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RICCI SOLITONS AND SYMMETRIES OF SPACETIME MANIFOLD OF GENERAL RELATIVITY

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ABSTRACT. The vector fields associated with Ricci solitons in Riemann manifold have been studied and the correspondence between these vector fields and symmetries of spacetime manifold of general relativity have been established. The examples of local Ricci soliton are given for Lorentzian signature and the case of Reisnerr-Nordström spacetime is explored as a soliton. The relationships between the symmetries of Petrov type D pure radiation fields and Ricci solitons have also established.

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1. INTRODUCTION

The construction of gravitational potentials satisfying Einstein's field equations is the principal aim of all investigations in gravitational physics and this has been often been achieved by imposing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. The geometrical symmetries of the spacetime are expressible through the vanishing of the Lie derivative of certain tensors with respect to a vector. This vector may be time-like, space-like or null. The role of symmetries in general theory of relativity has been introduced by Katzin, Levine and Davis in a series of papers ([14] - [17]). These symmetries, also known as collineations, were further studied by Ahsan ([1] - [5]), Ahsan and Ali [6] and Ahsan and Husain [7].

Recently geometric flows have become important tools in Riemannian geometry and general relativity. List [9] has studied a geometric flow whose fixed points corresponds to static Ricci flat spacetime which is nothing but Ricci flow pullback by a certain diffeomorphism. The association of each Ricci flat spacetime gives notion of local Ricci soliton in one higher dimension. The importance of geometric flow in Riemannian geometry is due to Hamilton who has given the flow equation and List generalized Hamilton's equation and extend it to spacetime for static metrics. He has given system of flow equations whose fixed points solve the Einstein free-scalar field system. This observation is useful for the correspondence of solutions of system i.e., Ricci soliton and symmetry property of spacetime, that how Riemannian space (or spacetime) with Ricci soliton deals different kind of symmetry properties.

Motivated by the role of symmetries and Ricci soliton, a comparative study of correspondence between the vector field associated with Ricci soliton and symmetries of spacetime is made with some examples in general relativity. In section 2 preliminaries are given. Examples of Ricci soliton and specially Reisnerr-Nordström soliton are discussed in detail in section 3. The relationship between Einstein spaces, Petrov type D pure radiation field and Ricci soliton has been discussed in section 4. Finally section 5 deals with the conclusion.

2. Preliminaries

So far more than twenty six different types of collineations have been studied and the literature on such collineations is very large and still expanding with results of elegance (cf., [5]). However, here we shall mention only those symmetry assumptions that are required for subsequent investigation and we have

(i) Motion A spacetime is said to admit motion if there exists a vector field ξ^a such that

$$\pounds_\xi g_{ij} = \xi_{i;j} + \xi_{j;i} = 0 \tag{2.1}$$

Equation (2.1) is known as Killing equation and vector ξ^a is called a Killing vector field (cf. [20]).

(ii) Conformal Motion (Conf M) If

$$\pounds_\xi g_{ij} = \sigma g_{ij}, \tag{2.2}$$

where σ is a scalar, then the spacetime is said to admit conformal motion.

(iii) Special Conformal Motion (SCM) A spacetime admits SCM if

$$\pounds_\xi g_{ij} = \sigma g_{ij},\ \sigma_{;ij} = 0 \tag{2.3}$$

(iv) Weyl Conformal Collineation (WCC) Infinitesimal transformation

$$'\xi^x = \xi^x + v^x dt \tag{2.4}$$

is called WCC if and only if

$$\pounds_\xi C^i_{jkl} = 0,\ (n > 3) \tag{2.5}$$

(v) Curvature Collineation (CC) A spacetime admits curvature collineation if there is a vector field ξ^i such that

$$\pounds_\xi R^i_{jkl} = 0, \tag{2.6}$$

where R^i_{jkl} is Riemann curvature tensor.

(vi) Ricci Collineation (RC) A spacetime is said to admit Ricci collineation if there is a vector field ξ^i such that

$$\pounds_\xi R_{ij} = 0, \tag{2.7}$$

where R_{ij} is the Ricci tensor.

(vii) Affine Collineation (AC) If

$$\pounds_\xi \Gamma^i_{jk} = \xi^i_{;jk} + R^i_{jmk}\xi^m = 0 \tag{2.8}$$

then the spacetime is said to admit an AC.

(viii) Conformal Collineation (Conf C) A spacetime admits Conf C if there is a vector ξ^a such that

$$\pounds_\xi \Gamma^i_{jk} = \delta^i_j \sigma_{;k} + \delta^i_k \sigma_{;j} - g_{jk} g^{il} \sigma_{;l} \tag{2.9}$$

(ix) Special Conformal Collineation (S Conf C) If

$$\pounds_\xi \Gamma^i_{jk} = \delta^i_j \sigma_{;k} + \delta^i_k \sigma_{;j} - g_{jk} g^{il} \sigma_{;l}\ ,\ \sigma_{;jk} = 0 \tag{2.10}$$

then the spacetime admits S Conf C along the vector field ξ^a.

(x) Ricci Soliton

A family $g_\lambda = g(\lambda; x)$ of Riemann metrics on a n-dimensional $(n \geq 3)$ smooth manifold M with parameter λ ranging in a time interval $J \subset$ R including zero is called a Ricci flow if the Hamilton Equations

$$\frac{\partial g_0}{\partial \lambda} = -2Ric_0 \tag{2.11}$$

of the Ricci flow (cf; [10], [11]) for $g_0 = g(0)$ and the Ricci tensor Ric_0 of the g_0 are satisfied. Corresponding to self similar solution of equation (2.11) is the notion of the Ricci soliton, defined as a metric g_0 satisfying the equation

$$-2Ric_0 = \pounds_\xi g_0 + 2kg_0 \tag{2.12}$$

for vector field ξ on V_n and a constant k. The Ricci soliton is said to be steady (static) if $k = 0$, shrinking if $k < 0$ and expanding if $k > 0$. The metric g_0 is called a gradient Ricci soliton if $\xi = \nabla\phi$ i.e., gradient of some function ϕ.

For n-dimensional Riemannian manifold equation (2.12) can be written in general as

$$R_{ij} - \frac{1}{2}\pounds_\xi g_{ij} = kg_{ij} \tag{2.13}$$

3. Reisnerr-Nordström soliton

Some of the examples of the spaces satisfying equation (2.13) are as follows:
(i) Natural extension of Einstein manifold.
(ii) Self similar (fixed point) solutions of the Ricci flow.
(iii) Positive Einstein manifolds such as round spheres.
(iv) Round cylinders $S^{n-1} \times$ R.
(v) Brayant soliton on R^n.

It may be noted that (iii) and (iv) are examples of gradient shrinking Ricci solitons and (v) is non-compact gradient steady soliton. There are lots of other examples but our main interest will be those which are related to general relativity.

For some vector field ξ Hamilton has expressed Ricci flow equation (2.11) by pulling back along a λ- dependent diffeomorphism as

$$\frac{\partial g_{ij}}{\partial \lambda} = -2R_{ij} + \pounds_{\xi} g_{ij} \tag{3.1}$$

where λ is flow parameter and $g_{ij}(\lambda; x)$ is Riemannian metric. Apart from Ricci flow List [9] has given equations

$$\frac{\partial g_{ij}}{\partial \lambda} = -2(R_{ij} - k_n^2 \nabla_i f \nabla_j f) \tag{3.2}$$

$$\frac{\partial f}{\partial \lambda} = \Delta f \tag{3.3}$$

where Δf represents Laplacian of function $f(\lambda; x)$, i.e., $\Delta f = g^{ij} \nabla_i f \nabla_j f$. The system (3.2) - (3.3) is a better approximation than Ricci flow. The fixed points of this system solve the static vacuum Einstein equations for arbitrary constant k_n^2. Choosing k_n as $\sqrt{\frac{n-1}{n-2}}$, $n \geq 3$ the spacetime metric $ds^2 = -e^{2f} dt^2 + e^{\frac{2f}{n-2}} d\Omega^2$ will be static. These fixed points are nothing but flat metrics.

The solutions of List flow system of equations obey the Einstein free scalar field system given as follows ([8])

$$R_{ij} - k_n^2 \nabla_i f \nabla_j f - \frac{1}{2} \pounds_{\xi} g_{ij} = A g_{ij} \tag{3.4}$$

$$\Delta f + \pounds_{\xi} f = 0 \tag{3.5}$$

where A is an scalar. If ξ vanishes, system (3.4) - (3.5) changes to

$$R_{ij} - k_n^2 \nabla_i f \nabla_j f = A g_{ij} \tag{3.6}$$

$$\Delta f = 0 \tag{3.7}$$

The terminology Einstein free-scalar field arises because for Lorentzian signature g_{ij}, above system (3.6) - (3.7) describe Einstein gravity with cosmological constant, coupled to a free scalar field. But here we will use the terminology without regard to signature of g_{ij}. The static vacuum Einstein terminology arises because, if g_{ij} has Euclidean signature and when $k = 0$ and the convensional choice $k_n = \sqrt{\frac{n-1}{n-2}}$ is made, system (3.6) - (3.7) imply that the metric on $\mathrm{R} \times M^n$, $n > 2$, given by

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = -e^{2f} dt^2 + e^{-\frac{2f}{n-2}} g_{ij} dx^i dx^j$$

is Ricci flat and $\frac{\partial}{\partial t}$ is hypersurface orthogonal Killing vector field.

Lemma 3.1. ([8]) If $(f,\ g_{ij})$ solves the Einstein free scalar field system (3.6) - (3.7) the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^{2k_n f} dt^2 + g_{ij} dx^i dx^j \tag{3.8}$$

is a local Ricci soliton on $\mathrm{R} \times M^n$ solving (2.13).

For Schwarzschild metric, Akbar and Woolger [8] have given that equation (3.6) can be applied on the metric g_{ij} of the form

$$ds^2 = g_{ij}dx^i dx^j \equiv dr^2 + h^2(r)d\Omega_k^2 \tag{3.9}$$

where $h(r)$ is a function of r and $d\Omega_k^2$ is an Einstein metric with scalar curvature R normalized to -1, 0 or 1.

For the integrability conditions (3.7), one can take

$$f'(r) = \frac{B}{[h(r)]^{n-1}} \text{ for } B = \text{constant} \tag{3.10}$$

This idea can be extended to Kerr-Newman, Kerr or Reisnerr-Nordström spacetime also. But for the Lorentzian signature, here we discuss Reisnerr-Nordström and Schwarzschild soliton.

For $B = 0$, There is a constant f-soliton $ds^2 = -dt^2 + g_{ij}dx^i dx^j$ for Einstein metric g_{ij} (cf., [8]). Now for the case $f'(r) \neq 0$, we will establish an example of a soliton which is solution of Einstein-free scalar field system and the Reisnerr-Nordström metric explains such soliton. The Reisnerr-Nordström metric is [19]

$$ds^2 = -\left(\frac{r^2 + e^2 - 2mr}{r^2}\right)dt^2 + \left(\frac{r^2}{r^2 + e^2 - 2mr}\right)dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{3.11}$$

Choosing the functions

$$f = \frac{1}{2}\log\left(\frac{r^2 - 2mr + e^2}{r^2}\right) \tag{3.12}$$

$$h = \sqrt{r^2 - 2mr + e^2} \tag{3.13}$$

and for 4-dimensional spacetime ($n = 3$), we have

$$k_n^2 = 2 \Rightarrow k_n = \sqrt{2} \tag{3.14}$$

Using equations (3.12) and (3.13), equation (3.11) takes the form as

$$ds^2 = -e^{2f}dt^2 + e^{-2f}d\Omega^2 \tag{3.15}$$

where

$$d\Omega^2 = dr^2 + h^2(r)(d\theta^2 + sin^2\theta d\phi^2) \tag{3.16}$$

Further using Lemma 3.1 and equations (3.12) - (3.16), we get the corresponding soliton

$$ds^{*2} = -e^{2\sqrt{2}f}dt^2 + e^{-2f}d\Omega^2$$

or,

$$ds^{*2} = -\left(\frac{r^2 + e^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2 + dr^2 + h^2(r)(d\theta^2 + sin^2\theta d\phi^2)$$

The Reisnerr-Nordström soliton is

$$ds^{*2} = -\left(\frac{r^2 + e^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2 + dr^2 + (r^2 - 2mr - e^2)(d\theta^2 + sin^2\theta d\phi^2) \tag{3.17}$$

Note: If we take $e = 0$ in equation (3.17), we get Schwarzschild soliton as

$$ds^2 = -\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2 + dr^2 + (r^2 - 2mr)(d\theta^2 + sin^2\theta d\phi^2) \tag{3.18}$$

4. Einstein spaces and Petrov type D gravitational fields

In this section, we shall discuss the role of Ricci solitons in the study of Einstein spaces and Petrov type D pure radiation fields and we have

(a) Einstein spaces

From equations (2.1) and (2.13), we have

$$\begin{aligned} 2R_{ij} &= \pounds_\xi g_{ij} + 2kg_{ij} \\ &= \xi_{i;j} + \xi_{j;i} + 2kg_{ij} \end{aligned} \tag{4.1}$$

Contracting this equation with g^{ij}, we get

$$R = \xi^i_{;i} + kn$$

which can be expressed as

$$div\xi = \nabla_i \xi^i = (R - kn) \tag{4.2}$$

where $R = g^{ij}R_{ij}$ is scalar curvature. From equations (4.1) and (4.2), we get

$$(n^{-1}Rg_{ij} - R_{ij}) = -\frac{1}{2}\pounds_\xi g_{ij} + n^{-1}(div\xi)g_{ij} \tag{4.3}$$

Now for g_{ij} to be Einstein metric i.e., $R_{ij} = \mu g_{ij}$ where μ can be chosen as $n^{-1}R$, equations (4.3) and (2.2) will give

Lemma 4.1. ([18]) The vector field ξ associated with Ricci soliton $(M,\ g)$ is conformally Killing if and only if $(M,\ g)$ is an Einstein manifold of dimension $(n \geq 3)$.

It is known that ([20])

$$\pounds_\xi \Gamma^i_{jk} = \frac{1}{2} g^{il}(\nabla_j \pounds_\xi g_{kl} - \nabla_l \pounds_\xi g_{jk} + \nabla_k \pounds_\xi g_{lj}) \tag{4.4}$$

If ξ is a conformally Killing vector field, then equations (2.2) and (4.4) lead to

$$\pounds_\xi \Gamma^i_{jk} = \delta^i_j \sigma_{;k} - g^{il} g_{jk} \sigma_{;l} + \delta^i_k \sigma_{;j} \tag{4.5}$$

Thus we have

Lemma 4.2. If in $(M,\ g)$ a vector field ξ is conformally Killing then it is also a conformal collineation vector field.

From Lemmas 4.1 and 4.2, we have the following

Theorem 4.3. A vector field ξ associated with a Ricci soliton $(M,\ g)$ is conformal collineation vector field if and only if M is Einstein manifold.

While the use of equation (2.3) leads to

Corollary 4.4. If $\sigma_{;jk} = 0$ the vector field ξ associated with a Ricci soliton $(M,\ g)$ is special conformal collineation vector field if and only if M is an Einstein manifold.

The Weyl conformal tensor is given by

$$\begin{aligned} C^i_{jkl} &= R^i_{jkl} + \frac{1}{2}(\delta^i_k R_{jl} - \delta^i_l R_{jk} + g_{jl} R^i_k - g_{jk} R^i_l) \\ &+ \frac{R}{(n-1)(n-2)}(\delta^i_l g_{jk} - \delta^i_k g_{jl}) \end{aligned} \tag{4.6}$$

We prefer 4-dimensional spacetime, so the component of Weyl conformal tensor are

$$C^i_{jkl} = R^i_{jkl} + \frac{1}{2}(\delta^i_k R_{jl} - \delta^i_l R_{jk} + g_{jl} R^i_k - g_{jk} R^i_l) + \frac{R}{6}(\delta^i_l g_{jk} - \delta^i_k g_{jl}) \tag{4.7}$$

where R^i_{jkl} and R are Riemann curvature tensor and scalar curvature tensor respectively. We know that ([20])

$$\pounds_\xi R^i_{jkl} = \nabla_j \pounds_\xi \Gamma^i_{kl} - \nabla_k \pounds_\xi \Gamma^i_{jl} \tag{4.8}$$

Using equation (4.5) in equation (4.8), we get

$$\pounds_\xi R^i_{jkl} = -2\delta^i_{[j} \nabla_{k]} \sigma_{;l} - 2(\nabla_{[j} \sigma^{;i}) g_{k]l} \tag{4.9}$$

On contraction we get

$$\pounds_\xi R_{kl} = -(n-2)\nabla_k \sigma_{;l} - g_{kl} \nabla_m \sigma^{;m} \tag{4.10}$$

Contracting equation (4.10) with g^{kl}, we get

$$\pounds_\xi R = -2\sigma^l_{;l} R - 2(n-1)\nabla_m \sigma^{;m}$$

or, for 4-dimensional spacetime

$$\pounds_\xi R = -2\phi R - 6\nabla_m \sigma^{;m} \tag{4.11}$$

where $\phi = div\sigma$.

Now taking the Lie derivative of equation (4.7) and using equations (4.9) - (4.11), we get

$$\pounds_\xi C^i_{jkl} = 0 \tag{4.12}$$

Thus from equation (2.5), we state

Lemma 4.5. Every conformally Killing vector is also Weyl conformal collineation vector for the spacetime of general relativity.

From Lemmas 4.1 and 4.5, we have

Theorem 4.6. A vector field ξ associated with Ricci soliton $(M,\ g)$ is Weyl conformal collineation vector if and only if M is an Einstein space.

(b) Petrov type D pure radiation fields

The study of Petrov type D gravitational fields is an important activity in general relativity as most of the physically significant metrics are of Petrov type D. The most familiar important members of this class are Schwarzschild exterior solution, Reisnerr-Nordström metric, Kerr and Gödel solutions. De Groote et al [13] have solved the problem of aligned Petrov type D pure radiation fields and have presented all Petrov type D pure radiation spacetimes with a shear-free and non-diverging geodesic principal null congruences. Recently, Ahsan and Ali [6] have studied the symmetries of Petrov type D pure radiation fields and have established a number of relations between different types of collineations. From Ahsan and Ali [6], we have

Lemma 4.7. In a pure radiation type D field conformal motion, special conformal motion and homothetic motion all degenerate to motion.

Thus by using Lemmas 4.1 and 4.7, we have the following

Theorem 4.8. Type D pure radiation fields do admit motion along a vector field ξ associated to Ricci soliton $(M,\ g)$ if and only if M is an Einstein space.

For Killing vector field ξ, equation (2.13) changes to

$$R_{ij} = kg_{ij} \tag{4.13}$$

Taking Lie derivative with respect to vector field ξ

$$\pounds_{\xi}R_{ij} = k\pounds_{\xi}g_{ij} = 0$$

Thus we have

Theorem 4.9. Type D PR fields admit Ricci collineation along a vector field ξ associated to Ricci soliton $(M,\ g)$ if and only if M is Einstein space.

Remark: A number of similar results can be obtained easily as motion implies approximately all symmetry properties of a spacetime (cf., [14]).

5. Conclusion

Reisnerr-Nordström soliton has been derived and Schwarzschild soliton is given as a special case. For Einstein spaces different kind of symmetry properties are established with the help of vector field associated with Ricci soliton. Further the idea is extended for type D pure radiation fields.

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