\delta + 2a\cos(2p-1)\delta \\ &\cdots\cdots\vdots\cdots\cdots \\ &= d_1 + 2a\cos 3\delta + 2a\cos 5\delta + \cdots + 2a\cos(2p-1)\delta \\ &= 2a\cos\delta + 2a\cos 3\delta + \cdots + 2a\cos(2p-1)\delta = a\frac{\sin 2p\delta}{sin\delta}. \end{aligned}$$

Theorem 3.10. *The following is valid for the diagonals drawn from the vertex of the semi-regular polygon $P_N^{a,\delta}$:*

(1) *There are $(k-1)n - (2k-1)$ diagonals which are within the interior angle $\gamma = \frac{(n-2)\pi}{n}$ of the corresponding regular polygon P_n^b which together form angle $\frac{\pi}{(k-1)n}$ and $2(k-2)$ diagonals of the polygon P_k^a constructed on sides of that regular polygon which form angle equal to angle d defined with(2.1) if the interior angle equal to angle L from (3.1) is corresponding to the vertex from which the diagonals are drawn.*
(2) *There are $(k-1)n - 5$ diagonals which together form angle $\frac{\pi-4\delta}{(k-1)n-4}$ if interior angle $\beta = \pi - 2\delta$ corresponds to the vertex.*

Proof.

(1) Let us consider that diagonals are constructed from vertex A_1 and that interior angle α defined in relation (3.1) is corresponding for the vertex. Let us denote with S_{A_1} number of diagonals which can be drawn from vertex A_1. Then we have $S_{A_1} = (k-1)n - 3$. It was earlier proven that we can construct $(k-2)$ diagonals $d_i, i = 1, 2, \ldots, k-2$, from the vertex A_1 of the polygon P_k^a constructed on the side $A_1A_2 = b$ of the regular polygon P_n^b, with interior angle $\gamma = \frac{(n-2)\pi}{n}$, for which $d_{k-2} = b$ and which together form angle δ.

Since there are two such polygons P_k^a constructed on sides $A_1A_2 = b$ and $A_1A_n = b$ with common vertex A_1 number of diagonals which together form angle δ defined in the relation (2.1), is $2(k-2)$.

Having in mind relation (3.1) number of diagonals within the interior angle γ of the corresponding regular polygon P_n^b,let us mark them $S_{A_1}^{\gamma}$, is

$$S_{A_1}^{\gamma} = (k-1)n - 3 - 2(k-2) = (k-1)n - (2k-1).$$

Those diagonals together form angle which we will mark with $\varepsilon_{A_1}^{\gamma}$, and for which the following is valid

$$\varepsilon_{A_1}^{\gamma} = \frac{\gamma}{S_{A_1}^{\gamma} + 1} = \frac{\frac{(n-2)\pi}{n}}{(k-1)n - (2k-1) + 1} = \frac{\pi}{(k-1)n}.$$

which was supposed to be proven.

(2) Let diagonals be drawn from the vertex B_1 with the corresponding interior angle $\beta = \pi - 2\delta$, then we have $S_{B_1}^{\beta} = (k-1)n - 3$ umber of drawn diagonals from the vertex B_1. Only two of those drawn diagonals form the angle d with the sides of the semi-regular polygon $P_{(k-1)n}^{a,\delta}$, and those are the first and the last diagonal.

Thus, other diagonals, and there are $S_{B_1} = (k-1)n - 5$ of them, within the angle $\psi = \beta - 2\delta = \pi - 4\delta$ together form angle e $\varepsilon_{B_1}^{\psi}$ and for them the following is valid

$$\varepsilon_{B_1}^{\psi} = \frac{\psi}{S_{B1} + 1} = \frac{\pi - 4\delta}{(k-1)n - 4}.$$

Example 2. Here is given tabular review of the values from the previous theorem, in which $i = 1, 2, .n$ and $j = 1, 2, ..(k-2)n$, for the convex semi-regular polygon $P_N^{a,\delta}$, for $n = 3, 4, 5$ and $k = 3, 4, 5$.(Table 2)

Table 2.

n	k	$P_{(k-1)n}^{a,\delta}$	$\delta \in \left(0; \frac{\pi}{(k-2)n}\right)$	γ	α	β	$S_{A_i}^{\gamma}$	$\varepsilon_{A_i}^{\gamma}$	$S_{B_j}^{\psi}$	$\varepsilon_{B_j}^{\psi}$
3	3	6	$\delta \in \left(0; \frac{\pi}{3}\right)$	$\frac{\pi}{3}$	$\frac{\pi}{3} + 2\delta$	$\pi - 2\delta$	1	$\frac{\pi}{6}$	1	$\frac{\pi}{2} - 2\delta$
	4	9	$\delta \in \left(0; \frac{\pi}{6}\right)$	$\frac{\pi}{3}$	$\frac{\pi}{3} + 4\delta$	$\pi - 2\delta$	2	$\frac{\pi}{9}$	4	$\frac{\pi}{5} - \frac{4}{5}\delta$
	5	12	$\delta \in \left(0; \frac{\pi}{9}\right)$	$\frac{\pi}{3}$	$\frac{\pi}{3} + 6\delta$	$\pi - 2\delta$	3	$\frac{\pi}{12}$	7	$\frac{\pi}{8} - \frac{1}{2}\delta$
4	3	8	$\delta \in \left(0; \frac{\pi}{4}\right)$	$\frac{\pi}{2}$	$\frac{\pi}{2} + 2\delta$	$\pi - 2\delta$	3	$\frac{\pi}{8}$	3	$\frac{\pi}{4} - \delta$
	4	12	$\delta \in \left(0; \frac{\pi}{8}\right)$	$\frac{\pi}{2}$	$\frac{\pi}{2} + 4\delta$	$\pi - 2\delta$	5	$\frac{\pi}{12}$	7	$\frac{\pi}{8} - \frac{1}{2}\delta$
	5	16	$\delta \in \left(0; \frac{\pi}{12}\right)$	$\frac{\pi}{2}$	$\frac{\pi}{2} + 6\delta$	$\pi - 2\delta$	7	$\frac{\pi}{16}$	11	$\frac{\pi}{12} - \frac{1}{3}\delta$
5	3	10	$\delta \in \left(0; \frac{\pi}{5}\right)$	$\frac{3\pi}{5}$	$\frac{3\pi}{5} + 2\delta$	$\pi - 2\delta$	5	$\frac{\pi}{10}$	5	$\frac{\pi}{6} - \frac{2}{3}\delta$
	4	15	$\delta \in \left(0; \frac{\pi}{10}\right)$	$\frac{3\pi}{5}$	$\frac{3\pi}{5} + 4\delta$	$\pi - 2\delta$	8	$\frac{\pi}{15}$	10	$\frac{\pi}{11} - \frac{4}{11}\delta$

Corollary 3.11. *$S^{\gamma}_{A_i} = S^{\psi}_{B_j}$, $i, j = 1, 2, \ldots, n$ is valid for every equilateral semi-regular polygon $P^{a,\delta}_{2n}$ with $2n$ sides.*

Proof. Let us presume that the equality is valid. Let us define semi-regular equilateral polygon for which that equality is valid. From $S^{\gamma}_{A_i} = S^{\psi}_{B_j}$ we have $(k-1)n - (2k-1) = (k-1)n - 5 \Leftrightarrow 2k - 1 = 5 \Rightarrow k = 3$. For $k = 3$ required semi-regular polygon is $(k-1)n = 2n$, $n \in \mathrm{N}$.

Corollary 3.12. *If $\varepsilon^{\gamma}_{A_i} = \varepsilon^{\psi}_{B_j}$, with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, (k-2)n$ polygon is regular.*

Proof. If the equality is valid then we have the equation from $\varepsilon^{\gamma}_{A_i} = \varepsilon^{\psi}_{B_j}$ and it $\frac{\pi}{(k-1)n} = \frac{\pi - 4\delta}{(k-1)n-4}$, from which we find that $\delta = \frac{\pi}{(k-1)n}$. According to Corollary (3.6) convex semi-regular equilateral polygon for that value of the angle δ is regular.

3.4. **Area of the semi-regular equilateral polygon $P^{a,\delta}_N$.** The following theorems show dependence of the area of the semi-regular polygon $P^{a,\delta}_N$ from the length of its sides a and angle δ defined by relation (2.1).

Theorem 3.13. *Area of the semi-regular polygon $P^{a,\delta}_N$ is given in the relation*

$$P^{a,\delta}_{(k-1)n} = \begin{cases} \frac{1}{4}na^2\Big(1 + 2\frac{\cos\frac{k}{2}\delta sin(\frac{k}{2}-1)\delta}{\sin\delta}\Big)^2 \cot\frac{\pi}{n} + nP^a_k & for \quad even \quad k \\ \frac{1}{4}na^2\Big(\frac{\sin(k-1)\delta}{\sin\delta}\Big)^2 \cot\frac{\pi}{n} + nP^a_k & for \quad odd \quad k \end{cases} \tag{3.14}$$

In which area of P^a_k edging isosceles polygon with side a and angle δ defined by (2.1).

Before we prove this Theorem let us prove lemma.

Lemma 3.14. *Area of edging polygon P^a_k with side a and angle δ is given in the relation*

$$P^a_k = \frac{a^2(k-2)\cot\delta}{4} - \frac{a^2}{4\sin\delta}\Big(\sum_{p=1}^{\lceil\frac{k}{2}\rceil-1}\cos(4p+1)\delta + \sum_{p=1}^{\lceil\frac{k-1}{2}\rceil}\cos(4p-1)\delta\Big). \tag{3.15}$$

Proof. Area of the isosceles polygon P^a_k can be expressed as sum of triangles' areas of which diagonals $d_i, i = 1, 2, \ldots, k-2$ are bases, that is, the following is valid

$$P^a_k = \sum_{i=1}^{k-2} p_{d_i} \tag{3.16}$$

where p_{d_i} is area of the triangle with base d_i, (figure 3). Since $p_{d_i} = \frac{d_i h_i}{2}$,in which h_i is height of the triangle drawn to the base d_i and $h_i = a\sin i\delta$ id using the relation (3.12) and (3.13) we find that for $k \geq 3, k \in \mathrm{N}$.

$$p_{d_{k-2}} = \begin{cases} \frac{a^2}{2}sin(k-2)\delta\Big(1 + 2\frac{\cos\frac{k}{2}\delta sin(\frac{k}{2}-1)\delta}{\sin\delta}\Big) & for \quad even \quad \mathrm{k} \\ \frac{a^2}{2}\frac{\sin(k-2)\delta sin(k-1)\delta}{\sin\delta} & for \quad odd \quad \mathrm{k} \end{cases} \tag{3.17}$$

So,

$$p_{d_i} = \begin{cases} \frac{a^2}{2}sin2p\delta\Big(1 + 2\frac{\cos(p+1)\delta sinp\delta}{\sin\delta}\Big) & for \quad \text{i=2p} \quad p \in \mathrm{N} \\ \frac{a^2}{2}\frac{\sin(2p-1)\delta sin2p\delta}{\sin\delta} & for \quad \text{i=2p-1} \quad p \in \mathrm{N} \end{cases} \tag{3.18}$$

On the basis of 3.17 and 3.18 equality 3.16 is transformed

$$P_k^a = \sum_{p=1}^{\lceil \frac{k}{2} \rceil - 1} \frac{a^2}{2} \sin 2p\delta (1 + 2\frac{\cos{(p+1)\delta} \sin p\delta}{\sin \delta}) + \sum_{p=1}^{\lceil \frac{k-1}{2} \rceil} \frac{a^2}{2} \frac{\sin{(2p-1)\delta} \sin 2p\delta}{\sin \delta}$$

$$= \frac{a^2}{2} \left(\sum_{p=1}^{\lceil \frac{k}{2} \rceil - 1} \sin 2p\delta \Big(1 + 2\frac{\cos{(p+1)\delta} \sin p\delta}{\sin \delta}\Big) + \sum_{p=1}^{\lceil \frac{k-1}{2} \rceil} \frac{\sin 2p\delta \sin(2p-1)\delta}{\sin \delta} \right)$$

$$= \frac{a^2}{2\sin\delta} \left(\sum_{p=1}^{\lceil \frac{k}{2} \rceil - 1} sin2p\delta (\sin\delta + 2\cos{(p+1)\delta} \sin p\delta) + \sum_{p=1}^{\lceil \frac{k-1}{2} \rceil} \sin 2p\delta \sin(2p-1)\delta \right)$$

After solving the equation we get the sought relation

$$P_k^a = \frac{a^2(k-2)\cot\delta}{4} - \frac{a^2}{4\sin\delta} \Big(\sum_{p=1}^{\lceil \frac{k}{2} \rceil - 1} \cos(4p+1)\delta + \sum_{p=1}^{\lceil \frac{k-1}{2} \rceil} \cos(4p-1)\delta \Big).$$

Let us return to proving the theorem.

Proof. Let us notice that area of the semi-regular polygon $P_N^{a,\delta}$ is equal to sum of area of the regular polygon P_n^b and area of all polygons P_k^a, constructed on each side of the regular polygon, that is

$$P_{(k-1)n}^{a,\delta} = P_n^b + nP_k^a. \tag{3.19}$$

Since the area of the regular polygon $P_n^b = \frac{1}{4}nb^2\cot(\frac{\pi}{n})$. with side b given in the equality

$$b = d_{k-2} = \begin{cases} a\Big(1 + 2\frac{\cos\frac{k}{2}\delta sin(\frac{k}{2}-1)\delta}{\sin\delta}\Big) & for \quad even \quad \text{k} \\ a\frac{\sin(k-1)\delta}{\sin\delta} & for \quad odd \quad \text{k} \end{cases} \tag{3.20}$$

Area of the semi-regular polygon $P_{(k-1)n}^{a,\delta}$ has the following form on the basis of the relations (3.15) and (3.19)

$$P_{(k-1)n}^{a,\delta} = \begin{cases} \frac{1}{4}na^2\Big(1 + 2\frac{\cos\frac{k}{2}\delta sin(\frac{k}{2}-1)\delta}{\sin\delta}\Big)^2 \cot\frac{\pi}{n} + nP_k^a & for \quad even \quad \text{k} \\ \frac{1}{4}na^2\Big(\frac{\sin(k-1)\delta}{\sin\delta}\Big)^2 \cot\frac{\pi}{n} + nP_k^a & for \quad odd \quad \text{k} \end{cases} \tag{3.21}$$

which proves the theorem.

Example 3. Formulas for area of the isosceles polygons, $k = 3$ and $k = 4$,if side a and angle δ are given, are as follows; for $k = 3$, $P_3^a = \frac{a^2\cot\delta}{4} - \frac{a^2}{4\sin\delta}\cos 3\delta = \frac{a^2}{4\sin\delta}(\cos\delta - \cos 3\delta) = \frac{a^2}{2}\sin 2\delta$.

Similarly, for $k = 4$, $P_4^a = \frac{2a^2\cot\delta}{4} - \frac{a^2}{4\sin\delta}(\cos 5\delta + \cos 3\delta) = \frac{a^2\cot\delta}{2} - \frac{a^2\cot\delta}{2}\cos 4\delta = \frac{a^2}{2}\cot\delta(1 - \cos 4\delta) = \cdots = \frac{a^2}{2}(2sin2\delta + \sin 4\delta)$.

Example 4. Let us determine area of the semi-regular $2n$ polygon with given side a and andle δ,defined in (2.1),

$P_{2n}^{a,\delta} = P_n^b + nP_3^a = \frac{1}{4}na^2(\frac{sin2\delta}{\sin\delta})^2 \cot\frac{\pi}{n} + \frac{na^2}{4\sin\delta}(\cos\delta - \cos 3\delta)$
$= na^2\cos\delta\left(\frac{\cos\delta\cos\frac{\pi}{n}+\sin\delta\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}}\right) = \frac{na^2}{\sin\frac{\pi}{n}}\cos\delta\cos(\frac{\pi}{n}-\delta).$
In which n is the number of sides of the regular polygon on which semi-regulr polygon is drawn.

Theorem 3.15. *Area of the isosceles polygon P_k^a, which given side a and angle δ, constructed on sides of the regular polygon P_n^b, is determined by relation*

$$P_k^a = \frac{a^2}{8}\frac{(k-1)\sin 2\delta - \sin 2(k-1)\delta}{\sin^2\delta} \tag{3.22}$$

za $k \geq 3$.

Proof. Let the polygon P_k^a be given, side a and diagonals d_i, $i = 1, 2, \ldots, k-2$, $k \geq 3$ drawn from the vertex A_j, $j = 1, 2, \ldots, n$, $A_j = B_1$, $B_k = A_{j+1}$,with $A_jA_{j+1} = b$ side of the regular polygons P_n^b. Let us note triangles $A_jB_{k-1}B_k$ basis of which are diagonals d_i. Let us introduce marks $P^1, P^2, P^3, \ldots, P^k$ for areas of triangles $A_jB_2B_3$, $A_jB_3B_4$, $\ldots$, $A_jB_{k-1}B_k$ respectively. It is easily proven that the following relations are valid for areas of those triangles

$$\begin{aligned}
P^1 &= \frac{a^2}{2}\sin 2\delta = P^{a,\delta} \\
P^2 &= \frac{a^2}{2}(2\sin 2\delta + \sin 4\delta) = P^{a,\delta} + P^{a,2\delta} \\
P^3 &= P^{a,\delta} + P^{a,2\delta} + P^{a,3\delta} \\
\cdots &\quad \cdots\cdots \\
P^i &= P^{a,\delta} + P^{a,2\delta} + P^{a,3\delta} + \cdots + P^{a,i\delta}.
\end{aligned}$$

,
$P^{a,i\delta} = \frac{a^2}{2}\sin 2i\delta$ is area of the isosceles triangle wirh side a and angle at the base $delta$, $i = 1, 2, \ldots, k-2$. Since the area of the enge polygon P_k^a is equal to sum of areas of the triangles $A_jb_{k-1}b_k$ is valid

$$P_k^a = (k-2)P^{a,\delta} + (k-3)P^{a,2\delta} + \cdots + P^{a,(k-3)\delta} + P^{(k-2)\delta}.$$

That is

$$P_k^a = \frac{a^2}{2}\Big[(k-2)\sin 2\delta + (k-3)\sin 4\delta + \cdots + 2\sin 2(k-3)\delta + \sin 2(k-2)\delta\Big].$$

Or

$$P_k^a = \frac{a^2}{2}\sum_{\nu=1}^{k-2}(k-\nu-1)\sin(2\nu\delta). \tag{3.23}$$

So, in order to prove the theorem it is necessary to determine sum of row (3.23). In order to calculate the sum of that row let us note the function

$$F(\theta) = \sum_{\nu=1}^{k-2}(k-\nu-1)\cos(\nu\theta) + i\sum_{\nu=1}^{k-2}(k-\nu-1)\sin(\nu\theta) \tag{3.24}$$

In which $\theta = 2\delta$. If replace $\cos\nu\theta = \frac{1}{2}(e^{i\nu\theta} + e^{-i\nu\theta})$ and $\sin\nu\theta = \frac{1}{2i}(e^{i\nu\theta} - e^{-i\nu\theta})$ into (3.24) we get the equality

$$F(\theta) = \sum_{\nu=1}^{k-2}(k-\nu-1)\frac{1}{2}(e^{i\nu\theta} + e^{-i\nu\theta}) + i\sum_{\nu=1}^{k-2}(k-\nu-1)\frac{1}{2i}(e^{i\nu\theta} - e^{-i\nu\theta}),$$

After arranging it it is transformed into

$$F(\theta) = \sum_{\nu=1}^{k-2}(k-1)e^{i\nu\theta} - \sum_{\nu=1}^{k-2}\nu e^{i\nu\theta}. \tag{3.25}$$

Since

$$\sum_{\nu=1}^{k-2}(k-1)e^{i\nu\theta} = (k-1)e^{i\theta}\frac{e^{i(k-2)\theta} - 1}{e^{i\theta} - 1} \tag{3.26}$$

and

$$\sum_{\nu=1}^{k-2}\nu e^{i\nu\theta} = \frac{1}{i}\frac{d}{d\theta}\Big(\sum_{\nu=1}^{k-2}e^{i\theta}\Big) = e^{i\theta}\frac{(k-2)e^{i(k-1)\theta} - (k-1)e^{i(k-2)\theta} + 1}{(e^{i\theta} - 1)^2} \tag{3.27}$$

Equality (3.25) can be rearranged into

$$F(\theta) = e^{i\theta}\frac{e^{i(k-1)\theta} - (k-1)e^{i\theta} + (k-2)}{(e^{i\theta} - 1)^2} \tag{3.28}$$

Having in mind that $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$ the last equality can be reduced into following form

$$F(\theta) = \frac{(k-1)\cos\theta - \cos(k-1)\theta - (k-2)}{4\sin^2\frac{\theta}{2}} + i\Big(\frac{(k-1)\sin\theta - \sin(k-1)\theta}{4\sin^2\frac{\theta}{2}}\Big). \tag{3.29}$$

By comparison of the initial equality (3.24) with (3.29) and by giving $\theta = 2\delta$, we find that the requested sum of row

$$\sum_{\nu=1}^{k-2}(k-\nu-1)\sin(2\nu\delta) = \frac{(k-1)\sin 2\delta - \sin 2(k-1)\delta}{4\sin^2\delta}. \tag{3.30}$$

On the basis of (3.23) and (3.30) we find that

$$P_k^a = \frac{a^2}{2}\sum_{\nu=1}^{k-2}(k-\nu-1)\sin(2\nu\delta) = \frac{a^2}{8}\frac{(k-1)\sin 2\delta - \sin 2(k-1)\delta}{\sin^2\delta} \tag{3.31}$$

that we wanted to prove.

Corollary 3.16. *Area of the semi-regular equilateral polygon $P_N^{a,\delta}$ constructed on the regular polygon P_n^B if side a and angle δ are given, defined in (2.1) is given in the relation*

$$P_{(k-1)n}^{a,\delta} = \frac{na^2}{4}\Big[\Big(1 + 2\frac{\cos(\frac{k}{2}\delta)\sin(\frac{k}{2}-1)\delta}{\sin\delta}\Big)^2\cot\frac{\pi}{n} + \frac{(k-1)\sin 2\delta - \sin 2(k-1)\delta}{2\sin^2\delta}\Big] \tag{3.32}$$

for even k, and

$$P^{a,\delta}_{(k-1)n} = \frac{na^2}{4}\Big[\Big(\frac{\sin{(k-1)\delta}}{\sin\delta}\Big)^2 \cot\frac{\pi}{n} + \frac{(k-1)\sin 2\delta - \sin 2(k-1)\delta}{2\sin^2\delta}\Big] \tag{3.33}$$

for odd k.

Proof. Using results of the theorems (Theorem 3.13) and equality (3.21) we get the required relations (3.32) and (3.33).

Example 5 Area of the semi-regular equilateral polygon with $2n$ sides, with given side a and angle δ is

$$\begin{aligned} P^{a,\delta}_{2n} &= \frac{na^2}{4}\Big[\Big(\frac{\sin 2\delta}{\sin\delta}\Big)^2 \cot\frac{\pi}{n} + \frac{2\sin 2\delta - \sin 4\delta}{2\sin^2\delta}\Big] \\ &= \frac{na^2}{4}\Big[4\cos^2\delta\cot\frac{\pi}{n} + \frac{2\sin 2\delta(1-\cos 2\delta)}{2\sin^2\delta}\Big] \\ &= \frac{na^2}{\sin\frac{\pi}{n}}\cos\delta\cos{(\frac{\pi}{n}-\delta)}. \end{aligned}$$

Theorem 3.17. *Convex regular polygon P^b_n and semi-regular equilateral polygon $P^{a,\delta}_N$ constructed on it cannot have equal sides.*

Proof. Let us presume opposite, that is, semi-regular and regular polygons have equal sides. Then we have $b = a$, in which a is side of the semi-regular polygon and b is side of the regular polygon on which it is constructed. Then on the basis of the equality (3.20) we have the equation $\cos\frac{k}{2}\delta\sin{(\frac{k}{2}-1)\delta} = 0$ if the edging polygon P^a_k has even number of sides and equation $\sin{(k-1)\delta} - \sin\delta = 0$ if k if k is odd number. We can see out of these equations that $\delta = \frac{(2p+1)\pi}{k}$ or $\delta = \frac{2s\pi}{(k-2)}$. On the basis of the theorem (3.4) and convexity of the semi-regular polygon we have that $\delta \in \Big(0; \frac{\pi}{(k-2)n}\Big); k,n \geq 3; k,n \in \mathrm{N}$ and the inequality

$$\begin{cases} 0 < \frac{(2p+1)\pi}{k} < \frac{\pi}{(k-2)n} \\ 0 < \frac{2s\pi}{(k-2)} < \frac{\pi}{(k-2)n}. \end{cases}$$

That is

$$\begin{cases} 0 < n < \frac{k}{(k-2)(2p+1)} \\ 0 < n < \frac{1}{2s} \end{cases}$$

From these inequality we can draw a conclusion that there are no values n and k in the set N for which the semi-regular polygon is convex, and the values of the angle δ are given in the relation $\delta = \frac{(2p+1)\pi}{k}$ or $\delta = \frac{2s\pi}{(k-2)}$.
We get the same results by similar procedure from the other equation

$$a = a\frac{\sin(k-1)\delta}{\sin\delta}$$

.

Thus, there is no angle δ defined with (2.1) for which the semi-regular polygon is convex and it has side which is equal to the side of the regular polygon on which it is constructed, which was meant to prove.

3.5. **Application.** By using results of the Lemma 3.14 and Theorem 3.15 we can define sum of rows,

$$\sum_{p=1}^{\lceil \frac{k}{2} \rceil -1} \cos(4p+1)\delta + \sum_{p=1}^{\lceil \frac{k-1}{2} \rceil} \cos(4p-1)\delta, k \in \mathrm{N}, k \geq 3$$

Which means that it is valid.

Corollary 3.18. *It is valid*

$$\sum_{p=1}^{\lceil \frac{k}{2} \rceil -1} \cos(4p+1)\delta + \sum_{p=1}^{\lceil \frac{k-1}{2} \rceil} \cos(4p-1)\delta = \frac{\sin 2(p-1)\delta - \sin 2\delta}{2\sin\delta}, k \in \mathrm{N}, k \geq 3 \quad (3.34)$$

Proof. If we equalize relations (3.15) and (3.31), after shortening and arranging we get tha the requested sum is

$$\sum_{p=1}^{\lceil \frac{k}{2} \rceil -1} \cos(4p+1)\delta + \sum_{p=1}^{\lceil \frac{k-1}{2} \rceil} \cos(4p-1)\delta = \frac{\sin 2(k-1)\delta - \sin 2\delta}{2\sin\delta}$$

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