Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.1, Issue2, pp.85-88

AREA OF SPHERICAL LUNE FORMED BY TWO RANDOM PLANES

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ABSTRACT. An analytical solution for the exact calculation of the area of spherical lune formed by two random planes intersected in the sphere is given.

2010 Mathematical Subject Classification:51M04, 51M25.
Keywords and phrases:Spherical lune; surface area; analytical solution; random planes.

1. INTRODUCTION

Generally, a spherical lune is the curved surface formed by revolving a semi-circular arc about its diameter by less than 360. Its area A_{lune} can be calculated exactly by the following well know equation ([1]):

$$A_{lune} = 2r^2\vartheta_{rad} \tag{1}$$

where, r is the radius of the sphere and ϑ_{rad} (in radians) is the angle between the two planes forming the spherical lune. The line of intersection of the two planes is strictly the diameter of the sphere.

In the present paper an analytical solution for the exact calculation of the area of spherical lune formed by two random planes intersected in the sphere is given. The solution is based on Archimedes of Syracuse (287-212 BC) who first shown that the projection of a sphere onto a circumscribing cylinder is area preserving.

2. THEORY

Consider that two random planes (Plane I and Plane II) intersect a sphere of radius r and that, the line of intersection of the two planes passes through the sphere as shown in Figure 1. It is noted that, the line of intersection does not necessarily coincides with the diameter of the sphere. The area of the spherical lune between these two planes arises by its radial projection to an imaginary cylinder that circumscribes the sphere. On this base, any intersection point Ω between Plane I or Plane II and the sphere is projected on the surface of the cylinder as shown in Figure 2 (point Ω'). The axis of the cylinder is considered parallel to the axis and as the diameter of the sphere is equal to the height of the cylinder, all points preserve their original coordinate. The total of the points Ω' on the cylindrical surface define the projected area of the spherical lune. Unrolling the cylinder, a $z - x$ diagram is obtained similar to the one given in Figure 1. The projected area in Figure 1 is equal to the original area of the spherical lune. If $x = f_1(z)$ and $x = f_2(z)$ are the functions representing the points of intersection between Plane I and the sphere and Plane II and the sphere respectively, then the area of the spherical lune is given as follows:

$$A_{lune} = \int_{z_L}^{z_R} [f_1(z) - f_2(z)]dz \tag{2}$$

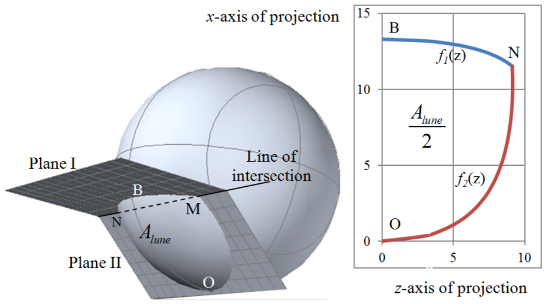

FIGURE 1. Left: Area of spherical lune formed by two random planes intersected in the sphere. Right: Half area of spherical lune (due to symmetry) as projected on $z - x$ plane (example).

Therefore, the problem is reduced to defining $x = f_1(z)$ and $x = f_2(z)$. Considering that the sphere, Plane I and Plane II are represented by Equations 3, 4 and 5 respectively, the limits of integration z_L and z_R are the two common points of these three geometric shapes (Equation 6). Here, the center of the sphere is considered the point C$(x_o, y_o, 0)$, Plane I is parallel to the $z - x$ plane and the lowest intersection point between Plane II and the sphere (which is the lowest intersection point, in general) is the point of origin of the Cartesian system of axes (Point O).

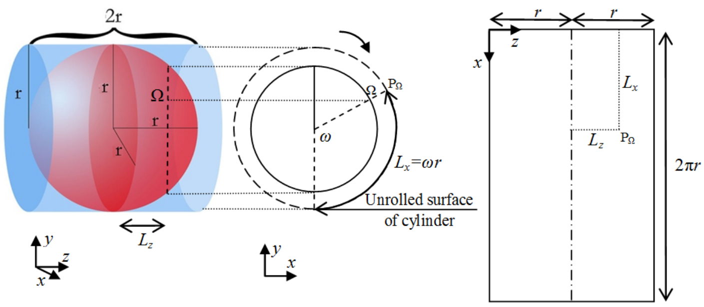

FIGURE 2. Projection of a random point of the sphere on a circumscribing cylinder.

$$(x - x_o)^2 + (y - y_o)^2 + z^2 = r^2 \tag{3}$$

$$y = h \tag{4}$$

$$y = -xtan\beta \tag{5}$$

$$z_{L,R} = \pm\sqrt{r^2 - \left(\frac{h}{-tan\beta} - x_o\right)^2 - (h-y_o)^2} \tag{6}$$

In order $f_1(z)$ and $f_2(z)$ to be defined, the sphere is cut by an infinite number of planes $(z = z_i)$ parallel to the $x - y$ plane. In essence, the two functions $f_1(z)$ and $f_2(z)$ represent the length of arc $O'P_{B'}$ and $O'P_{A'}$ respectively for a given $z-$ coordinate (Figure 3). Thus, it stands that:

$$f_1(z) = \left(\Omega_2\widehat{C}'D - O\widehat{C}F\right) r \tag{7}$$

$$f_2(z) = \left(\Omega_1\widehat{C}'E - O\widehat{C}F\right) r \tag{8}$$

where, geometrically, it can easily be shown that:

$$\Omega_2\widehat{C}'D = arcsin\left[\frac{(\Omega_2 D)}{(\Omega_2 C')}\right] = arcsin\left(\frac{\left|-\sqrt{r^2 - z^2 - (h-y_o)^2} + x_o\right| + x_o}{\sqrt{r^2 - z^2}}\right) \tag{9}$$

$$O\widehat{C}F = arcsin\left[\frac{(OF)}{(OC)}\right] = arcsin\left(\frac{x_o}{r}\right) \tag{10}$$

$$\Omega_1\widehat{C}'E = arcsin\left[\frac{(\Omega_1 E)}{(\Omega_1 C')}\right] = arcsin(\omega) \tag{11}$$

$$\omega = \frac{\left|\frac{2(x_o - y_o tan\beta) + \sqrt{4(x_o - y_o tan\beta)^2 - 4(1+tan^2\beta)(x_o^2 + y_o^2 + z^2 - r^2)}}{2(1+tan^2\beta)}\right| + x_o}{\sqrt{r^2 - z^2}} \tag{12}$$

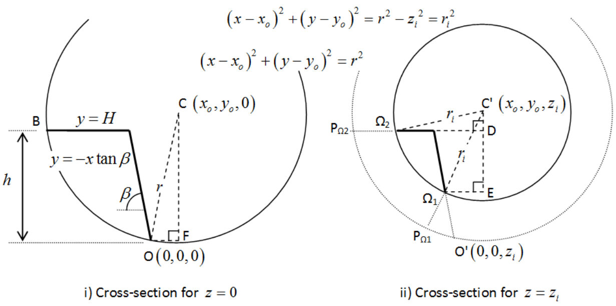

FIGURE 3. Cross-section of the geometry of the problem for $z = 0$ and for $z = z_i$.

The above procedure stands only for the geometry of Figure 3, where, the maximum width of the spherical lune on a $z - x$ plane is $|z_L| + |z_R|$; this is happening when $x_o \geq 0$. For other cases, it must be modified as needed. Finally, as regards to the area of the supplementary spherical lune, this derives from the subtraction of the surface area of the spherical lune as described herein from the respective spherical cap. The surface area of the spherical cap is given by the following well know equation ([1]):

$$A_{cap} = 2\pi r h_{cap} \tag{13}$$

where, h_{cap} is the height of the spherical cap.

3. Conclusions

An analytical solution for the exact calculation of the area of spherical lune formed by two random planes intersected in the sphere is given; the two planes could be any plane and do not necessarily pass from the center of the sphere. The area of the spherical lune is calculated by projecting it on a circumscribed cylinder based on the fact that the projection of a sphere onto a circumscribing cylinder is area preserving. In essence, the projected lune is represented by two mathematical functions and the area of lune derives from their integration.

References

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