



## INEQUALITIES WITH GUDERMANNIANS IN HYPERBOLIC TRIANGLE

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ABSTRACT. In paper [3] we presented some inequalities for hyperbolic triangles. In this paper we will use this inequalities to obtain some new inequalities in hyperbolic triangle using the gudermannian function.

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### 1. INTRODUCTION

The gudermannian function relates the circular functions and the hyperbolic functions without using complex numbers.

The gudermannian function is defined by:

$$gd\ x = \int_0^x \frac{dt}{\cosh t} = \arcsin(\tanh x) = 2 \arctan(e^x) - \frac{1}{2}\pi \quad (1)$$

The inverse of the gudermannian function is defined on the interval  $-\frac{\pi}{2} < x < \frac{\pi}{2}$  is given by

$$gd^{-1}x = \int_0^x \frac{dt}{\cos t} = \ln \left| \frac{1 + \sin x}{\cos x} \right| \quad (2)$$

Some properties of gudermannian function are:

$$\sin(gd\ x) = \tanh x \quad (3)$$

$$\sec(gd\ x) = \cosh x \quad (4)$$

$$\tan(gd\ x) = \sinh x \quad (5)$$

$$\csc(gd\ x) = \coth x \quad (6)$$

For more details on gudermannian function please see [1] and [2].

Next, let us consider the expansion in Mac Laurin series of the following functions:

$$gd\ x = x - \frac{1}{3}x^3 + \frac{1}{24}x^5 - \frac{61}{5040}x^7 + \dots \quad (7)$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots \quad (8)$$

$$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \dots \quad (9)$$

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In paper [1], we obtain the following results:

$$\cosh a + \cosh b \leq 2 \left( \cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2} \right) \quad (10)$$

$$3 \cosh \frac{a+b+c}{3} \leq \frac{\cosh a + \cosh b + \cosh c}{3} \quad (11)$$

$$\cosh c \leq \cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2} \quad (12)$$

$$\frac{\sinh a}{a} + \frac{\sinh b}{b} < \frac{4}{3} + \frac{2}{3} \left( \cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2} \right) \quad (13)$$

$$6 \sinh \frac{a+b+c}{6} \leq \sinh a + \sinh b + \sinh c \quad (14)$$

In this paper we will work with this relations ((10)-(14)) and using the first terms of the expansions (7), (8) and (9) we will obtain new inequalities for hyperbolic triangle using the gudermannian function.

## 2. MAIN RESULTS

**Theorem 2.1** *In any hyperbolic triangle, the following inequality hold:*

$$(gd a)^2 + (gd b)^2 \leq 2 \left( \cosh^2 \frac{a}{2} + \sinh^2 \frac{b}{2} \right) \quad (15)$$

*Proof.* Using (4), the inequation (10) became:

$$\sec(gd a) + \sec(gd b) \leq 2 \left( \cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2} \right) \quad (16)$$

In relation (9) if we replace  $x$  with the function  $gd x$  then one obtains:

$$\sec(gd x) = 1 + \frac{1}{2}(gd x)^2 + \frac{5}{24}(gd x)^4 + \frac{61}{720}(gd x)^6 + \dots \quad (17)$$

Using (16) and the first two terms of (17), we get:

$$1 + \frac{1}{2}(gd a)^2 + 1 + \frac{1}{2}(gd b)^2 \leq 2 \left( \cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2} \right) \quad (18)$$

After easy computations in inequation (18) we get:

$$\frac{1}{2}(gd a)^2 + \frac{1}{2}(gd b)^2 \leq 2 \left( \cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2} - 1 \right)$$

And then, because  $\cosh^2 \frac{b}{2} - 1 = \sinh^2 \frac{b}{2}$ , one obtains the assertion of the theorem. ■

**Theorem 2.2** *In any hyperbolic triangle, the following inequality hold:*

$$\left( gd \frac{a+b+c}{3} \right)^2 \leq \frac{2(\cosh a + \cosh b + \cosh c) - 18}{9} \quad (19)$$

*Proof.* Using (4), the inequation (11) became:

$$\left( \sec \left( gd \frac{a+b+c}{3} \right) \right) \leq \frac{\cosh a + \cosh b + \cosh c}{9} \quad (20)$$

Using first two terms of (17) one obtains:

$$1 + \frac{1}{2} \left( gd \frac{a+b+c}{3} \right)^2 \leq \frac{\cosh a + \cosh b + \cosh c}{9} \quad (21)$$

After easy computations in inequation (21) we obtains:

$$\frac{1}{2} \left( gd \frac{a+b+c}{3} \right)^2 \leq \frac{\cosh a + \cosh b + \cosh c}{9} - 1 \quad (22)$$

Multiplying relation (22) with 2, one obtains the assertion of the theorem . ■

**Theorem 2.3** *In any hyperbolic triangle, the following inequality hold:*

$$(gd c)^2 \leq 2 \left( \cosh^2 \frac{a}{2} + \sinh^2 \frac{b}{2} \right) \quad (23)$$

*Proof.* Using (4), the inequation (12) became:

$$\sec(gd c) \leq \cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2} \quad (24)$$

Taking first two terms of (17) and replacing in (24), one obtains:

$$1 + \frac{1}{2} (gd c)^2 \leq \cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2} \quad (25)$$

We know that  $\cosh^2 \frac{b}{2} - 1 = \sinh^2 \frac{b}{2}$ .

Using this and also after easy computations one obtains the inequality (23) and in this way the proof of theorem is done.

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**Theorem 2.4** *In any hyperbolic triangle, the following inequality hold:*

$$\frac{gd a}{a} + \frac{gd b}{b} + \frac{(gd a)^3}{3a} + \frac{(gd b)^3}{3b} \leq \frac{4}{3} + \frac{2}{3} \left( \cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2} \right) \quad (26)$$

*Proof.* Using (5), the inequation (13) became:

$$\frac{\tan(gda)}{a} + \frac{\tan(gdb)}{b} < \frac{4}{3} + \frac{2}{3} \left( \cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2} \right) \quad (27)$$

Using the first two terms of the expansion in Mac Laurin functions

$$\tan(gdx) = gdx + \frac{1}{3} (gdx)^3 + \frac{2}{15} (gdx)^5 + \frac{17}{315} (gdx)^7 + \dots \quad (28)$$

and replacing in (27), one obtains:

$$\frac{gd a}{a} + \frac{gd b}{b} + \frac{(gd a)^3}{3a} + \frac{(gd b)^3}{3b} \leq \frac{4}{3} + \frac{2}{3} \left( \cosh^2 \frac{a}{2} + \cosh^2 \frac{b}{2} \right) \quad (29)$$

After easy computation in (28), we obtains the inequality (26), so the theorem is proved. ■

**Theorem 2.5** *In any hyperbolic triangle, the following inequality hold:*

$$6 \left( gd \frac{a+b+c}{6} \right) + 2 \left( gd \frac{a+b+c}{6} \right)^3 \leq \sinh a + \sinh b + \sinh c \quad (30)$$

*Proof.* Using (14) and (28), we get:

$$6 \left( \left( gd \frac{a+b+c}{6} \right) + \frac{1}{3} \left( gd \frac{a+b+c}{6} \right)^3 \right) \leq \sinh a + \sinh b + \sinh c \quad (31)$$

After easy computation in (31) we get the assertion of the theorem. ■

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