Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.1, Issue2, pp.66-70

A PURELY SYNTHETIC PROOF OF THE GENERALIZED DROZ-FARNY THEOREM

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ABSTRACT. We will present a purely synthetic proof of the Generalized Droz-Farny Theorem using the notion of cross ratio.

2010 Mathematical Subject Classification:51M04, 51M25.
Keywords and phrases:cross ratio, signed distances.

1. GENERALIZED DROZ-FARNY THEOREM

In 1899, Arnold Droz-Farny published without proof the following remarkable theorem.

Theorem 1.1 (Droz-Farny [2]). If any pair of perpendicular lines Δ_1 and Δ_2 passes through the orthocenter H of a triangle ABC and meets the three lines containing the sides BC, CA, AB at A_1 and A_2; B_1 and B_2; C_1 and C_2 respectively, then the midpoints A_0, B_0, C_0 of the segments A_1A_2, B_1B_2, C_1C_2 are collinear.

The author of the first proof of Theorem 1.1 remains unknown. It is only known that in 1952, Simon T. Kao generalized the theorem as follows

Theorem 1.2 (Simon T. Kao [7]). If any pair of perpendicular lines Δ_1 and Δ_2 passes through the orthocenter H of a triangle ABC and meets the three lines containing BC, CA, AB at A_1 and A_2; B_1 and B_2; C_1 and C_2 respectively, then the points A_0, B_0, C_0 which divide the segments A_1A_2, B_1B_2, C_1C_2 in the same ratio are collinear.

Shortly after that, in 1953, two proofs of Theorem 1.2 were published [9]: an analytic proof by M. Perisastri and a projective proof by O. J. Ramler.

There are many proofs of Theorem1.1 [1, 3, 4, 5, 6, 8, 10, 11, 12], amongst which it is worth mentioning a purely synthetic proof of Jean-Loui Ayme published in 2004 in the Forum Geometricorum Journal [1]. In this paper, we present a purely synthetic proof of Theorem 1.2 using the notion of cross ratio.

As in [13], the signed distances from the point A to the point B denoted by $\overline{AB}$.

2. A purely synthetic proof of Theorem 1.2

In the case where triangle ABC is right-angled, Theorem 1.2 is obvious.

Now we shall consider the case where triangle ABC does not have a right angle. Let us go through the lemmas needed for proving this theorem.

Lemma 2.1. If any pair of perpendicular lines Δ_1 and Δ_2 pass through the orthocenter H of a triangle ABC and Δ_1 meets the two lines containing AB and AC at P and Q respectively; Δ_2 meets the line containing BC at M, then $\dfrac{\overline{MB}}{\overline{MC}} = \dfrac{\overline{HP}}{\overline{HQ}}$.

Proof. Let AE and HF be the lines passing through A, H and parallel to PQ and BC respectively (Figure 1).

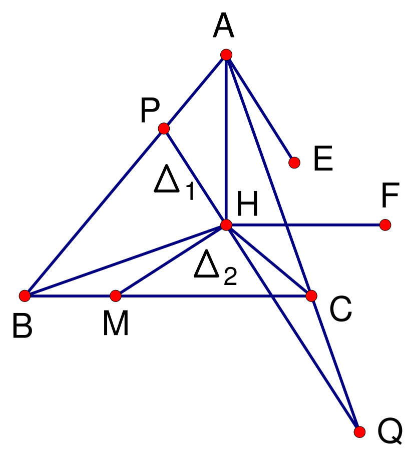
It is clear that

$$\frac{\overline{MB}}{\overline{MC}} = H(BCMF) \text{ and } \frac{\overline{HP}}{\overline{HQ}} = A(PQHE) = A(QPEH) \tag{2.1}$$

On the other hand, since HB, HC, HM and HF are perpendicular to AQ, AP, AE and AH respectively, we have

$$H(BCMF) = A(QPEH) \tag{2.2}$$

From (2.1) and (2.2), $\dfrac{\overline{MB}}{\overline{MC}} = \dfrac{\overline{HP}}{\overline{HQ}}$ (proven).



(Figure 1)

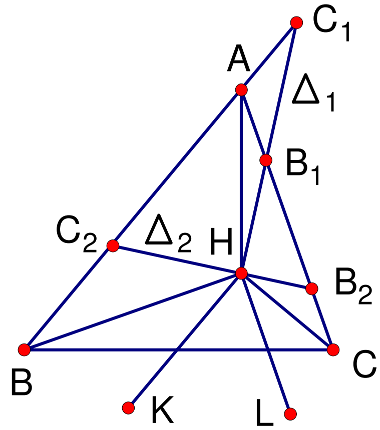
Lemma 2.2. If any pair of perpendicular lines Δ_1 and Δ_2 pass through the orthocenter H of a triangle ABC and meet the two lines containing AC and AB at B_1 and B_2; C_1 and C_2 respectively, then $\dfrac{\overline{BC_1}}{\overline{BC_2}} = \dfrac{\overline{CB_1}}{\overline{CB_2}}$.

Proof. Let HK and HL be the lines passing through H and parallel to AB and AC respectively (Figure 2).

It is clear that

$$\frac{\overline{BC_1}}{\overline{BC_2}} = H(C_1C_2BK) \text{ and } \frac{\overline{CB_1}}{\overline{CB_2}} = H(B_1B_2CL) \tag{2.3}$$

On the other hand, since HC_1, HC_2, HB and HK are perpendicular to HB_2, HB_1, HL and HC respectively, we have $H(C_1C_2BK) = H(B_2B_1LC)$.



(Figure 2)

From this, notice that $H(B_2B_1LC) = H(B_1B_2CL)$, we get

$$H(C_1C_2BK) = H(B_1B_2CL) \tag{2.4}$$

From (2.3) and (2.4), $\dfrac{\overline{BC_1}}{\overline{BC_2}} = \dfrac{\overline{CB_1}}{\overline{CB_2}}$ (proven).

Lemma 2.3. Let $A_1, B_1, C_1; A_2, B_2, C_2$ be two triplets of points which lie on the lines Δ_1 and Δ_2 respectively and satisfy the condition $\dfrac{\overline{B_1A_1}}{\overline{B_1C_1}} = \dfrac{\overline{B_2A_2}}{\overline{B_2C_2}}$. Let A_0, B_0, C_0 be the points that divide the segments A_1A_2, B_1B_2, C_1C_2 in the same ratio. Then A_0, B_0, C_0 are collinear.

Proof. We shall assume that Δ_1 and Δ_2 are not parallel, and omit the easy case when Δ_1 and Δ_2 are parallel.

Let K, L be the points on lines B_1A_0, B_1C_0 respectively such that

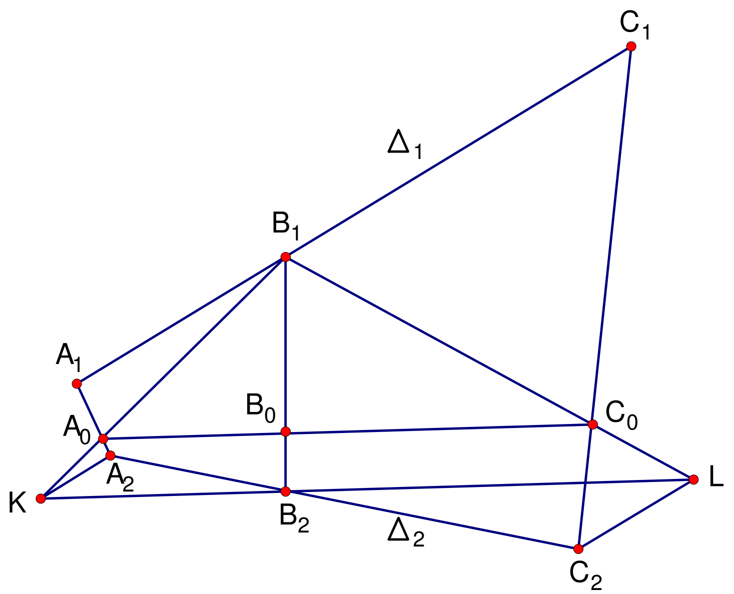
$$\frac{\overline{A_0B_1}}{\overline{A_0K}} = \frac{\overline{A_0A_1}}{\overline{A_0A_2}} = \frac{\overline{B_0B_1}}{\overline{B_0B_2}} = \frac{\overline{C_0C_1}}{\overline{C_0C_2}} = \frac{\overline{C_0B_1}}{\overline{C_0L}} \text{ (Figure 3)} \tag{2.5}$$

From (2.5), by Thales' Theorem

$$KA_2 \parallel A_1C_1 \parallel LC_2 \tag{2.6}$$

From (2.5) and (2.6), notice that $\dfrac{\overline{B_1A_1}}{\overline{B_1C_1}} = \dfrac{\overline{B_2A_2}}{\overline{B_2C_2}}$, by Thales' Theorem, we get

$$\frac{\overline{KA_2}}{\overline{LC_2}} = \frac{\overline{KA_2}}{\overline{B_1A_1}}.\frac{\overline{B_1A_1}}{\overline{B_1C_1}}.\frac{\overline{B_1C_1}}{\overline{LC_2}} = \frac{\overline{A_0A_2}}{\overline{A_0A_1}}.\frac{\overline{B_2A_2}}{\overline{B_2C_2}}.\frac{\overline{C_0C_1}}{\overline{C_0C_2}} = \frac{\overline{B_2A_2}}{\overline{B_2C_2}} \tag{2.7}$$



(Figure 3)

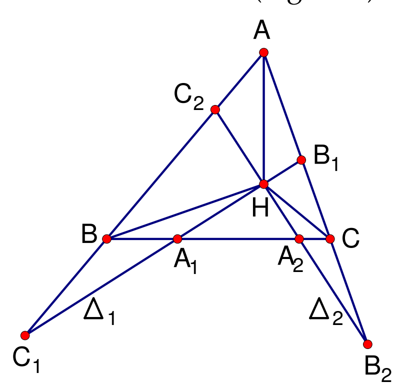
From (2.6) and (2.7), by Thales' Theorem we have

$$K, B_2, L \text{ are collinear.} \tag{2.8}$$

From (2.5) and (2.8), we easily deduce A_0, B_0, C_0 are collinear (proven).

Remark 2.4. If we use the concept of vector, we will have a shorter proof of lemma 2.3. However, that proof is not purely synthetic.

Now we return to our proof of Theorem 1.2 (Figure 4).



(Figure 4)

By Lemma 2.1, we have

$$\frac{\overline{A_1B_1}}{\overline{A_1C_1}} = \frac{\overline{HB_1} - \overline{HA_1}}{\overline{HC_1} - \overline{HA_1}} = \frac{\dfrac{\overline{HB_1}}{\overline{HA_1}} - 1}{\dfrac{\overline{HC_1}}{\overline{HA_1}} - 1} = \frac{\dfrac{\overline{C_2A}}{\overline{C_2B}} - 1}{\dfrac{\overline{B_2A}}{\overline{B_2C}} - 1} = \frac{\overline{C_2A} - \overline{C_2B}}{\overline{B_2A} - \overline{B_2C}} . \frac{\overline{B_2C}}{\overline{C_2B}} = \frac{\overline{AB}}{\overline{AC}} . \frac{\overline{CB_2}}{\overline{BC_2}} \tag{2.9}$$

Similarly

$$\frac{\overline{A_2B_2}}{\overline{A_2C_2}} = \frac{\overline{AB}}{\overline{AC}}.\frac{\overline{CB_1}}{\overline{BC_1}} \tag{2.10}$$

From (2.9) and (2.10), by Lemma 2.2, $\frac{\overline{A_1B_1}}{\overline{A_1C_1}} = \frac{\overline{A_2B_2}}{\overline{A_2C_2}}$.

Thus, by Lemma 2.3, A_0, B_0, C_0 are collinear (proven).

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