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DECOMPOSITION THEOREMS ON CR-SUBMERSIONS OF KAEHLER MANIFOLDS

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ABSTRACT. In this paper we consider the Riemannian submersions of CR-submanifolds of Kaehler and obtain product theorem on such submersion by imposing conditions on CR-submanifold and its distribution.

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1. Introduction

The study of submersion π of a Riemannian manifold M onto a Riemannian manifold B was initiated by O'Niell [10]. A submersion gives rise to two distributions on M which we call horizontal and vertical distributions of which the vertical distribution is always integrable and gives rise to fibres of the submersion which are closed submanifolds of M. A. Bejancu [1] defined a special class of submanifolds of an almost Hermitian manifold, of which both the class of complex submanifolds as well as totally real submanifolds become particular class. He named this special class of submanifolds as CR-submanifolds. On a CR-submanifold M of a Kaehler manifold $\bar{M}$ with almost complex structure J there exist two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M, $\mathcal{D}$ being invariant under J and $\mathcal{D}^\perp$ being totally real and always integrable [2], [3], [5]. Kobayshi [9] observed this similarity between the total space of the submersion $\pi : M \to B$ and the CR-submanifold M of a Kaehler manifold $\bar{M}$ in terms of distributions. He considered the submersion $\pi : M \to B$ of a CR-submanifold M of a Kaehler manifold onto an almost Hermitian manifold B such that the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M become respectively the horizontal and vertical distributions required by the submersion π and π restricted to $\mathcal{D}$ becomes a complex isometry [9], that is, $\pi_* o J = J^{'} o \pi_*$ on $\mathcal{D}$, where J and $J^{'}$ are complex structure of M and B respectively. He has proved that in such a situation B becomes a Kaehler manifold and obtained a relation between the holomorphic sectional curvatures of $\bar{M}$ restricted to $\mathcal{D}$ and B.
Motivated by the importance of submersions, in this paper we have investigated the CR-submersions of Kaehler manifolds. Section 2 deals with some of known results given by Kobayashi [9] and O'Niell [10], which are needed for further investigations. The geometry of CR-submanifold has been studied in Section 3 and a number of decomposition theorems have been proved here.

2. Preliminaries

In this section we recall some basic definitions and extract part of O'Niell's paper [10] and Kobayshi's paper [9] which we shall use in the sequel.

Let $\bar{M}$ be a Kaehler manifold of real dimension $2n$ with an almost complex structure J and Hermitian metric g. Then

$$(\bar{\nabla}_X J)Y = 0,\ \forall\ X,\ Y \in \chi(\bar{M}), \tag{2.1}$$

where $\bar{\nabla}$ being the Riemannian connection on $\bar{M}$ and $\chi(\bar{M})$ is the Lie algebra of vector fields on $\bar{M}$.

Let M be an m-dimensional submanifold of $\bar{M}$. The Riemannian connection $\bar{\nabla}$ on $\bar{M}$ induces the Riemannian connection ∇ and $\nabla^{\perp}$ on M and in the normal bundle ν of M in $\bar{M}$ respectively. These connections are connected by Gauss and Wiengarten formulae

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.2}$$

$$\bar{\nabla}_X N = \tilde{A}_N X + \nabla^{\perp}_X N, \tag{2.3}$$

for any $X,\ Y \in \chi(M)$ and $N \in \nu$. h and $\tilde{A}_N$ are the second fundamental form and Wiengarten map respectively and obey the following relation

$$g(h(X,Y),N) = g(\tilde{A}_N X, Y). \tag{2.4}$$

Definition 2.1 The submanifold M of $\bar{M}$ is said to be a CR-submanifold if there exist two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on M such that $\mathcal{D}$ is invariant under the almost complex structure J, i.e., $J\mathcal{D} = \mathcal{D}$ and $\mathcal{D}^{\perp}$ is totally real, i.e., $J\mathcal{D}^{\perp} \subset \nu$, where ν is the normal bundle of M in $\bar{M}$. The distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are called horizontal and vertical distributions respectively.

In what follows we shall always take $J\mathcal{D}^{\perp} = \nu$ so that if dim $\mathcal{D} = 2p$, dim $\mathcal{D}^{\perp} = q$ then dim $M = 2(p+q)$. It is known that the distribution $\mathcal{D}$ is integrable if and only if

$$h(X, JY) = h(JX, Y),\ \forall\ X,\ Y \in \mathcal{D}. \tag{2.5}$$

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Definition 2.2 A CR-submanifold M of a Kaehler manifold $\bar{M}$ is said to be totally geodesic if $h = 0$ identically on M.

Definition 2.3 A CR-submanifold M of a Kaehler manifold $\bar{M}$ is said to be foliate if the horizontal distribution $\mathcal{D}$ is integrable.

Definition 2.4 A CR-submanifold M of a Kaehler manifold $\bar{M}$ is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H,\ \forall\ X,\ Y \in \chi(M), \tag{2.6}$$

where $H = \dfrac{1}{m}(\text{trace } h)$.

For the theory of submersion we follow O'Niell [10] and Kobayshi [9].

Let B be an almost Hermitian manifold with Hermitian metric g_*. Let $\pi : M \to B$ be a Riemannian submersion of a CR-submanifold M onto B such that

(i) $\mathcal{D}^\perp$ is the Kernel of π_*, that is $\pi_* \mathcal{D}^\perp = \{0\}$,

(ii) $\pi_* : \mathcal{D}_p \to T_{\pi(p)}B$ is a complex isometry for all $p \in M$, where $T_{\pi(p)}B$ is the tangent space of B at $\pi(p)$,

(iii) J interchanges $\mathcal{D}^\perp$ and ν.

We call such submersions as CR-submersions.

For a vector field X on M, we set

$$X = \mathcal{H}X + \mathcal{V}X, \tag{2.7}$$

where $\mathcal{H}$ and $\mathcal{V}$ denote the horizontal and vertical part of X.

To relate the geometry of M to that of B we make the special choice of vector fields on M and name them as basic vector fields.

Definition 2.5 A vector field X on M is said to be basic if

(i) X is horizontal, i.e., $X \in \mathcal{D}$ and

(ii) X is π-related to a vector field on B, that is, there exists a vector field X_* on B such that $(\pi_* X)_p = (X_*)_{\pi(p)}$ for every $p \in M$.

We have the following lemma for the basic vector fields.

Lemma 2.1 [10] Let X and Y be basic vector fields on M. Then

(i) $g(X,Y) = g_*(X_*, Y_*) \circ \pi$,

(ii) $\mathcal{H}[X,Y]$ is basic and corresponds to $[X_*, Y_*]$,

(iii) $\mathcal{H}(\nabla_X Y)$ is basic and corresponds to $\nabla^*_{X_*} Y_*$, where ∇^* is a Riemannian connection on B.

(iv) $[V, X] \in \mathcal{D}^\perp, \forall\ V \in \mathcal{D}^\perp$.

For a covariant differentiation operator ∇^* we define a corresponding operator $\tilde{\nabla}^*$ for the basic vector fields of M by

$$\tilde{\nabla}^*_X Y = \mathcal{H}(\nabla_X Y). \tag{2.8}$$

Then $\tilde{\nabla}^*_X Y$ is basic vector field and we have

$$\pi_*(\tilde{\nabla}^*_X Y) = \nabla^*{}_{X_*} Y_*. \tag{2.9}$$

Define a tensor field C by

$$\nabla_X Y = \mathcal{H}(\nabla_X Y) + C(X,Y), \text{ for all } X,\ Y \in \mathcal{D} \tag{2.10}$$

where $C(X,Y)$ is the vertical part of $\nabla_X Y$, i.e., $\mathcal{V}\nabla_X Y = C(X,Y)$. If in particular X, Y are basic, then

$$\nabla_X Y = \tilde{\nabla}^*_X Y + C(X,Y). \tag{2.11}$$

It has been observed in [9] that C is skew symmetric and

$$C(X,Y) = \frac{1}{2}\mathcal{V}[X,Y]. \tag{2.12}$$

For $X \in \mathcal{D}$ and $V \in \mathcal{D}^\perp$, we define an operator $\mathcal{A}$ on M by

$$\nabla_X V = \mathcal{V}(\nabla_X V) + \mathcal{A}_X V,\ \mathcal{A}_X V = \mathcal{H}(\nabla_X V) \tag{2.13}$$

Since $[V, X] \in \mathcal{D}^\perp$ for $V \in \mathcal{D}^\perp$, we have

$$\mathcal{H}(\nabla_V X) = \mathcal{H}(\nabla_X V) = \mathcal{A}_X V.$$

The operator C and $\mathcal{A}$ are related by

$$g(C(X,Y),V) = -g(\mathcal{A}_X V, Y),\ \forall\ X,\ Y \in \mathcal{D},\ V \in \mathcal{D}^\perp \tag{2.14}$$

Also, for $X \in \mathcal{D}$ and $V \in \mathcal{D}^\perp$, we define the operator $\mathcal{T}$ on M by

$$\nabla_V X = \mathcal{T}_V X + \mathcal{H}(\nabla_V X),\ \mathcal{T}_V X = \mathcal{V}(\nabla_V X). \tag{2.15}$$

The operator C is introduced by Kobayshi [9] while the operator $\mathcal{T}$ and $\mathcal{A}$ are due to O'Niell [10] and are called fundamental tensors of the submersion π. The operator $\mathcal{A}$ coincide with C for horizontal vector fields. The alternative of $\mathcal{T}$ for vertical vector fields will be denoted by L which we will introduce in next section.

3. Geometry of CR-submanifolds

In this section we study the geometry of CR-submanifold M of a Kaehler manifold $\bar{M}$ where the submersion is $\pi : M \to B$.

Let $\pi : M \to B$ be a CR-submersion and let $X, Y \in \mathcal{D}$, then from (2.1) we have

$$\bar{\nabla}_X JY = J\bar{\nabla}_X Y,$$

which on using (2.2), (2.7), (2.8) and (2.9) yields

$$\tilde{\nabla}^*_X JY + h(X,JY) + C(X,JY) = J\tilde{\nabla}^*_X Y + JC(X,Y) + Jh(X,Y) \tag{3.1}$$

Comparing vertical and normal parts, we have

Lemma 3.1 Let X, Y be horizontal vector fields on M. Then

(i) $C(X,JY) = Jh(X,Y)$, (3.2)

(ii) $h(X,JY) = JC(X,Y)$. (3.3)

Next, we have

Lemma 3.2 Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$ and π be a CR-submersion. Then

$$\mathcal{A}_{JX}V = J\mathcal{A}_X V,$$

for any $X, Y \in \mathcal{D}$ and $V \in \mathcal{D}^{\perp}$.

Proof. Let X be a basic vector field and Y in $\mathcal{D}$, $V \in \mathcal{D}^{\perp}$. Then using (2.1), (2.2), (2.13) and Lemma 2.1 (iv), we have

$$\begin{aligned} g(\mathcal{A}_{JX}V, Y) &= g(\mathcal{H}\nabla_{JX}V, Y) \\ &= g(\nabla_{JX}V, Y) \\ &= g([JX, V] + \nabla_V JX, Y) \\ &= g(\nabla_V JX, Y) \end{aligned}$$

$$\begin{aligned} g(\mathcal{A}_{JX}V, Y) &= g(\bar{\nabla}_V JX, Y) \\ &= -g(\nabla_V X, JY) \\ &= -g(\mathcal{H}(\nabla_V X), JY) \\ &= -g(\mathcal{A}_X V, JY) \\ &= g(J\mathcal{A}_X V, Y)\ , \end{aligned}$$

which proves the result.

From above lemma and (2.14) it follows that

$$C(JX, Y) = -C(X, JY). \tag{3.4}$$

Which inturn proves the following identity

$$C(JX, JY) = C(X, Y). \tag{3.5}$$

Proposition 3.1 Let M be a $\mathcal{D}$-totally geodesic CR-submanifold of a Kaehler manifold $\bar{M}$. Then the horizontal distribution of a CR-submersion is integrable and totally geodesic.

Proof. Proof follows from Lemma 3.1, Lemma 3.2 and (2.14).

Definition 3.1 A smooth distribution S on a Riemannian manifold M is said to be totally umbilical if there exists a vector field H in $S^{\perp}$ such that

$$\nabla_X Y = g(X, Y)H, \text{ for all } X,\ Y \in S,$$

where $S^{\perp}$ is the orthogonal complement of S according the decomposition $TM = S \oplus S^{\perp}$. In this case, H is called the mean curvature vector of S.

We recall the following from [13].

Let g be a Riemannian metric tensor on the manifold $M = B \times F$ and assume that the canonical distribution $\mathcal{D}_B$ and $\mathcal{D}_F$ intersect perpendicularly everywhere. Then g is the metric tensor of

(**i**) a usual product of Riemannian manifolds if and only if $\mathcal{D}_B$ and $\mathcal{D}_F$ are totally geodesic foliations.

(**ii**) a twisted product $B \times_f F$ if and only if $\mathcal{D}_B$ is totally geodesic foliation and $\mathcal{D}_F$ is totally umbilical foliation.

(**iii**) a doubly twisted product ${}_fB \times_g F$ if and only if $\mathcal{D}_B$ and $\mathcal{D}_F$ are totally umbilic foliations.

Theorem 3.1 Let M be a $\mathcal{D}$-totally geodesic CR-submanifold of a Kaehler manifold $\bar{M}$. Let π be a CR-submersion of M onto an almost Hermitian manifold B. Then M is a locally product manifold $M_1 \times M_2$, where M_1 is an invariant submanifold and M_2 is a totally real submanifold of $\bar{M}$.

Proof. Since M is totally geodesic, then Lemma 3.1 yields $C(X, Y) = 0$, for X, $Y \in \mathcal{D}$ and thus from the defining equation of C i.e., (2.10) it follows that $\nabla_X Y \in \mathcal{D}$, that is, $\mathcal{D}$ is parallel. Further, since on a CR-submanifold M of a Kaehler manifold $\bar{M}$, $\mathcal{D}^\perp$ is integrable and totally geodesic, from a result of A. Bejancu [Theorem 4.4 p-9] $\mathcal{D}^\perp$ is also parallel. Since both the distributions are parallel, then by De Rham's theorem M is the product manifold $M_1 \times M_2$, where M_1 is the integral submanifolds of $\mathcal{D}$ and M_2 is integral submanifold of $\mathcal{D}^\perp$. This completes the proof of the theorem.

As a consequence of above theorem and Theorem 1.3 of Kobayshi [9], we have

Corollary 3.1 Let M be a $\mathcal{D}$-totally geodesic CR-submanifold of a Kaehler manifold $\bar{M}$. If $\pi : M \to B$ be a CR-submersion from M onto an almost Hermitian manifold B, then

$$\bar{H}(X) = H^*(X),\ X \in \mathcal{D},$$

where $\bar{H}$ and H^* are respectively the holomorphic sectional curvatures of $\bar{M}$ and B.

In particular, if $\bar{M}$ is a space of constant holomorphic sectional curvature c, then so is B.

Definition 3.2 [1] A CR-submanifold M is said to be CR-totally geodesic if $h(X, V) = 0$, for any $X \in \mathcal{D}$ and $V \in \mathcal{D}^\perp$.

For the submersion of CR-totally geodesic submanifold, we have the following;

Theorem 3.2 Let M be a CR-totally geodesic foliate submanifold of a Kaehler manifold $\bar{M}$ and let $\pi : M \to B$ be a CR-submersion of M onto an almost Hermitian manifold B. Then M is a locally product $M_1 \times M_2$, where M_1 is an invariant submanifold and M_2 is totally real submanifold of $\bar{M}$.

Proof. From the proof of the Theorem 3.1, it follows that $\mathcal{D}$ is parallel. Also M being CR-totally geodesic, we have

$$h(X,V)=0,\ \text{for any } X\in\mathcal{D}, V\in\mathcal{D}^{\perp}. \tag{3.5}$$

For any vector fields $V,W\in\mathcal{D}^{\perp}$, from (2.1) we have

$$-\tilde{A}_{JW}V+\nabla^{\perp}_{V}JW=J\nabla_{V}W+Jh(V,W) \tag{3.6}$$

To complete the proof of the theorem we need the following result whose proof will be demonstrated at the end of the proof of the theorem.

Lemma 3.3 A CR-submanifold M of a Kaehler manifold $\bar{M}$ is CR-totally geodesic if and only if $\tilde{A}_N V\in\mathcal{D}^{\perp}$ for each $V\in\mathcal{D}^{\perp}$ and $N\in\nu$.

If we admit this lemma in (3.6) and put $JW=N$, we see that $\tilde{A}_{JW}V\in\mathcal{D}^{\perp}$. And thus equating vertical part we have $-\tilde{A}_{JW}V=Jh(V,W)$, which then yields

$$\nabla^{\perp}_{V}JW=J\nabla_{V}W. \tag{3.7}$$

Hence (3.7) proves that $\nabla_V W\in\mathcal{D}^{\perp}$, i.e., $\mathcal{D}^{\perp}$ is parallel and thus proves the theorem.

Proof of the Lemma. Let M be a CR-totally geodesic, then by using Definition 3.1 and (2.4) we get

$$g((\mathcal{A}_N V,X)=0$$

for each $X\in\mathcal{D}, V\in\mathcal{D}^{\perp}$ and $N\in\nu$. This implies that $\mathcal{A}_N V\in\mathcal{D}^{\perp}$.

Conversely, suppose that $\mathcal{A}_N V\in\mathcal{D}^{\perp}$ for any $V\in\mathcal{D}^{\perp}$ and $N\in\nu$. Let $\{N_1,\ N_2,...,N_{2n-m}\}$ be a local orthonormal frame of ν, when $2n=\dim\bar{M}$ and $m=\dim M$. Then for any $X\in\mathcal{D}$, we have

$$0=g(\mathcal{A}_{N_p}V,X)=g(h(V,X),N_p),\ 1\le p\le 2n-m. \tag{3.8}$$

Since $h(V,X)\in\nu$, from (3.8) it follows that $h(V,X)=0$. And hence M is CR-totally geodesic.

For $U,V\in\mathcal{D}^{\perp}$, we define a tensor field L by

$$\nabla_U V=\hat{\nabla}_U V+L(U,V), \tag{3.9}$$

where $L(U,V)$ is the horizontal part of $\nabla_U V$, i.e., $\mathcal{H}(\nabla_U V)=L(U,V)$ and $\hat{\nabla}_U V=\mathcal{V}(\nabla_U V)$. Since it is observed that $\mathcal{D}^{\perp}$ is always integrable, we get $L(U,V)=L(V,U)$.

The operators $\mathcal{T}$ and $\mathcal{A}$ are related by

$$g(\mathcal{T}_U X,V)=-g(L(U,V),X), \tag{3.10}$$

for any $U,\ V\in\mathcal{D}^{\perp},\ X\in\mathcal{D}$.

From (3.9) we see that the operator L is the second fundamental form of fibres in M and hence the fibres of the submersion π are totally geodesic if and only if $L(U,V)=0$.

We have

Proposition 3.2 Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$ and π be a CR-submersion of M onto an almost Hermitian manifold B. Then the fibres of π are totally geodesic submanifold of M if and only if M is CR-totally geodesic.

Proof. From (2.1) it follows that

$$-\tilde{\mathcal{A}}_{JV}U + \nabla^{\perp}_{U}JV = JL(U,V) + Jh(U,V),$$

for any $U,\ V \in \mathcal{D}^{\perp}$. Using (2.7) and comparing horizontal and vertical parts we get

$$\mathcal{H}(\tilde{\mathcal{A}}_{JV}U) = -JL(U,V) \tag{3.11}$$

and

$$\mathcal{V}(\tilde{\mathcal{A}}_{JV}U) = -Jh(U,V). \tag{3.12}$$

From (3.11) it follows that the fibres are totally geodesic if and only if $\tilde{\mathcal{A}}_{JV}U \in \mathcal{D}^{\perp}$, which by Lemma 3.3 proves that the fibres are totally geodesic if and only if M is CR-totally geodesic.

Theorem 3.3 Let M be a $\mathcal{D}$-totally geodesic CR-submanifold of a Kaehler manifold $\bar{M}$ and $\pi : M \to B$ be a CR-submersion of M onto an almost Hermitian manifold B. Then M is a locally product manifold $M_1 \times M_2$ if and only if fibres are totally geodesic submanifold of M, where M_1 is an invariant submanifold and M_2 is a totally real submanifold of $\bar{M}$.

Proof. From the proof of the Theorem 3.1, $\mathcal{D}$ is parallel. Further for any $U, V \in \mathcal{D}^{\perp}$ and $X \in \mathcal{D}$, on using (2.7) and (3.9) we have

$$g(\nabla_U V, X) = g(L(U,V), X).$$

From which it follows that $\mathcal{D}^{\perp}$ is parallel if and only if fibres are totally geodesic. Which then proves the theorem.

Theorem 3.4 Let M be a CR-submanifold of Kaehler manifold $\bar{M}$ and $\pi : M \to B$ be a CR-submersion of M onto an almost Hermitian manifold B. Then M is a locally twisted product manifold $M_1 \times M_2$ if and only if

(i) $\mathcal{T}_V X = -g(X, L(V,V))\, \|V\|^{-2}\, V$ and
(ii) $C(X, JY) = 0$,

for any $X,\ Y \in \mathcal{D}, V \in \mathcal{D}^{\perp}$, where M_1 is an invariant submanifold and M_2 is totally real submanifold of $\bar{M}$.

Proof. For any $V,\ W \in \mathcal{D}^{\perp},\ X \in \mathcal{D}$, from (2.1), we have

$$\begin{aligned} g(\nabla_V W, X) &= g(-\tilde{A}_{JW}V, JX) \\ &= -g(\mathcal{H}(\tilde{\mathcal{A}}_{JW}V), JX) \\ &= g(JL(V,W), JX) \\ &= g(L(V,W), X) \\ &= -g(\mathcal{T}_V X, W). \end{aligned}$$

This implies that $\mathcal{D}^{\perp}$ is totally umbilical if and only if

$$\mathcal{T}_V X = -X(f)V, \tag{3.13}$$

where f is some function on M.

On taking inner product with V in (3.13) it follows that

$$X(f) = -g(\mathcal{T}_V X, V)\ \|V\|^{-2}.$$

Using (3.10), we have

$$X(f) = g(L(V,V), X)\ \|V\|^{-2}.$$

Putting it in (3.13), we get

$$\mathcal{T}_V X = -g(L(V,V), X)\ \|V\|^{-2}\ V.$$

The proof then follows from Lemma 3.1.

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